Twisted period integrals and applications

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Goal of this talk

The goal is to understand the following questions together.

- Why do we study twisted period integrals of automorphic forms? Examples, motivations and models from geometry.
- What is the twisted Gan-Gross-Prasad conjecture on unitary groups? An explicit yet not well-understood example.
- What are our theorems and applications? New for GL_2.
- How to prove theorems? Trace formulas, branching laws, and new difficulties. Could we prove the full conjecture? New questions.

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Analogs:

- Waldspurger formula (1985) is a twisted version of Hecke's works on L-functions on GL_2 ($+\ \mathrm{GL}_1$) using Mellin transform (ICM 1936).
- Thanks to the twist by a quadratic extension $K/\mathbb{Q}$ (you can vary), the Waldspurger formula is useful for a lot of different things.
- The proofs are completely different (no Mellin transform in general).

Thanks for the invitation!

Geometric model: quadratic twists of elliptic curves

Elliptic curve over $\mathbb{Q}$

$$A : y^2 = x^3 + ax + b$$

For squarefree $d \in \mathbb{Q}^\times$, consider quadratic twist

$$A^{(d)} : dy^2 = x^3 + ax + b$$

$K = \mathbb{Q}(\sqrt{d})$ quadratic field.

$$(A^{(d)})_K \cong (A)_K, (x, y) \mapsto (x, \sqrt{d}y)$$

$$L(s, A^{(d)}) = L(s, A \otimes \eta_{K/\mathbb{Q}}).$$

Question

How does twisting affect ranks and L-functions? Is there a constant d such that $L(A^{(d)}, 1/2) \neq 0$ and $A^d(\mathbb{Q})$ finite?

Application: BSD conjecture and Goldfeld conjecture.

Slogan: twisted L-functions are related to twisted period integrals.

Geometric model: spheres and Waldspurger formula

Consider lattice points

$$X_n = \{(\frac{x}{\sqrt{n}}, \frac{y}{\sqrt{n}}, \frac{z}{\sqrt{n}}) | (x, y, z) \in \mathbb{Z}^3, x^2 + y^2 + z^2 = n\} \subseteq \mathbb{S}^2$$

(Gauss 1801) $|X_n|$ are class numbers. $X_n = \emptyset$ when $n = 7 \mod 8$.

Question

What are the limiting statistical properties of X_n when $n \to +\infty$?

Theorem (Duke (1988), Duke–Schulze-Pillot (1990))

When square-free $n \to +\infty$ with $n \neq 7 \mod 8$, X_n equidistribute in $\mathbb{S}^2$.

$$\frac{1}{|X_n|} \sum_{p \in X_n} \phi(p) \to \int_{\mathbb{S}^2} \phi(y) dy, \quad \forall \phi \in \mathcal{S}(\mathbb{S}^2).$$

Geometric model: spheres and Waldspurger formula

- Use spherical harmonic on $\mathbb{S}^2 = SO(3)/SO(2)$ to reduce to the case ϕ is a harmonic polynomial of degree $m \geq 1$.
- Theta series
$$\sum_{n\geq 0} \sum_{p\in X_n} \phi(p) q^n$$
is modular form of weight $3/2 + m$.
- Theta lift to a modular form f of weight $2 + 2m$.
- **Transfer of integrals**: twisted $\mathbb{Q}(\sqrt{-n})$-toric period integrals of f = central L-values **(Waldpurger formula)**.
- Apply Iwaniec's subconvex Bound for central L-values.

Theorem (Waldspurger formula on toric period integrals)

Let E/F be a quadratic extension of number fields, and π cuspidal automorphic representation on $G = \mathrm{PGL}_{2,F}$.

$$P_H(f) := \int_{[H]} f(h)dh, \quad f \in \pi, \quad H = (\mathrm{Res}_{E/F}\, \mathbb{G}_{m,E})/\mathbb{G}_{m,F} \leq G.$$

Then we have

$$|P_H(f)|^2 \sim \frac{L(1/2, f)L(1/2, f \otimes \eta_{E/F})}{L(1, f, Ad)}.$$

For refined applications on statistic properties of lattice points, see Equidistribution, L-functions and ergodic theory: on some problems of Yu. Linnik (ICM 2006, P. Michel, A. Venkatesh).

Geometric model: from linear algebra to unitary groups

Geometry may be understood via group actions as homogeneous spaces.

- $GL_n(\mathbb{R})$ acts on $\mathbb{R}^n$. $\mathbb{R}^n - 0 = GL_n(\mathbb{R})/P_n$ (mirabolic).
- $SO(n+1)$ acts on $\mathbb{S}^n$. $\mathbb{S}^2 = SO(n+1)/SO(n)$.
- $U(n,1)$ acts on the hyperbolic unit ball $\mathbb{D}^n = U(n,1)/U(n) \times U(1)$.

$$\mathbb{D}^n = \{|x_0|^2 + \cdots + |x_{n-1}|^2 - |x_n|^2 < 0\} \subseteq \mathbb{CP}^n.$$

Our focus: unitary groups and ball quotients. Motivations:

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Our focus: unitary groups and ball quotients. Motivations:

- study statistical properties of arithmetic submanifolds of ball quotients, using test functions and their period integrals.
- study automorphic forms on GL_n by descent to unitary groups, with nicer symmetry and (algebraic) geometry.

We won't say anything about geometry and arithmetic, but focus on L-functions and harmonic analysis of trace formulas.

Magical model: base change and descent

To study automorphic forms, use magic tools from Langlands program.

- Langlands transfers to other G'.
- Langlands transfers from nice G.

Unfortunately, no purely geometric explanations or proofs over $\mathbb{Q}$ and $\mathbb{Q}_p$.
Proofs: harmonic analysis and integrals of test functions (trace formulas).
Example: Jacquet-Langlands to inner forms.

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Example (symmetric powers for a modular form f)

(Langlands) The existence of $Sym^r f$ on GL_{r+1} $(r \geq 1)$ would imply the Ramanujan and Sato-Tate conjecture. Note $Sym^2 f = Ad^0(f)$.

Example (quadratic base change for a modular form f)

The base change to a real quadratic field K, is a Hilbert modular form f_K. The associated Asai L-functions $L(s, As_{K/\mathbb{Q}}(f_K))$ are related to Hilbert modular surfaces over $\mathbb{Q}$. Here

$$As_{K/\mathbb{Q}}(f_K) = Sym^2 f \oplus (\wedge^2 f \otimes \eta_{K/\mathbb{Q}}).$$

Unitary groups: base change, descent and Asai

E/F quadratic extension of number fields. V hermitian space over E.

Theorem (Arthur (2013), Mok, Kaletha-Minguez-Shin-White..)

There exists a base change (must be isobaric, $n = \dim V$)

$$\mathrm{BC}_{E/F} : \mathcal{A}_{cusp}(U(V))) \to \mathcal{A}(GL_{n,E})$$

An isobaric $\Pi \in \mathcal{A}(GL_{n,E})$ is from unitary groups, iff each cuspidal factor of Π is conjugate self-dual of parity $(-1)^{n-1}$. Here we use poles of Asai L-functions at $s = 1$ to detect conjugate self-duality.

local analogs and local-global compatibilities are known.

- The quadratic base change $GL_{n,F} \to GL_{n,E}$ is already known by Arthur-Clozel by trace formula (1980s).
- The base change of unitary groups (and other classical groups) is known only in the last 10 years.

Unitary groups: base change, descent and Asai

E/F quadratic extension of number fields. V hermitian space over E. Involution c on W_E from $s \in W_F - W_E$. For irreducible $\rho \in \mathrm{Rep}(W_E)$,

$$\mathrm{Ind}_{W_E}^{W_F}(\rho \otimes \rho^c) = \mathrm{As}^+_{E/F}(\rho) \oplus \mathrm{As}^-_{E/F}(\rho).$$

For $\mathrm{As} = \mathrm{As}^+$, s acts by $v \otimes w \mapsto w \otimes s^2 v$, and $As|_{W_E} \cong \rho \otimes \rho^c$.

- ρ is conjugate self-dual iff $\rho^c \cong \rho^\vee$ iff $\exists \epsilon = \pm 1, [\mathrm{As}^\epsilon_{E/F}(\rho)]^{W_F} \neq 0$.
- $\epsilon = +1$: conjugate orthogonal, $\epsilon = -1$: conjugate sympletic.
- When $\rho \in \mathrm{Rep}(W_F)$, $\mathrm{As}_{E/F}(\rho) \cong Sym^2(\rho) \oplus (\wedge^2 \rho \otimes \eta_{E/F})$.

Relative Langlands program over $\mathbb{Q}$: more examples?

Relative Langlands program (1960s-2010s-2020s-):

- Main global problem: relate L-functions to period integrals, in a family of test functions: renormalization and convergence.
- Main local problem: study branching laws (from algebra to analysis). Match local periods / local relative characters.
- Use non-reductive period integrals beyond spherical varieties [BZSV].

Explicit examples over $\mathbb{Q}$? Only understood in a few cases:

- $- \otimes -$ for $GL_n \times GL_m$, Jacquet-Piatetski-Shapiro-Shalika (1983).
- **Asai=twisted** $- \otimes -$ on $Res_{L/E} GL_{n,L}$: Flicker (1988), Beuzart-Plessis (2018). If $L = E \times E$, recover $- \otimes -$ for $GL_n \times GL_n$.

Knowledge on formulations / proofs, is limited in higher dimension.

Twisted GGP: set up

Twisted GGP conjecture (2023) is a nice example to

- test general ideas on twisted period integrals.
- find concrete applications similar to Waldspurger formulas.

Set up:

- E/F quadratic extension of number fields.
- For simplicity, consider Π_0 on $\mathrm{GL}_{n,E}$ with a descent to π_0 on $\mathrm{U}(V)$.
- V a skew-Hermitian space over E, isometry group $\mathrm{U}(V)$.

Magical model: Weil representation for $\mathrm{U}(V)$

The unitary group

$$\mathrm{U}(V) \subseteq \mathrm{Sp}(V)$$

forms a reductive dual pair with $Z_{\mathrm{Sp}(V)}(\mathrm{U}(V)) = Z(\mathrm{U}(V)) = \mathrm{U}(1)$.

- Fix a character $\mu : \mathbb{A}_E^\times \to \mathbb{C}^1$ lifting $\eta_{E/F}$.
- Leray splitting (Bruhat decompositions of Siegel parabolics)

$$i_\mu : \mathrm{U}(V) \to \widetilde{\mathrm{Sp}}(V).$$

Weil representation of theta functions

$$i_\mu : \mathrm{U}(V) \to \widetilde{\mathrm{Sp}}(V) \curvearrowright \omega_{\mu,\psi}.$$

Twisted GGP: K/F and base change

- K/F quadratic extension.
- $L = E \otimes_F K$ **biquadratic extension of** F.
- $H = U(V) \to G = U(V_K)$, base change $H_1 = GL_{n,E} \to G' = GL_{n,L}$.
- Consider the base change Π on $\mathrm{GL}_{n,L}$.
- The Asai E-motive is conjugate-orthogonal:

$$As_{L/E}(\Pi) = \mathrm{Sym}^2(\Pi_0) \oplus (\wedge^2(\Pi_0) \otimes \eta_{L/E}).$$

The twisted period integral

$$P_V \in \mathrm{Hom}_{\mathrm{U}(V)}(\pi, \omega).$$

$$P_V(\varphi, \phi) := \int_{[U(V)]} \varphi(h)\overline{\phi}(h)dh, \quad \varphi \in \pi \in \mathcal{A}_{cusp}(U(V_K)), \quad \phi \in \omega.$$

- The base change of $H \curvearrowright \omega$ is $H_1 = GL_{n,E} \curvearrowright \omega' = \mathcal{S}(\mathbb{A}_E^n)$.

Twisted GGP: conjecture

Conjecture (Twisted Gan-Gross-Prasad conjecture, global)

For tempered and hermitian $\Pi \in \mathcal{A}_{cusp}(GL_{n,L})$, the following are equivalent

1. $L(1/2, As_{L/E}(\Pi) \otimes \mu^{-1}) \neq 0$.
2. $P_V \neq 0$ *on a descent* $\pi \in \mathcal{A}_{cusp}(U(V_K))$ *of* Π *for some skew-hermitian space* V *over* E.

Main results and applications (K/F, E/F and $L = E \otimes_F K$)

Theorem (Lu-Wang-Z.)

Choose K and E such that for any place v of F,

- *If v is ramified in E (resp. K), then v is split in K (resp. E).*
- *If $v|2$ or $v|\infty$, then v is split in L.*

For any Π supercuspidal at two finite places split in L, and $\mathrm{Ram}(\Pi, \psi, \mu) \cap \mathrm{Inert}(L/F) = \varnothing$, the twisted GGP conjecture holds.

Theorem (Lu-Wang-Z.)

The refined conjecture also holds, assuming the local unramified computation at $v \in \mathrm{Inert}(L/F)$ which is true when $n \le 3$.

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Applications:

- Base change of classical modular forms to $\mathrm{GL}_{2,L}$.
- Non-negativity of central L-values (predicted by RH).
- New arithmetic applications to Asai motives and elliptic curves.
- (potential) statistical properties of L-values.

Twisted GGP: refined version

From ψ, obtain measures:

- Tamagawa measure on $\mathrm{U}(V)$: $dh = \prod_v dh_v$. Used for periods.
- Tamagawa measure on $\mathrm{U}(V_K)$: $dg = \prod_v dg_v$. Used for Petersson inner product $\langle -, - \rangle_{\mathrm{Pet}} = \prod_v \langle -, - \rangle_{\mathrm{Pet},v}$ on π.
- Self-dual measure on $\mathbb{L}$: $dx = \prod_v dx_v$. Used for inner product $\langle -, - \rangle_\omega = \prod_v \langle -, - \rangle_{\omega,v}$ on the Weil representation $\omega = \mathcal{S}(\mathbb{L}(\mathbb{A}))$.

Twisted GGP: refined version

Integral of matrix coefficients for each place v of F:

$$\mathcal{I}_v(\varphi_v, \varphi'_v, \phi_v, \phi'_v) = \int_{H(F_v)} \langle h_v.\varphi_v, \varphi'_v \rangle_{\mathrm{Pet},v} \langle \phi_v, \phi'_v \rangle_{\omega,v} \, dh_v$$

Let $\Delta^*_{U(V)} = \prod_{i=1}^n L(i, \eta^i_{E/F}), \Delta^*_{U(V_K)} = \prod_{i=1}^n L(i, \eta^i_{L/K})$,

$$\mathcal{L}(1/2, \pi) = \frac{\Delta^*_{U(V_K)}}{\Delta^*_{U(V)}} \frac{1}{L(1, \Pi, Ad)} L(1/2, As_{L/E}(\Pi) \otimes \mu^{-1}).$$

Conjecture (Unramified computation)

For unramified (π, μ, ψ, v), spherical norm one φ_v and ϕ_v,

$$\mathcal{I}_v(\varphi_v, \varphi'_v, \phi_v, \phi'_v) = \mathcal{L}(1/2, \pi_v)\mathrm{vol}(H(O_{F_v})).$$

This is independent of measures.

Twisted GGP: refined version

Assuming this local unramified computation (similar to Ichino-Ikeda), form the normalized integral

$$\mathcal{I}_v^\sharp = \mathcal{L}(1/2, \pi_v)^{-1}\mathcal{I}_v.$$

Conjecture (Refined version)

As elements in Hom$(\pi \otimes \overline{\pi} \otimes \overline{\omega} \otimes \omega^\vee, \mathbb{C})$,

$$P_V \otimes \overline{P_V} = \mathcal{L}(1/2, \pi)\frac{1}{|S_\Pi|}\prod_v \mathcal{I}_v^\sharp.$$

Here $|S_\Pi|$ is 4 ($K = F \times F$), 2 ($K \neq E$ field), 1 ($K = E$ field).
The case $n = 1$ is known by theta lifting.

Remarks

The result may be the first example to go beyond classical period integrals with a family of twists (change μ and K), even for GL_2.
The proof is based on comparison of new relative trace formulas.

- There is no direct relation of twisted GGP conjecture to GGP conjecture, as we are considering a family of different L-functions.
- **Use normal elements to understand orbits** for involution θ: $\{x\theta(x) = \theta(x)x\}$.
- **Use the third quadratic extension** $F \subseteq M = L^{\sigma_{E/F} \otimes \sigma_{K/F} = id}$.
- **Geometric decompositions** into non-classical orbital integrals with Fourier transforms.
- Local relative characters are not well-understood.
- Unramified computations are quite different.

Proof: base change from F to E

We outline the proof, focusing on essential differences to Bessel GGP conjecture on $U_n \times U_{n+1}$ and Fourier-Jacobi GGP conjecture on $U_n \times U_n$.

- The conjecture is true when $E = F \times F$ by works of Flicker.
- If $E \neq F \times F$, we can base change everything to E.

Example

Fix a splitting $V^\vee = \mathbb{L} \oplus \mathbb{L}^\vee$ into Lagrangians over F. Consider the Schrodinger model for $\mathrm{U}(V)$

$$\omega = \mathcal{S}(\mathbb{L}(\mathbb{A})).$$

- If $E = F \times F$, $\mathbb{L} = F^n$ with action of $\mathrm{U}(V) = GL_n(F)$. $\omega = \mathcal{S}(\mathbb{A}_F^n)$
- (general) $\omega =$ twist of the theta series realization for $GL_n(\mathbb{A})$ on $\mathbb{A}^n$.

To characterize the base change (of period integrals), we need to match and compare a lot of integrals of test functions (always hard).

Proof Step 0: new relative trace formulas on period integrals

- $H = U(V) \to G = U(V_K)$.
- $f \in S(G(\mathbb{A}))$, $\phi_1, \phi_2 \in S(\mathbb{L}(\mathbb{A}))$

$$J(f, \phi_1 \otimes \phi_2) = \int_{[H]} \int_{[H]} K_f(h_1, h_2)\overline{\Theta(h_1, \phi_1)}\Theta(h_2, \phi_2)\, dh_1\, dh_2.$$

- $H_1 = Res_{E/F}GL_n \to G' = Res_{L/F}GL_n \leftarrow H_2 = Res_{K/F}GL_n$.
- $f' \in S(G'(\mathbb{A}))$, $\phi' \in S(\mathbb{A}_E^n)$

$$I(f', \phi') = \int_{[H_1]} \int_{[H_2]} K_{f'}(h_1, h_2)\overline{\Theta(h_1, \phi')}\eta_{L/K}(h_2)^{n+1}\, dh_1\, dh_2.$$

Here

- $K_f(h_1, h_2) = \sum_{\gamma \in G(F)} f(h_1^{-1}\gamma h_2)$ is the kernel function on $L^2([G])$.
- $\overline{\Theta(h_1, \phi')} = \sum_{z \in E^n} \phi'(zg)|\det g|^{1/2}\mu(\det g)$ is also related to μ.

The supercuspidal condition of Π is used for simplifying trace formulas.

Proof Step 0: analysis of trace formulas

We need to prove things like $I = J$ by comparing the geometric sides.

$$\sum_{\Pi} I_{\Pi} = I \overset{??}{=} \sum_{a} I_a, \quad \sum_{\pi} J_{\pi} = J \overset{??}{=} \sum_{a} J_a.$$

- $J_\pi(f, \phi_1 \otimes \phi_2) = \sum_{\varphi \in OB(\pi)} P_V(f.\varphi, \phi_1)\overline{P_V(\varphi, \phi_2)}$.
- $I_\Pi(f', \phi') = \sum_{\varphi' \in OB(\Pi)} P_{As} \times P_\eta$ directly related to Asai L-values.
- $J_{\pi,v}(f_v, \phi_{1,v} \otimes \phi_{2,v}) = \sum_{\varphi_v \in OB(\pi_v)} \mathcal{I}_v(f_v.\varphi_v, \varphi_v, \phi_1, \phi_2)$.

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1. To prove the conjecture, show $I_{\Pi} \neq 0$ iff $J_{\pi} \neq 0$ for some π on V.
2. To prove the refined conjecture, show $J_{\pi} = \mathcal{L}(1/2, \pi) \prod_v J^{\sharp}_{\pi_v}$. Note $I_{\Pi} = \mathcal{L}(1/2, \pi) \prod_v I^{\sharp}_{\Pi,v}$. Study $0 \neq I_{\Pi,v} \overset{?}{=} J_{\pi,v}$, $I_{\Pi} \overset{?}{=} J_{\Pi}$.

Question: what are geometric sides? orbital integrals I_a, J_a, orbits?

$$I_a = \prod_{v} I_{a,v}, \quad J_a = \prod_{v} J_{a,v}.$$

We are not in the classical set up of relative trace formulas.

Proof Step 1: enough normal representatives

Definition

Consider a group G and a subgroup $H \leq G$ as the fixed subgroup by an involution θ. Then $g \in G$ is called θ-normal if $g\theta(g) = \theta(g)g$.

Hope: any $H \times H$-orbit of G contains a θ-normal element g.

Then we can consider the action of g on the geometric space after partial Fourier transforms. The stabilizer of g acts on Weil representations nicely.

- work with $R \subseteq G$, a representative of regular-semisimple $H \times H$-orbits of G, i.e. any $x \neq y \in R$ have different $H \times H$-orbits.
- On GL side, there are enough normal regular-semisimple representatives (proved by Lie algebra).
- On unitary side, we only consider test functions supported on orbits with normal representatives.

Proof Step 1: orbits under commutating involutions

Question (geometry of twist)

How do you describe $GL_n(E)\backslash GL_n(L)/GL_n(K)$?

1. θ_1 and θ_2 **commuting involutions** of a group G.
2. H_1 (resp. H_2) fixed subgroups of G by θ_1 (resp. θ_2).
3. $H_1 \curvearrowright G/H_2 \hookrightarrow G^{\theta_2 = (-)^{-1}}$, $\xi \mapsto \gamma = \xi\theta_2(\xi)^{-1}$.

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Proposition

1. Let $\delta = \gamma\theta_1\theta_2(\gamma) = \gamma\theta_1(\gamma)^{-1} \in G$. Then $\theta_1(\delta)\delta = id$.
2. $\gamma \in G^{\theta_2=(-)^{-1}}$ is *normal* if γ commutes with $\theta_1\theta_2(\gamma) = \theta_1(\gamma)^{-1}$.
3. **Check**: γ is normal, iff $\theta_1\theta_2(\delta) = \delta$.

So M is important. Use $GL_n(M) \to GL_n(E)\backslash GL_n(L)/GL_n(K)$ as representatives.

Proof Step 1: normal regular elements have nice stabilizers

Lemma (GL side)

If γ is $\theta_1\theta_2$-normal and the centralizer $C(\delta) = C(\gamma\theta_1\theta_2(\gamma))$ is a torus, then $Stab_{H_1}(\gamma) \subseteq H_1^{\theta_2 = id}$.

Unitary: normal regular-semi-simple representatives of $H(F)\backslash G(F)/H(F)$ has stabilizers in $H(F) \subseteq H(F) \times H(F)$ hence acts on $V^\vee$.

Proof Step 2: decomposition under nice representatives

Partial Fourier transforms (changing Lagrangians after doubling see-saw):

$$\ddagger : \mathcal{S}(\mathbb{L}) \otimes \mathcal{S}(\mathbb{L}) \to \mathcal{S}(V^\vee)$$

For $z = x + y, x \in \mathbb{L}, y \in \mathbb{L}^\vee$,
$(\phi_1 \otimes \phi_2)^\ddagger(z) = \int_{\mathbb{L}} \phi_1(x' + x)\phi_2(x - x')\psi(-2\mathrm{Tr}_{E/F} \langle x', y\rangle)dx'$.

$$\dagger : \mathcal{S}(E^n) \to \mathcal{S}(F_n) \otimes \mathcal{S}(F^{n,-})$$

$\phi'^\dagger(x, y) = \int_{F_n} \phi'(x + jw^\vee)\psi(w^\vee jy)dw^\vee$.
Here $j \in E^\times$ is pure imaginary of trace zero and $jF^{n,-} = F^n$.

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Here $j \in E^\times$ is pure imaginary of trace zero and $jF^{n,-} = F^n$.

Theorem

For $\phi = (\phi_1 \otimes \phi_2)^\ddagger$, $\overline{\Theta(g, \phi_1)}\Theta(g, \phi_2) = \sum_{v \in V(F)} \phi(vg) = \Theta_\phi(g)$. Assume test functions are good (hence decomposable with normal support), then

$$J(f, \phi_1 \otimes \phi_2) = J(f, \phi) = \sum_{[\xi, z]} \mathrm{Orb}.$$

Proof Step 2: local orbital integrals

Orbital spaces: $G(F) \times V^\vee$ and $G'(F)/H_2(F) \times F_n \times F^{n,-}$.

Definition (local orbital integrals)

On regular-semisimple normal $[\zeta, z] \in G(F_v) \times V_v^\vee$, define orbital integral of $f_v \in S(G(F_v))$, $\phi_{1,v} \otimes \phi_{2,v} \in S(\mathbb{L}(F_v)) \otimes S(\mathbb{L}(F_v))$

$$Orb = \int_{H(F_v)} \int_{H(F_v)} f_v(h_1^{-1}\zeta h_2)(\overline{\omega_v(h_2^{-1}h_1)\phi_{1,v}} \otimes \phi_{2,v})^\ddagger (h_2^{-1}z)\, dh_1\, dh_2.$$

Definition (local orbial integrals, v inert in E and K)

On regular-semisimple normal $[\gamma, x, y] \in GL_n(E_v) \times F_n \times F^{n,-}$, define

$$\int_{GL_n(E_v)} \widetilde{f}_v'(g^{-1}\gamma\overline{g})\overline{(\omega_\mu'(g).\phi_v')^\dagger(x,y)}dg.$$

Proof Step 3: fundamental lemmas and transfers

Question

To compare geometric side, how do you prove local FL and transfers for sufficiently many good test functions?

We use Lie algebras and a delicate choice of nice normal (sub)-representatives to do reductions.

- The reduction is quite complicated, even if K/F is split.

We use that if v is ramified in E (resp. K), then v is split in K (resp. E). Finally, we reduce to Jacuqet-Rallis transfers, fundamental lemmas and new twisted fundamental lemmas (v inert in E and K, Π_v unramified).

Proof Step 4: comparison of relative trace formulas for transfers, using strongly compatible representatives

We choose compatible representatives and make the comparison using Step 2, 3.

Proposition (Proposition 5.2, Wang's thesis)

Let $k \geq 0$ be an integer, and let v_2 be a nonarchimedean split place of F. Suppose $\delta \in GL_n(F)$ is a regular semisimple element which is the norm of some element of $GL_n(E)$. Then there exists $\gamma \in GL_n(E)$ with $\gamma\overline{\gamma} = \delta$

- And γ is Kottwitz at all inert places v of F, and k-Kottwitz at v_2.

The definition of Kottwitz is for reductions.
We have a generalization to general E and K (via combinatorics).

Proof Step 5: spectral results

Such a comparison of RTFs leads to global comparison $I = J$.
By a density argument, splitting places of F and K are enough to imply many spectral identities from geometric comparison.

- We use the local twisted GGP conjecture (proved in tempered case by Nhat Hoang Le) to decompose the spectral distribution J_π into local spectral distributions $J_{\pi,v}$, up to a constant C.
- We prove together with non-vanishing of local characters and local spectral identities.

Hence the global comparison leads to the proof.
Refined version are proved, if we assume local unramified computations.
Compute $C = \mathcal{L}(1/2, \pi)$ by suitable test functions.

After proof: new features and questions

1. How to prove transfer exists in ramified-ramified situation and archimedean situation? We need to consider orbital integrals with paritial Fourier transforms, which are not studied before.
2. Unramified computation for $n \geq 4$? It seems a hard problem, but we are trying to use theta liftings.
3. If $E = F \times F$ and K/F non-split, our comparisons of trace formulas via base change are still non-trivial. Any applications?
4. Arithmetic twisted GGP conjecture and the discovery of a twisted AFL (Z.). Applications to the Bloch-Kato conjecture, in progress.

In general, consider $H \to \mathrm{Res}_{K/F} H_K$ with a Weil representation ω of H. What are meanings of these period integrals? Ex: $H = Sp_{2n}$.

Thank you for your attention!