On Arithmetic Fundamental Lemmas and Arithmetic Transfers

In honor of Professor Shouwu Zhang on the occasion of his 60th birthday

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Schedule

Schedule

Gross–Zagier–Zhang formula: origins

- A elliptic curve over $\mathbb{Q}$ with conductor N. Then $N \geq 11$. Choose a modular uniformization $\varphi : X_0(N) \to A$ over $\mathbb{Q}$.
- $F = \mathbb{Q}(\sqrt{-d})$ an imaginary quadratic field with odd discriminant.
- Heegner condition: assume there is an ideal $\mathfrak{N}$ of O_F s.t. $O_F/\mathfrak{N} = \mathbb{Z}/N\mathbb{Z}$. This implies $L(A_F, 1) = 0$.

We get a Heegner point

$$P_F := (\mathbb{C}/O_F \to \mathbb{C}/\mathfrak{N}^{-1}) \in X_0(N)(H_F),$$

$$P_{F,A} := \mathrm{Tr}_{H_F/F}\varphi(P_F) \in A(F).$$

Theorem (Gross—Zagier, 1980s)

$$L'(A_F, 1) \simeq \langle P_{F,A}, P_{F,A} \rangle_{\mathrm{NT}}.$$

Gross–Zagier–Zhang formula: use representation theory

- Uniformization — the automorphic representation π of E appears in the cohomology of $X_0(N)$.
- Heegner point — the embedding $G^\flat = \mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m \to G = \mathrm{GL}_2$ gives an embedding of Shimura varieties $M^\flat \to M = X_0(N)$ over F.
- Neron–Tate height — arithmetic intersection of 0-cycles on curves.

Reformulation and generalization to Shimura curves over totally real fields:

Gross–Zagier–Zhang formula

Let F/F_0 be any CM quadratic extension of a totally real number field.

Theorem (The Gross-Zagier-Zhang formula, S. Zhang, YZZ)

Let $\pi \in \mathcal{A}(\mathrm{GL}_{2,F_0})$ be a Hilbert modular form of parallel weight 2 appearing in the cohomology of a Shimura curve $M = \mathrm{Sh}_B$ over F_0. Assume Hom$_{\mathbb{A}_F^\times}(\pi_{\mathbb{B}}, 1) \neq 0$ (non-zero periods), then $L(\pi_F, \frac{1}{2}) = 0$ and

$$L'(\pi_F, \frac{1}{2}) \simeq \left\langle M_F^\flat, M_F^\flat \right\rangle_{M,\mathrm{NT},\pi_f}.$$

The local Hom $\neq 0$ is precisely described (Saito–Tunnell) and $\dim$ Hom ≤ 1.

Applications

Applications:

- Let f be any new form of weight 2, then $L'(f,1) \geq 0$ (non-negativity predicted by RH).
- For $\sigma \in \mathrm{Gal}(\mathbb{Q}^{\mathsf{alg}}/\mathbb{Q})$, $\mathrm{ord}_{s=1}L(f,s) = 1 \iff \mathrm{ord}_{s=1}L(f^\sigma,s) = 1$.
- BSD conjecture for (principal polarized) modular GL_2-type abelian variety A/F_0 when $r_{\mathrm{an}}(A) \leq 1$.
- Gauss class number problem.
- ...

Similar formulas for modular forms of weight $2k \geq 2$ is also discovered by S. Zhang on modular curves.
Generalizations to Shimura varieties?

Unitary Shimura varieties and GGP cycles

- A hermitian space V over F/F_0, with sign $(n,1)_{\varphi_0}, (n+1,0)_{\varphi\neq\varphi_0}$.
- A decomposition $V = V^\flat \oplus Fe$ with $(e,e) = 1$.

We have an embedding of unitary Shimura varieties over F

$$M^\flat = \mathrm{Sh}_{\mathrm{U}(V^\flat),K^\flat} \to M = \mathrm{Sh}_{\mathrm{U}(V),K}.$$

They are related to moduli spaces of abelian varieties with compatible polarizations, O_F-actions and level structures.

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Consider the diagonal cycle

$$\Delta : M^\flat \to M^\flat \times M.$$

As $\dim M^\flat = n-1 = (\dim(M^\flat \times M)-1)/2$, we can take arithmetic self-intersection of Δ inside $M^\flat \times M$ (Beilinson–Bloch height pairing).

The arithmetic Gan–Gross–Prasad (AGGP) conjecture

Conjecture (AGGP conjecture for unitary groups)

For a tempered cuspidal automorphic representation π *of* $\mathrm{U}(V^\flat) \times \mathrm{U}(V)$ *appearing in the cohomology of* $M^\flat \times M$ *with* $\mathrm{Hom}_{\mathrm{U}(\mathbb{V}^\flat)(\mathbb{A}_{F_0})}(\pi, 1) \neq 0$ *(in particular* $L(\pi, \frac{1}{2}) = 0$*), we have*

$$L'(\pi, \frac{1}{2}) \simeq \left\langle M_0^\flat, M_0^\flat \right\rangle_{M^\flat \times M, \mathrm{BB}, \pi_f}.$$

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- It's better to use the language of incoherent Shimura varieties for nearby $\mathbb{V} > 0$ (Milne-Shih, Gross, Talyor-Semplincr).
- $(-)_0$ is a Hecke type modification to cohomological trivial cycles.
- The condition local Hom $\neq 0$ is precisely described by local GGP conjectures and $\dim$ Hom ≤ 1.

Application: Beilinson–Bloch–Kato conjecture of hermitian Rankin—Selberg motives of $\mathrm{GL}_n \times \mathrm{GL}_{n+1}$ when $r_{\mathrm{an}} \leq 1$ ([LTXZZ, DZ]).

The arithmetic fundamental lemma (AFL)

Under the relative trace formula (RTF) approach of W. Zhang, **the AGGP conjecture is reduced to its analogs over local fields ($\mathbb{Q}_p, \mathbb{R}$)**:

- AFL and AT on *Rapoport–Zink spaces* $\mathcal{N}$ (formal schemes over $\mathbb{Z}_p$).

Theorem (AFL, W. Zhang ($p \geq n+1$), Z. (any p))

For regular semi-simple elements $\gamma \in S_{n+1} \leftrightarrow g \in \mathrm{U}(\mathbb{V}_{n+1})$,

$$\partial\mathrm{Orb}(\gamma, 1_{S_{n+1}(\mathbb{Z}_p)}) = -\mathrm{Int}(\mathcal{N}_n, (1 \times g)\mathcal{N}_n)_{\mathcal{N}_n \times \mathcal{N}_{n+1}} \log p.$$

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- The nearby unitary group $\mathrm{U}(\mathbb{V}_n)$ acts on $\mathcal{N}_n$.
- $\Delta_p : \mathcal{N}_n \to \mathcal{N}_n \times \mathcal{N}_{n+1}$ is the local analog of $M^\flat \to M^\flat \times M$ over good primes.
- For orbital integrals, we consider the action of $\mathrm{GL}_n(\mathbb{Q}_p)$ on the symmetric space $S_{n+1} = \mathrm{GL}_{n+1}(\mathbb{Q}_{p^2})/\mathrm{GL}_{n+1}(\mathbb{Q}_p)$.
- We have AFLs over any p-adic field $F_v/F_{0,v}$ (Mihatsch-Zhang, Z.).

Arithmetic transfers (AT)

For applications in practice, it is natural and necessary to consider ramification and formulate similar **AT identities**.

- Let V be any $\mathbb{Q}_{p^2}/\mathbb{Q}_p$ hermitian space of dimension $n+1$.
- Let L be a hermitian lattice in V such that $p(L^\vee/L) = 0$.
- A decomposition $L = L^\flat \oplus \mathbb{Z}_{p^2} e$.

We have the local diagonal cycle $\Delta_{L^\flat \to L} : \mathcal{N}_{L^\flat} \to \mathcal{N}_{L^\flat} \times \mathcal{N}_L$.

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Theorem ($\mathrm{AT}(L^\flat \to L)$, Z.)

$$\partial \mathrm{Orb}(\gamma, 1_{S(L,L^\vee)}) = -\widetilde{\mathrm{Int}}(\mathcal{N}_{L^\flat}, (1\times g)\mathcal{N}_{L^\flat})_{\mathcal{N}_{L^\flat}\times \mathcal{N}_L} \log p.$$

New: $\mathcal{N}_{L^\flat} \times \mathcal{N}_L$ is singular. To define $\widetilde{\mathrm{Int}}$, we use a small resolution and do derived intersection of strict transforms of cycles in K-groups.

Main goals today

- I will recall the twisted GGP conjectures and formulate an arithmetic analog for the twisted diagonal cycle (for real quadratic algebra E_0/F_0)
$$\Delta^{E_0} : X \to X \times^{E_0} X.$$
If $E_0 = F_0 \times F_0$ is split, it recovers Y. Liu's RTF approach to Fourier–Jacobi AGGP conjectures.
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- I will formulate the twisted AFL (+ some ATs) in a uniform way.
- The new geometric side is very rich. I will introduce new **mirabolic special divisors** ("quantization" of RZ spaces for non-reductive groups). Via pullbacks, they recover the notion of Kulda–Rapoport cycles. I will introduce Galois involutions of RZ spaces.
- I will prove the twisted AFL by globalization. FL and some transfers could be formulated and proved similarly.

Schedule

1. Beilinson–Bloch conjecture and Bloch–Kato conjecture
2. Gross–Zagier–Zhang formula and AGGP conjectures
3. Twisted GGP and Twisted AGGP conjectures: $X \to X \times^{E_0} X$
4. Twisted AFL, AT and mirabolic special divisors

Twisted GGP conjecture: GL case

- Let $E/F, K/F$ be two quadratic extensions of number fields.
- Let μ be a Hecke character of $\mathbb{A}_E^\times$ s.t. $\mu|_{\mathbb{A}_F^\times} = \eta_{E/F}$.
- Let V be a hermitian space over E/F, and $\omega_V = \omega_{V,\psi,\mu}$ be the Weil representation of $\mathrm{U}(V)(\mathbb{A}_F)$.

Assume $E = K$. Let Π be a cuspidal automorphic representation of $\mathrm{GL}(V)(\mathbb{A}_F)$. We consider the period integral

$$\mathcal{P}_V(f,\phi) = \int_{\mathrm{U}(V)} f(g)\bar{\phi}(g)dg,\ \ f \in \Pi,\ \phi \in \omega_V.$$

Assume $\mathrm{Hom}_{\mathrm{U}(V_v)}(\Pi_v, \omega_{V_v}) \neq 0$ for all places v of F, which is precisely described by local twisted GGP conjectures.

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Conjecture (Twisted GGP conjecture)

The following things are equivalent.

1. $P_V \neq 0.$
2. *The central Rankin–Selberg L-value* $L(\frac{1}{2}, \Pi \times {}^\sigma\Pi^\vee \times \mu^{-1}) \neq 0.$

RTF approach to twisted GGP conjecture

Theorem (D. Wang)

Assume V is split, E/F is unramified and splits at infinity, Π_v is unramified over non-split places of E/F and is supercuspidal for at least two split places of E/F. Then the twisted GGP conjecture is true.

The (refined) general case (allowing general K/F, ramifications) is a joint work in progress (Lu-Wang-Z.). We have a RTF approach.

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$$G' = \mathrm{Res}\,\mathrm{GL}_{n,E\otimes_F K}, H_1 = \mathrm{Res}\,\mathrm{GL}_{n,E}, H_2 = \mathrm{Res}\,\mathrm{GL}_{n,K}.$$

For $f' \in \mathcal{S}(G'(\mathbb{A}_F)), \phi' \in \mathcal{S}(\mathbb{A}_{E^n})$, we study the global distribution

$$\mathcal{J}(f', \phi', s) := \int_{[H_1]} \int_{[H_2]} K_{f'}(h_1, h_2)\overline{\Theta_\mu(h_1, \phi')}dh_1 dh_2,\ s \in \mathbb{C}.$$

$K_{f'}(x, y) = \sum_{\gamma\in G(F)} f'(x^{-1}\gamma y)$, $\Theta_\mu(h_1, \phi') = \sum_{v\in E^n} \mu(h_1)|h_1|^s \phi'(vh_1)$.

How do you twist the diagonal cycle?

Given a smooth proper variety X over a field F. We have the diagonal cycle

$$\Delta : X \to X \times X.$$

We know $\chi(X) = \langle \Delta, \Delta \rangle_{X \times X}$. Let E/F be a (quadratic) extension. We expect a twisted diagonal cycle (at least over E):

$$\Delta^E : X \to X \times^E X?$$

Idea: take Weil restriction. If $E = F \times F$, we obtain the diagonal cycle. For Shimura varieties, we consider the twisted diagonal cycle

$$\mathrm{Sh}_G \to \mathrm{Sh}_{\mathrm{Res}_{E/F} G}$$

Its cohomology class is related to the plectic conjecture.

Twisted AGGP conjecture: cycles

- Let $F_0/\mathbb{Q}$ be totally real. Let F/F_0 (resp. E_0/F_0) be a CM quadratic (resp. real quadratic) extension.
- Choose $\mu : \mathbb{A}_F^\times \to \mathbb{C}$ lifting η_{F/F_0} of weight one ($\mu_\infty(z) = \sqrt{z\bar{z}}/z$).
- A hermitian space V over F/F_0, with sign $(n-1,1)_{\varphi_0}, (n,0)_{\varphi \neq \varphi_0}$.

We introduce **the twisted diagonal cycle** over the CM field $E = FE_0$:

$$\Delta_K : X = \mathrm{Sh}_{\mathrm{U}(V),K} \to X \times^{E_0} X := \mathrm{Sh}_{\mathrm{U}(V_{E_0}),K_{E_0}}.$$

$$\dim(X) = n-1 = \dim(X \times^{E_0} X)/2.$$

Twisted AGGP conjecture: special divisors and distributions

For $\phi \in \mathcal{S}(\mathrm{U}(V)(\mathbb{A}_f))^K$, we have Kudla's generating series of special divisors associated to ϕ:

$$Z_\phi \to \mathrm{Sh}_{\mathrm{U}(V),K}.$$

For $F \in \mathcal{H}_{K_{E_0}} = \mathcal{S}(K_{E_0} \backslash \mathrm{U}(V)(\mathbb{A}_{E_0,f})/K_{E_0})$, we consider the Hecke translation of the twisted diagonal cycle $R(F) * X$.
We are interested in the arithmetic functional

$$\mathcal{I}(F,\phi) := \langle (R(F) * X)_0, (Z_\phi)_0 \rangle_{X \times^{E_0} X, \mathrm{BB}}.$$

$(-)_0$ are certain odd Hecke type projectors.

Twisted AGGP conjecture

Let Π be a tempered cuspidal automorphic representation of $\mathrm{U}(V)(\mathbb{A}_{E_0})$ appearing in the cohomology of $X \times^{E_0} X$.
Consider the twisted Asai automorphic L-function

$$L(\Pi, s) := L(s, \Pi, As^+ \otimes \mu^{-1}).$$

$(\mathrm{Ind}_{WD_E}^{WD_F}(M \otimes ({}^{\sigma_{E/F}}M)) = As^+ \oplus As^-)$.
Assume $\mathrm{Hom}_{\mathrm{U}(\mathbb{V})(\mathbb{A}_{F_0})}(\pi, \omega_{\mathbb{V}}) \neq 0$. By local twisted GGP conjectures, we expect $L(\Pi, \frac{1}{2}) = 0$.

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Assume $\mathrm{Hom}_{\mathrm{U}(\mathbb{V})(\mathbb{A}_{F_0})}(\pi, \omega_{\mathbb{V}}) \neq 0$. By local twisted GGP conjectures, we expect $L(\Pi, \frac{1}{2}) = 0$.

Conjecture (Twisted AGGP)

$\mathcal{I}(F, \phi) \neq 0$ for some $(F, \phi) \in \Pi_f^{K_{E_0}} \times \mathcal{S}(\mathrm{U}(V)(\mathbb{A}_f)^K)$ if and only if $L'(\Pi, \frac{1}{2}) \neq 0$.

A refined version is work in progress. If $K = F_0 \times F_0$, then we recover Y. Liu's refined Fourier–Jacobi AGGP conjectures.

Twisted Fourier–Jacobi cycles and a RTF approach

As in [Liu], consider the Albanase A_X of X. The character μ gives an abelian variety A_μ with CM by a number field M_μ.
For $F \in \mathcal{H}_{K_{E_0}}$ and $\phi : A_X \to A_\mu$, we introduce **the twisted Fourier–Jacobi cycle**

$$\mathrm{FJ}^K(F, \phi) := R(F) * X \times \phi(A_X) \to X \times^{E_0} X \times A_\mu.$$

These cycles are expected to bound Selmer groups for the automorphic motive associated to $As_{E/F}(\Pi) \otimes \mu^{-1}$.
We may reformulate the arithmetic intersection problem using twisted Fourier–Jacobi cycles.
By a RTF approach, we may reduce the twisted AGGP conjecture to local twisted AFL and ATs.

Schedule

Quadratic extensions

Let p be an odd prime. Let F/F_0 be an unramified quadratic extension of p-adic local fields. Let K/F_0 be a quadratic extension of F_0.

- By Galois theory, there is a third quadratic extension F'/F_0 inside $F \otimes_{F_0} K$ fixed by $\sigma_{F/F_0} \otimes \sigma_{K/F_0}$.
- (Split case) If $K = F_0 \times F_0$, then $F' \cong F$ is the conjugated diagonal F inside $F \otimes_{F_0} K \cong F \times F$.
- (GL case) If $K = F$, then $F' \cong F_0 \times F_0$.

The analytic side: spaces and orbits

Consider **the symmetric space**

$$S_{n,F'} = \{\gamma \in \mathrm{GL}_n(F') | \gamma \sigma_{F'/F_0}(\gamma) = id\}.$$

- (Split case $K = F_0 \times F_0$) $S_{n,F'} = S_n$.
- (GL case $K = F$) $S_{n,F'} \cong \mathrm{GL}_n(F_0)$.

Let $V_0' = F_0^n \times (F_0^n)^*$. We consider the natural action of $h \in \mathrm{GL}_n(F_0)$ on

$$(\gamma, u') \in S_{n,F'} \times V_0'.$$

The analytic side: orbital integrals

Consider $f' \in \mathcal{S}(S_{n,F'}), \phi' \in \mathcal{S}(V_0')$. Consider a regular semi-simple orbit $(\gamma, u') \in S_{n,F'} \times V_0'$.

Definition (Derived orbital integral)

For $s \in \mathbb{C}$, define $\mathrm{Orb}(\gamma, u', f' \otimes \phi', s)$ as

$$\int_{h \in \mathrm{GL}_n(F_0)} f'(h^{-1}\gamma h)\phi'(h^{-1}.u')(-1)^{\mathrm{val}(\det h)}|\det h|^s dh.$$

Let $F'_{\mathrm{std}} := 1_{S_{n,F'}(O_{F_0})} \times 1_{O_{V_0'}}$. The fundamental derived orbital integral is

$$\partial\mathrm{Orb}((\gamma, u'), F'_{\mathrm{std}}) = \omega(\gamma, u')\frac{d}{ds}|_{s=0}\mathrm{Orb}((\gamma, u'), F'_{\mathrm{std}}, s).$$

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Here we normalize the Haar measure such that $\mathrm{GL}_n(O_{F_0})$ has volume 1. And $\omega(\gamma, u') \in \{\pm 1\}$ is a transfer factor.

Unitary side: $\mathrm{U}(V) \to \mathrm{U}(V_K)$

Let V be the split hermitian space over F/F_0 of dimension n. Set $V_K := V \otimes_{F_0} K$. Consider the hermitian symmetric space

$$\mathrm{U}(V_K)^- := \{g \in \mathrm{U}(V_K) | g\sigma_{KF/F}(g) = \mathrm{id.}\}$$

We consider the natural action of $h \in \mathrm{U}(V)$ on

$$(g, u) \in \mathrm{U}(V_K)^- \times V.$$

- (Split case $K = F_0 \times F_0$) $\mathrm{U}(V_K) = \mathrm{U}(V) \times \mathrm{U}(V), \mathrm{U}(V_K)^- \cong \mathrm{U}(V)$.
- (GL case $K = F$) $\mathrm{U}(V_K) = \mathrm{GL}(V), \mathrm{U}(V_K)^- = \mathrm{Herm}^{split}(V)$.

Fundamental Lemmas

There exists a matching of semi-simple orbits

$$\mathrm{GL}_n(F_0)\backslash[S_{n,F'} \times V_0'] \leftrightarrow (\mathrm{U}(V)\backslash[\mathrm{U}(V_K)^- \times V]) \coprod (\mathrm{U}(\mathbb{V})\backslash[\mathrm{U}(\mathbb{V}_K)^- \times \mathbb{V}]).$$

Choose a self-dual lattice L in V. Set $F_{\mathrm{std}} = 1_{\mathrm{U}(L\otimes O_K)^-} \times 1_L$. In the RTF approach, we reduce related matchings of orbital integrals at good places (by D. Wang when $E = K$) to following fundamental lemmas.

Fundamental Lemmas

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Theorem (FL)

For any regular semisimple $(g,u)\in \mathrm{U}(V_K)^-\times V$ *matching* (γ,u'), *we have*

$$\mathrm{Orb}((\gamma,u'),F'_{\mathrm{std}}) = \mathrm{Orb}((g,u),F_{\mathrm{std}}).$$

If regular semi-simple (γ,u') matches $(g,u)\in \mathrm{U}(\mathbb{V}_K)^-\times\mathbb{V}$, then $\mathrm{Orb}((\gamma,u'),F'_{\mathrm{std}})=0$. How about $\partial\mathrm{Orb}((\gamma,u'),F'_{\mathrm{std}})$?

Objects: Basic triples

For any $O_{\breve{F}_0}$-scheme S where p is locally nilpotent, consider triples (X, ι, λ) over S where

- X is a formal p-divisible O_{F_0}-module of height $2n$ and dimension n over S.
- $\iota : O_F \to \mathrm{End}(X)$ is an action satisfying *Kottwitz condition* of signature $(n-1, 1)$ on Lie$X$.
- $\lambda : X \cong X^\vee$ is a O_F-linear principal polarization.

There is a basic triple (unique up to quasi-isogeny) over $\mathbb{F}_p^{\mathrm{alg}}$ (e.g. from supersingular elliptic curves). Choose one and call it the framing triple $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$.

Unitary RZ space and divisors

The **unitary Rapoport-Zink space**

$$\mathcal{N}_n \to \operatorname{Spf} O_{\breve{F}_0}$$

is the moduli space classifying basic triples $(X, \iota, \lambda, \rho)$ with a quasi-isogeny $\rho : X \to \mathbb{X}$ over the reduction $S \otimes \mathbb{F}_p^{\mathsf{alg}}$.

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Consider the baisc triple $\mathbb{E}$ of signature $(1, 0)$, which has a canonical lifting $\mathcal{E}$. We have $\mathbb{V} = \mathrm{Hom}^{\circ}_{O_F}(\mathbb{E}, \mathbb{X})$.
The unitary group $g \in \mathrm{U}(\mathbb{V})(F_0) \cong \mathrm{Aut}^{\circ}(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ acts on $\mathcal{N}_n$ by changing the framing ρ.

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The **unitary Rapoport-Zink space**

$$\mathcal{N}_n \to \operatorname{Spf} O_{\breve{F}_0}$$

is the moduli space classifying basic triples $(X, \iota, \lambda, \rho)$ with a quasi-isogeny $\rho : X \to \mathbb{X}$ over the reduction $S \otimes \mathbb{F}_p^{\mathrm{alg}}$.
Consider the baisc triple $\mathbb{E}$ of signature $(1, 0)$, which has a canonical lifting $\mathcal{E}$. We have $\mathbb{V} = \mathrm{Hom}^{\circ}_{O_F}(\mathbb{E}, \mathbb{X})$.
The unitary group $g \in \mathrm{U}(\mathbb{V})(F_0) \cong \mathrm{Aut}^{\circ}(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ acts on $\mathcal{N}_n$ by changing the framing ρ.
For $u \in \mathbb{V} - 0$, we have the **Kudla–Rapoport divisor**

$$\mathcal{Z}(u) \to \mathcal{N}_n$$

as the lifting locus of $u : \mathbb{E} \to \mathbb{X}$ to $\mathcal{E} \to X$.

Special divisors and twisted AFL

Similar to global situation, we have the twisted Rapoport–Zink space with $\mathrm{U}(\mathbb{V}_K)$-action

$$\mathcal{N}_{n,K} \to \operatorname{Spf} O_{\breve{K}}$$

parameterizing formal O_K-modules with O_{KF}-actions, O_{KF}/O_K-linear polarization and $(-) \otimes O_K$-signature $(1, n-1)$.
$\mathcal{N}_n$ (resp. $\mathcal{N}_{n,K}$) is formally smooth over $\operatorname{Spf} O_{\breve{K}}$ of dimension $n-1$ (resp. $2n-2$) with natural action of $\mathrm{U}(\mathbb{V})$ (resp. $\mathrm{U}(\mathbb{V}_K)$). Consider the twisted diagonal cycle

$$\Delta^K = \mathcal{N}_n \to \mathcal{N}_{n,K}.$$

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Theorem (Twisted AFL, Z.)

Assume K/F_0 is unramified. For any regular semisimple $(g, u) \in \mathrm{U}(\mathbb{V})^- \times \mathbb{V}$, matching (γ, u'), we have

$$\partial \mathrm{Orb}((\gamma, u'), F'_{\mathrm{std}}) = -\mathrm{Int}(g\mathcal{N}_n, \mathcal{Z}(u))_{\mathcal{N}_{n,K}} \log q.$$

The embedding when $K = F$

- We may use the twisted AFL ($+$ AT) to establish the twisted AGGP conjectures when the global quadratic extensions are unramified.
- If $K = F_0 \times F_0$, then $\mathcal{N}_{n,K} = \mathcal{N}_n \times \mathcal{N}_n$ and the twisted AFL recovers the AFL related to original AGGP conjectures.
- We also formulate the twisted AFL when K/F_0 is ramified, which is a work in progress.

We now discuss new constructions on Rapoport–Zink spaces in the case $K = F$, which in particular leads to the proof of twisted AFL.
From now on $K = F$. Then $\mathcal{M}_n := \mathcal{N}_{n,F}$ is the moduli of basic triples (X, ι, ρ) (forget the data of polarization). The twisted diagonal cycle is the forgetful map

$$\mathcal{N}_n \to \mathcal{M}_n.$$

Involution on RZ spaces

Consider the polarization $\lambda_{\mathbb{X}} : \mathbb{X} \to \mathbb{X}^\vee$. We introduce an involution

$$\sigma : \mathcal{M}_n \to \mathcal{M}_n$$

$$(X, \iota, \rho) \to (X^\vee, \sigma_{F/F_0} \circ \iota^\vee, \lambda_{\mathbb{X}}^{-1} \circ (\rho^\vee)^{-1} : X^\vee \to \mathbb{X})$$

Proposition

The action of $\mathrm{U}(\mathbb{V}_K)$ on $\mathcal{M}_n$ is σ-equivariant. And $\mathcal{N}_n$ is the derived fixed locus of σ

$$\mathcal{M}_n^{\sigma=1} = \mathcal{N}_n.$$

We have similar constructions for Weil restrictions of general RZ spaces.

New special divisors on RZ spaces

We introduce new special divisors on $\mathcal{M}_n$ which doesn't exist globally. They are related to RZ spaces of (non-reductive) mirabolic subgroups of GL_n.

$$\mathbb{V} = \mathrm{Hom}^{\circ}_{O_F}(\mathbb{E}, \mathbb{X}), \mathbb{V}^* = \mathrm{Hom}^{\circ}_{O_F}(\mathbb{X}, \mathbb{E}).$$

Definition (Mirabolic special divisors)

- For non-zero $u \in \mathbb{V}$, $\mathcal{Z}(u)$ is the lifting locus of $u : \mathbb{E} \to \mathbb{X}$.
- For non-zero $u^* \in \mathbb{V}^*$, $\mathcal{Z}^*(u^*)$ is the lifting locus of $u^* : \mathbb{X} \to \mathbb{E}$.

Basic geometry of mirabolic special divisors

Proposition (Z.)

- $\mathcal{Z}(u)$ *and* $\mathcal{Z}^*(u^*)$ *are Cartier divisors on* $\mathcal{M}_n$.
- *(The pullback formula)* $\mathcal{Z}(u)|_{\mathcal{N}_n} = \mathcal{Z}(u)$.
- *We have the duality*

$$\sigma(\mathcal{Z}^*(u^*)) = \mathcal{Z}(u), \; \lambda_{\mathbb{X}}^{-1} \circ (u^*)^\vee \circ \lambda_{\mathbb{E}} = u.$$

- *If* $(u, u^*) = 1$, *then* $\mathcal{Z}(u) \cap \mathcal{Z}(u^*) \cong \mathcal{M}_{n-1}$.

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- Using these new divisors, we show a key reduction of the twisted AFL.
- We expect a twisted local arithmetic theta correspondence using these cycles. We could use them to reformulate (more) KR conjectures.
- Motivated by relative Langlands, we hope to develop a general theory of parabolic version RZ spaces. See also the mirabolic Satake equivalence.

Proof: a global method via modularity

- We globalize the local quadratic extensions to global quadratic extensions $F/F_0, E_0/F_0$.
- We globalize the difference of geometric side and the analytic side to a Hilbert automorphic form Dif. On analytic side, we consider

$$\mathbb{J}_\alpha(\Phi', s) := \int_{\mathrm{GL}_n(\mathbb{A}_F)} \sum_{(\gamma, u') \in S_{n,F'}(\alpha) \times V_0'} \Phi'(g^{-1}.(\gamma, u'))|g|_{F_0}^s \eta(g) dg.$$

Using modularity and vanishing of primitive Fourier coefficients, we show $Dif = 0$.
By explicit computations, we verify the twisted AFL if $n \leq 2$ or (g, u) is simple.
By the twisted FL and non-vanishing of local orbital integrals away from the distinguished local places, we obtain the desired twisted AFL from $Dif = 0$.

Happy birthday, Shouwu!