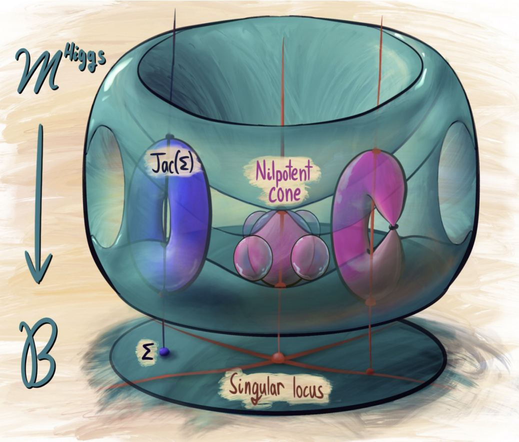
Monoids, spaces and fundamental lemmas without induction

Zhiyu Zhang

Stanford University

11/28/2025



Elliot Kienzle, Steven Rayan, Figure 8, arxiv:2201.12689

Goal of this talk (Lie algebra - Group - Monoid)

- Why do we like monoids in representation theory and geometry?
- What are our new constructions of spaces and cycles?
- What are our main global and local theorems? Applications?
- How to prove theorems? Conceptual ideas, new features and technical complexities. Could we use these ideas in other contexts?

Imprecise points (why):

- **Group theory may be more essential than pure linear algebra**, beyond GL_n. Harmonics on spaces, e.g. $\mathbb{S}^{2n-1} = U(n)/U(n-1)$.
- Groups: over $\mathbb{C}$ (classical) → over curves and discs (**21th century**) .
- Lie algebras simplify things? **not always**!
- **Study functions on groups**: automorphic forms, Hecke algebras.

Goal of this talk (Lie algebra - Group - Monoid)

We have a new multiplicative approach via monoids.

Definition

A monoid M is a set with an associative multiplication and an identity element. The units of M form a group $M^\times$.

Clean results on classical questions and new generalizations. Today:

- Endoscopic FL (Wang) and twisted variants (crucial for Langlands).
- **relative Hecke FL (Wang-Z.), AFL (Wang-Yun-Z.)**..
- We construct **suitable spaces of integral Hecke correspondences**. Over $\mathbb{Z}_p$, similar ideas may be applied, see e.g. Lee-Madapusi.
- **Applications: trace formulas, Langlands program, L-functions..**
- Relative FL is crucial in the proof of GGP conjecture for L-functions of unitary groups (hence its applications to Bloch-Kato conjecture).

Our methods:

- multiplicative Hitchin fibrations over $\mathbb{F}_q^{alg}$, with Fr (descent to $\mathbb{F}_q$) (Wang), some relative variants (Wang-Z.), after Frenkel-Ngo 2010.
- Pushforwards of sheaves $\overset{Tr(Fr_*)}{\rightsquigarrow}$ integrals of functions on groups!

Goal of this talk (Lie algebra - Group - Monoid)

We are studying questions for a reductive group G. Choose reductive M:

$$G \to M \curvearrowleft \mathbb{G}_m, \quad M \to M^{ab} = \mathbb{A}_M, \quad Z = Z(M^{\times})^o \curvearrowright M \curvearrowleft G.$$

M^{ab} **is a toric** Z**-variety**, $M \to M^{ab}$ is Z-equivariant.

Example

$G = GL_n \to M = Mat_{n\times n}, \quad M^{ab} = \mathbb{A}^1, \quad Z = \mathbb{G}_m.$

- $[\mathbb{A}_M/Z] = [\mathbb{A}^1/\mathbb{G}_m]$ classifies line bundle L with a section s. If s is generically non-zero, we obtain a divisor D (boundary divisor).
- $\mathbb{G}_m \subseteq Z \curvearrowright M \to M^{ab}$: flat deformation of G to simpler linear objects.
- Action of G on M: Cartan and Hecke algebras $(M(O) \cap G^{sc}(F))$.
- Geometry of orbital integrals $= \{g \in G(F)/G(O) | g^{-1}\gamma g \in M(O)\}$.

Goal of this talk (Lie algebra - Group - Monoid)

We are studying questions for a reductive group G. Choose reductive M:

$$G \to M \curvearrowleft \mathbb{G}_m, \quad M \to M^{ab} = \mathbb{A}_M, \quad Z = Z(M^\times)^o \curvearrowright M \curvearrowleft G.$$

M^{ab} **is a toric** Z**-variety**, $M \to M^{ab}$ is Z-equivariant.

Example

$G = GL_n \to M = Mat_{n\times n}, \quad M^{ab} = \mathbb{A}^1, \quad Z = \mathbb{G}_m.$

- $[\mathbb{A}_M/Z] = [\mathbb{A}^1/\mathbb{G}_m]$ classifies line bundle L with a section s. If s is generically non-zero, we obtain a divisor D (boundary divisor).
- $\mathbb{G}_m \subseteq Z \curvearrowright M \to M^{ab}$: flat deformation of G to simpler linear objects.
- Action of G on M: Cartan and Hecke algebras $(M(O) \cap G^{sc}(F))$.
- Geometry of orbital integrals $= \{g \in G(F)/G(O) | g^{-1}\gamma g \in M(O)\}$.

A fresh new framework for many questions, offering avenues of inquiry worth pursuing for many years to come.
Multiplicative mirror symmetry (Gallego), multiplicative affine Springer fibers (Bouthier, Chi, Ong), affine Lusztig varieties (He), Braverman-Kazhdan-Ngo on L-functions (Tate thesis on $M = \mathbb{A}^1$)...

Motivation of FL: Langlands program

Question

Is arithmetic geometry over $\mathbb{Q}$ *hard? Rational points? Is a set non-empty?*

Conjecture (Langlands program)

Polynomial equations over $\mathbb{Q}$ *are related to automorphic forms on* $G = \mathrm{GL}_n, \mathrm{U}_n, \mathrm{SO}_{2n+1}, \mathrm{Sp}_{2n}$, *in particular modular forms.*

local-global, mysterious and important, see e.g. Fermat Last Theorem.

Question

How to study Langlands program and automorphic forms over $\mathbb{Q}$*?*

Motivation of FL: Langlands program

Question

Is arithmetic geometry over $\mathbb{Q}$ hard? Rational points? Is a set non-empty?

Conjecture (Langlands program)

Polynomial equations over $\mathbb{Q}$ are related to automorphic forms on $G = \mathrm{GL}_n, \mathrm{U}_n, \mathrm{SO}_{2n+1}, \mathrm{Sp}_{2n}$, in particular modular forms.

local-global, mysterious and important, see e.g. Fermat Last Theorem.

Question

How to study Langlands program and automorphic forms over $\mathbb{Q}$?

Do analysis, compare global trace formulas and prove theorems.

- Local Langlands for $GL_n(\mathbb{Q}_p)$ and classical groups (no local proof).
- Lifting automorphic forms on classical groups (no geometric way).
- L-functions of Shimura varieties (no local proof).

They all deeply rely on solutions of tricky local harmonic analysis questions on groups: fundamental lemma (FL) and smooth transfer.

- Lie algebra FL proved by Ngo (2010). Via complicated analysis, Ngo's work implies FL and transfer on groups.

Monoids

- G semisimple group. Chevalley map

$$G^{sc} \to \mathfrak{c}_{G^{sc}} = G^{sc} /\!\!/ G \cong \mathbb{A}^r$$

with a Steinberg quasi-section.

- A reductive monoid M with $G^{sc} = (M^\times)^{der}$.
- Commutative monoid called abelianization (via $g_1 h g_2^{-1}$, often $\cong \mathbb{A}^{r_M}$)

$$\mathbb{A}_M = M /\!\!/ (G^{sc} \times G^{sc}).$$

Monoids

- G semisimple group. Chevalley map

$$G^{sc} \to \mathfrak{c}_{G^{sc}} = G^{sc} /\!\!/ G \cong \mathbb{A}^r$$

with a Steinberg quasi-section.

- A reductive monoid M with $G^{sc} = (M^\times)^{der}$.
- Commutative monoid called abelianization (via $g_1 h g_2^{-1}$, often $\cong \mathbb{A}^{r_M}$)

$$\mathbb{A}_M = M /\!\!/ (G^{sc} \times G^{sc}).$$

- Assumption: M is normal and integral. $M \to \mathbb{A}_M$ is very flat (flat+ dominant + integral fibers).
- Chevalley map

$$M \to c_M = M /\!\!/ \mathrm{Ad}(G) = c_{G^{sc}} \times \mathbb{A}_M.$$

- The torus $Z = Z(M^\times)^0$ acts on $\mathbb{A}_M$, which is a toric variety.

Toy case: $G = \mathrm{PGL}_2, G^{sc} = \mathrm{SL}_2, M = \mathrm{Mat}_{2\times 2}$.

Universal monoid and regular parts

It is essential to enlarge the torus part Z of M to make deformations.

Example (The universal monoid)

As $Z(G^{sc})$ may not be connected, we do a z-extension

$$G_+^{sc} = (G^{sc} \times T^{sc})/Z(G^{sc}) \to G^{sc}.$$

(Vinberg) There is a universal monoid $\mathrm{Env}(G^{sc})$ with unit group G_+^{sc} and abelianization $\mathbb{A}_G = \mathbb{A}^r$. Any very flat M is of the form $\mathrm{Env}(G^{sc}) \times_{\mathbb{A}_G} \mathbb{A}_M$.

Universal monoid and regular parts

It is essential to enlarge the torus part Z of M to make deformations.

Example (The universal monoid)

As $Z(G^{sc})$ may not be connected, we do a z-extension

$$G_{+}^{sc} = (G^{sc} \times T^{sc})/Z(G^{sc}) \to G^{sc}.$$

(Vinberg) There is a universal monoid $\mathrm{Env}(G^{sc})$ with unit group G_{+}^{sc} and abelianization $\mathbb{A}_G = \mathbb{A}^r$. Any very flat M is of the form $\mathrm{Env}(G^{sc}) \times_{\mathbb{A}_G} \mathbb{A}_M$.

- regular $M^{reg} \subseteq M$: $\dim \mathrm{Stab}_{Ad(G)}(x)$ is minimal. Allow $x \notin M^{\times}$.
- (If $G \neq G^{sc}$) strong regular M^{srs}: $\mathrm{Stab}_{Ad(G)}(x)$ is a connected torus.

(Vinberg) divisor $\mathcal{B}_M \subseteq c_M =$ the complement of $c_M^{\times} = c_{G^{sc}} \times \mathbb{A}_M^{\times}$. Note $x \in M^{\times}$ iff $[x] \in c_M^{\times}$. discriminant divisor $\mathcal{D}_M \subseteq c_M$ for regularity.

Example

$M = Mat_{n\times n}$. $\mathcal{B}_M$: no zero eigenvalues. $\mathcal{D}_M$: distinct eigenvalues.

Geometrization of trace formulas

- X a smooth projective geometrically connected curve, e.g. $X = \mathbb{P}^1$.
- "divisor" $D : X \to [\mathbb{A}_M/Z] \rightsquigarrow$ a Z-torsor L and $s \in H^0(X, L(\mathbb{A}_M))$.

$$[M/(Ad(G) \times Z)] \overset{Chevalley}{\to} [c_M/Z] \to [\mathbb{A}_M/Z] \to BZ.$$

Multiplicative Hitchin Fibration

$$h = h_{G,M,D} : \mathcal{M} = Map_D(X, [M/(G \times Z)]) \to \mathcal{A} = Map_D(X, [c_M/Z]).$$

Geometrization of trace formulas

Proposition

Over a large open locus of $a \in c_M$, $h^{-1}(a)$ is proper and (modulo symmetry of stabilizer P_a) encodes information of trace formulas.

- Smooth $h^{-1}(a)$ is easier to compute (evidence!). But we have to deal with some singular fibers (important + interesting).
- Check: global h has locally finite type + proper algebraic geometry. $\mathcal{M}$ may be non-smooth (!) but has a local model of singularity.
- Then we can apply decomposition theorem and supports to study singular fibers (after Ngo). By continuity of perverse sheaves on D and a, we may use simple cases to imply general cases!

Example for SL_2

$$[M/(G\times Z)] = [\mathrm{Mat}_{2\times 2}/(\mathrm{SL}_2\times\mathbb{G}_m)]$$

Fix "divisor" $D : X \to [\mathbb{A}_M/Z]$, a line bundle L with $0 \neq s \in H^0(X, L^2)$. $\mathcal{M}$ classifies a $G\times Z$-torsor with an equivariant map to M, namely (E, φ)

- a rank 2 vector bundle E over X with $\det E \cong O_X$.
- a twisted endomorphism $\varphi : E \to E\otimes L$ with $\det\varphi = s$.

$\mathcal{A} = H^0(X, L)\oplus H^0(X, L^2)$ is the Hitchin base. Recover Higgs bundles:

$$h : \mathcal{M}\to\mathcal{A}, \quad (E,\varphi)\mapsto (Tr(\varphi), Tr(\wedge^2\varphi))$$

Example

$(a, b)\in\mathcal{A}$, $a^2 - 4b \neq 0$. Smooth spectral curve $Y_{a,b}\to X$ defined by $y^2 + ay + b = 0$. The fiber $h^{-1}(a, b)$ (if not empty), is the Prym variety $Ker(J(Y_{a,b})\to J(X))^0$ (an abelian variety of fixed dimension).

Adelic orbital integrals

G acts on S over a local field F_v (e.g. $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$).

$$\mathrm{Orb}(G.x, f) := \int_{y \in G.x} f(y)dy, \ \forall f \in \mathcal{S}(S), x \in S.$$

Allow twist by a character η on G trivial on G_x.

Example (Tate thesis)

$\mathbb{G}_m$ acts on $\mathbb{A}^1$. $f_p = 1_{\mathbb{Z}_p}$, $x = 1 \in \mathbb{A}^1$.

$$\zeta_p(s) = \int_{\mathbb{G}_m} f_p(t.1)|t|^s dt = (1 - p^{-s})^{-1}.$$

$$\zeta_\infty(s) = \int_{\mathbb{G}_m} f_\infty(t.1)|t|^s dt = \pi^{-s/2}\Gamma(s/2).$$

$$\zeta_\infty(s)\zeta(s) = \prod_p \zeta_p(s) = \int_0^\infty y^{s/2} \sum_{n=1}^\infty e^{-n^2\pi y} dy.$$

Local Hitchin densities (Frenkel-Langlands-Ngo)

The local Hitchin map $h : [S/G] \to S /\!\!/ G$.
$S /\!\!/ G$ "=" regular G-orbits on S (assuming a Kostant type section $a \to x_a$).
$f = 1_\Omega$ for a K-stable compact open subset $\Omega \subseteq S$ ("integral model").

$$h : \Omega \to \Omega /\!\!/ K.$$

Proposition (counting volume)

$$\mathrm{Orb}(a, f) = \#\{x \in \Omega | h(x) = h(x_a)\}.$$

Affine Springer fiber $AS_x = \{g \in G/K | g^{-1}.x \in \Omega\}$.
local evaluation map

$$ev : AS_x \to K\backslash\Omega, \quad g \mapsto g^{-1}.x$$

detects orbital integrals.

Relative set ups: Jacquet-Rallis case

Relative story: $H_1 \to G \leftarrow H_2$. $H_1 \times H_2 \curvearrowright G \longleftrightarrow H_1 \curvearrowright X = G/H_2$.

- F'/F quadratic extension. V n-dimensional hermitian space over F'.
- $V^\sharp = V \oplus F'e$. $H = U(V) \to G_{tot} = U(V) \times U(V^\sharp) \leftarrow H = U(V)$.

$$H^U \overset{Ad}{\curvearrowright} G^U = U(V^\sharp). \qquad \text{(Unitary side)}$$

- $H' = GL_{n,F'} \to G'_{tot} = GL_{n,F'} \times GL_{n+1,F'} \overset{\eta}{\leftarrow} H'_2 = GL_{n,F} \times \mathrm{GL}_{n+1,F}$.

$$H^S = \mathrm{GL}_{n,F_0} \overset{Ad}{\curvearrowright} S_{n+1} = \{\gamma \in GL_{n+1,F} | \gamma\overline{\gamma} = 1\}. \qquad \text{(Symmetric side)}$$

Relative set ups: Jacquet-Rallis case

Relative story: $H_1 \to G \leftarrow H_2$. $H_1 \times H_2 \curvearrowright G \longleftrightarrow H_1 \curvearrowright X = G/H_2$.

- F'/F quadratic extension. V n-dimensional hermitian space over F'.
- $V^\sharp = V \oplus F'e$. $H = U(V) \to G_{tot} = U(V) \times U(V^\sharp) \leftarrow H = U(V)$.

$$H^U \overset{Ad}{\curvearrowright} G^U = U(V^\sharp). \qquad \text{(Unitary side)}$$

- $H' = GL_{n,F'} \to G'_{tot} = GL_{n,F'} \times GL_{n+1,F'} \overset{\eta}{\leftarrow} H'_2 = GL_{n,F} \times \mathrm{GL}_{n+1,F}$.

$$H^S = \mathrm{GL}_{n,F_0} \overset{Ad}{\curvearrowright} S_{n+1} = \{\gamma \in GL_{n+1,F} | \gamma\overline{\gamma} = 1\}. \qquad \text{(Symmetric side)}$$

Base change to F': conjugacy action of $H^{std} = GL_n$ on $G^{std} = GL_{n+1}$.

- For a very flat G^{std}-monoid M^{std}, study (there are sections!)
  $M^{std}/H^{std} \to M^{std} /\!\!/ H^{std} \to M^{std}_H /\!\!/ H^{std} = (Hit \sim GL_n /\!\!/ GL_n)$.

Relative Hitchin Fibration ($H^{std} = \mathrm{Stab}(e, e^*) \leq G^{std}$)

$$\mathcal{M}^{std} = Map_D(X, [M^{std}/(H^{std} \times Z)]) \to \mathcal{A}^{std}.$$

Deformed quotient stacks

Global twist: $X' \to X$ etale double cover of curves over $k = \mathbb{F}_q$. $F \subseteq F'$.

Let S be a k-scheme. Let $\underline{M}^{std}(S)$ be the groupoid of $(\mathcal{E}, x, e, e^\vee)$ where

- $\mathcal{E}$ a vector bundle on $X \times S$ of rank $n+1$.
- ϕ a section in $M^{std} \times^{G^{std}} \mathcal{E}(S)$, generically a unit (integral Hecke).
- $e : \mathcal{O}_S \to \mathcal{E}$, $e^\vee : \mathcal{E} \to \mathcal{O}_S$.

$\mathbb{G}_m$-equivariant map (preimage $\underline{M}_1^{std} \cong [M^{std}/H^{std}]$)

$$\underline{M}^{std} \to \mathbb{A}^1, (\mathcal{E}, \phi, e, e^\vee) \to e^\vee \circ e \in \mathcal{O}_S.$$

Twisted Relative Hitchin Fibration

$$\mathcal{M}^S \xrightarrow{h} \mathcal{A} \xleftarrow{h'} \mathcal{M}^U.$$

- $\{(\mathcal{E}^U, \phi^U, e^U)\}$: U_{n+1}-torsor $\mathcal{E}^U$, $\phi^U \in M^U \times^{U_{n+1}} \mathcal{E}^U$, $e^U : O_{X'} \to V_{\mathcal{E}^U}$.
- $\{(\mathcal{E}^S, \phi^S, e^S, e^{S,\vee})\}$: GL_{n+1}-torsor $\mathcal{E}^S$, $\phi^S \in M^S \times^{GL_{n+1}} \mathcal{E}^S$.
- (Technical point) replace $\underline{M}^{std}$ by $\underline{M}^{std}/(Z \times \mathbb{G}_m)$, use Z-torsor L (with a section s_L of $\mathbb{A}_M \times_Z L$) and line bundle $\mathcal{D}$ on X. Get

$$\phi^U \in M^U \times^{Z^U \times G^U} \mathcal{E}^U, e^U : \mathcal{D}_{X'}^{-1} \to V_{\mathcal{E}^U}.$$

- Forgetting vectors and covectors produces usual mH-fibrations

$$\mathcal{M}^U \to \mathcal{M}^{U,Hit} \to \mathcal{A}^{Hit} \leftarrow \mathcal{M}^{S,Hit} \leftarrow \mathcal{M}^S.$$

Note $Z^U = Z^S, \mathbb{A}^U = \mathbb{A}^S$. $\mathcal{A}_{L,\mathcal{D}} = \mathcal{A}_L^{Hit} \times \mathcal{A}''_{L,\mathcal{D}}$.

Relative affine Springer fibers for $a \in \mathcal{A}_v^{srs}(k)$

For any closed point v of X, have local Hitchin maps

$$\mathcal{M}_v^S \xrightarrow{\chi} \mathcal{A}_v \xleftarrow{\chi'} \mathcal{M}_v^U$$

Relative multiplicative affine Springer fibers with local evaluation maps

Relative affine Springer fibers for $a \in \mathcal{A}_v^{srs}(k)$

For any closed point v of X, have local Hitchin maps

$$\mathcal{M}_v^S \overset{\chi}{\to} \mathcal{A}_v \overset{\chi'}{\leftarrow} \mathcal{M}_v^U$$

Relative multiplicative affine Springer fibers with local evaluation maps

$$\mathcal{M}_v^S(a) \to S_{n+1}(F_v)/_{Ad} GL_{n+1}(O_v'), \quad U_{n+1}(F_v)/_{Ad} U_{n+1}(O_v) \leftarrow \mathcal{M}_v^U(a).$$

$$\mathcal{M}_v^S(a) = \{g \in GL_n(F_v')/GL_n(O_v') | g^{-1}\gamma_a g \in M^S(O_v)\}.$$

$$\mathcal{M}_v^U(a) = \{g \in U_n(F_v)/U_n(O_v) | g^{-1} g_a^U g \in M^U(O_v)\}.$$

Use section $a \to \gamma_a^S \in S_{n+1}(F_v), a \to g_a^U \in U_{n+1}(F_v)$ (assuming g_a^U exists).

- Satake (no Lie algebra version): Sat_λ on $\mathrm{GL}_{n+1}(F_v')/\mathrm{GL}_{n+1}(O_v')$.
- IC_λ^S, IC_λ^U: pullback of Sat_λ via $S_{n+1} \to GL'_{n+1}, U_{n+1} \to GL'_{n+1}$.

Relative affine Springer fibers for $a \in \mathcal{A}_v^{srs}(k)$

For any closed point v of X, have local Hitchin maps

$$\mathcal{M}_v^S \xrightarrow{\chi} \mathcal{A}_v \xleftarrow{\chi'} \mathcal{M}_v^U$$

Relative multiplicative affine Springer fibers with local evaluation maps

$$\mathcal{M}_v^S(a) \to S_{n+1}(F_v)/_{Ad}GL_{n+1}(O'_v), \quad U_{n+1}(F_v)/_{Ad}U_{n+1}(O_v) \leftarrow \mathcal{M}_v^U(a).$$

$$\mathcal{M}_v^S(a) = \{g \in GL_n(F'_v)/GL_n(O'_v) | g^{-1}\gamma_a g \in M^S(O_v)\}.$$

$$\mathcal{M}_v^U(a) = \{g \in U_n(F_v)/U_n(O_v) | g^{-1}g_a^U g \in M^U(O_v)\}.$$

Use section $a \to \gamma_a^S \in S_{n+1}(F_v), a \to g_a^U \in U_{n+1}(F_v)$ (assuming g_a^U exists).

- Satake (no Lie algebra version): Sat_λ on $\mathrm{GL}_{n+1}(F'_v)/\mathrm{GL}_{n+1}(O'_v)$.
- IC_λ^S, IC_λ^U: pullback of Sat_λ via $S_{n+1} \to GL'_{n+1}, U_{n+1} \to GL'_{n+1}$.

May choose M such that $M(O_v) \cap \mathrm{SL}_{n+1}(F) = \bigcup_{\mu \le \lambda} K\omega^\mu K$.

Example (For split v, $g_a = (\gamma, u, u^*)$)

$\mathcal{M}_v^U(a)(k) = \mathcal{M}_v^S(a)(k) = \{\Lambda \subseteq V_v | \gamma_t\Lambda_t \subseteq \Lambda_t, u \in \Lambda, u^* \in \Lambda^*\}$. $\Lambda_t = \wedge^t\Lambda$.

Theorems: matching of relative fibrations

Theorem (Hecke FL, Wang-Z)

Assume $p > 2n + 2$ (to make the theory of monoids nice). Then

$$\#_{IC^S_\lambda \otimes L_\eta} \mathcal{M}^S_v(a)(k) = \#_{IC^U_\lambda} \mathcal{M}^U_v(a)(k).$$

- $\mathcal{F} \to \#_{\mathcal{F}}(-)$ is the function-sheaf dictionary.
- L_η is the local system from X'/X.
- $\mathcal{M}^S_v(a) = \coprod_i \mathcal{M}^S_v(a), i = \mathrm{val}(f^\vee_n)$ sees $\Delta(a)\mathrm{Orb}^\eta(a, f_\lambda)$.
- Magic of sheaf theory: are two traces of Fr_q the same? Sufficient to treat k_n-points: $Tr(A^n), n >> 0$ determines $Tr(A^n), n \geq 0$.

No induction on n. No pure local proof except $n = 1$ (Li-Rapoport-Zhang, Hecke AFL, Proposition 7.2.1, 7.3.1).

Theorems: matching of relative fibrations

Theorem (Hecke FL, Wang-Z)

Assume $p > 2n+2$ (to make the theory of monoids nice). Then

$$\#_{IC_\lambda^S \otimes L_\eta} \mathcal{M}_v^S(a)(k) = \#_{IC_\lambda^U} \mathcal{M}_v^U(a)(k).$$

- $\mathcal{F} \to \#_{\mathcal{F}}(-)$ is the function-sheaf dictionary.
- L_η is the local system from X'/X.
- $\mathcal{M}_v^S(a) = \coprod_i \mathcal{M}_v^S(a), i = \mathrm{val}(f_n^\vee)$ sees $\Delta(a)\mathrm{Orb}^\eta(a, f_\lambda)$.
- Magic of sheaf theory: are two traces of Fr_q the same? Sufficient to treat k_n-points: $Tr(A^n), n >> 0$ determines $Tr(A^n), n \geq 0$.

No induction on n. No pure local proof except $n = 1$ (Li-Rapoport-Zhang, Hecke AFL, Proposition 7.2.1, 7.3.1).

Theorem (Global matching, Wang-Z)

On a dense open subset U of $\mathcal{A}^\heartsuit$, $Rh_*(IC_\lambda^S \otimes L_\eta) \cong Rh'_*(IC_\lambda^U)$ (up to ss).

Idea of proof

- (Step 1) global matching: computations on a smaller open where the disc is multiplicity free. This is similar to $O_v[t]/(f(t))$ being smooth. Stratified smallness of h on U, implies matching of simple perverse sheaves on the whole U.
- (Step 2) product formula: $\prod_v \mathcal{M}_v^U(a) \cong \mathcal{M}^U(a)$ over k. Use Beauville-Laszlo descent of $(L, \mathcal{D}, \mathcal{E}, e, e^\vee)$. Note that $\mathcal{M}_v^U(a)$ is empty or a point, $v >> 0$, so the product is finite.
- (Step 3) approximation of moduli spaces: $\mathcal{M}_v^U(a) \cong \mathcal{M}_v^U(a'), a' \to a$.

Idea of proof

- (Step 1) global matching: computations on a smaller open where the disc is multiplicity free. This is similar to $O_v[t]/(f(t))$ being smooth. Stratified smallness of h on U, implies matching of simple perverse sheaves on the whole U.
- (Step 2) product formula: $\prod_v \mathcal{M}_v^U(a) \cong \mathcal{M}^U(a)$ over k. Use Beauville-Laszlo descent of $(L, \mathcal{D}, \mathcal{E}, e, e^\vee)$. Note that $\mathcal{M}_v^U(a)$ is empty or a point, $v >> 0$, so the product is finite.
- (Step 3) approximation of moduli spaces: $\mathcal{M}_v^U(a) \cong \mathcal{M}_v^U(a'), a' \to a$.
- (Step 4) nice globalization for fixed a and v: use very ample D. by step 1-2, just need non-vanishing simple matching terms for $u \neq v$. Let $Z \subseteq U$ be defined by (1) a smaller residue disc at v, (2) for $u \neq v$, the disc is 0 or multiplicity free. $Z(K) \neq \varnothing$ for large extension K/k. By the density theorem, to prove matching of traces of powers of Frob, we just need to prove it for large K.
- (Step 5) integral section and nonvanishing: WLOG $K = k, Z(k) \neq \varnothing$. To show non-vanishing, construct a F_u-point (hence $\mathrm{Orb}_u(a) \neq 0$), i.e. integral section g_a (similar to [Yun, Lemma 5.2.1-5.2.2]). Search in $Z \subseteq U$ without changing v-term, an integral section is constructed by hand away from v and then glue with a_v.

Short break: analogs on affine Grassmannians

Geometric $Gr_G \leftrightarrow G(\mathbb{F}_q^{alg}((t)))/G(\mathbb{F}_q^{alg}[[t]])$.

- affine Springer fibers

$$Hk \to Gr_G \times Gr_G, \quad Gr_G \overset{(id,\gamma)}{\to} Gr_G \times Gr_G.$$

$$AS(\gamma, \mu) = (id, \gamma) Gr_G \cap Hk_\mu \subseteq Gr_G$$

Short break: analogs on affine Grassmannians

Geometric $Gr_G \leftrightarrow G(\mathbb{F}_q^{alg}((t)))/G(\mathbb{F}_q^{alg}[[t]])$.

- affine Springer fibers

$$Hk \to Gr_G \times Gr_G, \quad Gr_G \overset{(id,\gamma)}{\to} Gr_G \times Gr_G.$$

$$AS(\gamma,\mu) = (id,\gamma)Gr_G \cap Hk_\mu \subseteq Gr_G$$

- affine Deligne-Lusztig varieties (with a Fr descent to k)

$$Hk \to Gr_G \times Gr_G, \quad Gr_G \overset{(id,b\circ Fr)}{\to} Gr_G \times Gr_G.$$

$$ADLV(b,\mu) = (id, b\circ Fr)(Gr_G) \cap Hk_\mu \subseteq Gr_G.$$

Short break: analogs on affine Grassmannians

Geometric $Gr_G \leftrightarrow G(\mathbb{F}_q^{alg}((t)))/G(\mathbb{F}_q^{alg}[[t]])$.

- affine Springer fibers

$$Hk \to Gr_G \times Gr_G, \quad Gr_G \overset{(id,\gamma)}{\to} Gr_G \times Gr_G.$$

$$AS(\gamma,\mu) = (id,\gamma)Gr_G \cap Hk_\mu \subseteq Gr_G$$

- affine Deligne-Lusztig varieties (with a Fr descent to k)

$$Hk \to Gr_G \times Gr_G, \quad Gr_G \overset{(id,b\circ Fr)}{\to} Gr_G \times Gr_G.$$

$$ADLV(b,\mu) = (id, b\circ Fr)(Gr_G) \cap Hk_\mu \subseteq Gr_G.$$

- a space R over k, and an integral Hecke $Hk \to R \times R$. Our shtuka construction (μ=a type of modification of legs)

$$Sht_{R,\mu} = (id, Fr)R \cap Hk_\mu \to R.$$

Motivation: arithmetic trace formulas

Similar to trace formulas, arithmetic trace formulas aim to solve questions in arithmetic geometry, especially on volumes / sizes of special cycles.

- Applications: Gross-Zagier formulas and BSD conjecture for motives.

We work on Shimura varieties Sh_G for a classical group G over $\mathbb{Q}$, roughly moduli spaces of $\mathbb{Q}$-motives with additional structures.

- Over $\mathbb{C}$, by Hodge realization Sh_G =a moduli of $\mathbb{Q}$-Hodge structures.
- Over $\mathbb{Z}_p$, by integral p-adic Hodge realization Sh_G is related to "moduli of $\mathbb{Z}_p$-Hodge structures" (Rapoport-Zink $\mathcal{N}_{G_p} \to \mathrm{Isoc}_{G_p}$).

Motivation: arithmetic trace formulas

Similar to trace formulas, arithmetic trace formulas aim to solve questions in arithmetic geometry, especially on volumes / sizes of special cycles.

- Applications: Gross-Zagier formulas and BSD conjecture for motives.

We work on Shimura varieties Sh_G for a classical group G over $\mathbb{Q}$, roughly moduli spaces of $\mathbb{Q}$-motives with additional structures.

- Over $\mathbb{C}$, by Hodge realization Sh_G =a moduli of $\mathbb{Q}$-Hodge structures.
- Over $\mathbb{Z}_p$, by integral p-adic Hodge realization Sh_G is related to "moduli of $\mathbb{Z}_p$-Hodge structures" (Rapoport-Zink $\mathcal{N}_{G_p} \to \mathrm{Isoc}_{G_p}$).

Question (global distribution)

$(R(f)Sh_{H_1}, Sh_{H_2})_{Sh_G} =?$ *for a regular Hecke operator* f *on* Sh_G.

By a geometric product formula (uniformization), we are led to

Question (matching of local terms)

$(g\mathcal{N}_{H_{1,p}}, \mathcal{N}_{H_{2,p}})_{\mathcal{N}_{G_p}} =?$ *for a regular automorphism* g *on* $\mathcal{N}_{G_p} \to \mathrm{Isoc}_{G_p}$.

Motivation: arithmetic fundamental lemmas (AFL)

Theorem (AFL, W. Zhang, Mihatsch-Zhang, Z.)

$(g\mathcal{N}_{U_n}, \mathcal{N}_{U_n})_{\mathcal{N}_{U_n} \times \mathcal{N}_{U_{n+1}}}$ *is an explicit derived orbital integral, $p > 2$.*

Application: p-adic Gross-Zagier for $\rho_n \otimes \rho_{n+1}$ (Disegni-Zhang).

Theorem (Twisted AFL, Z.)

$(g\mathcal{N}_{U_n}, \mathcal{N}_{U_n}, \mathcal{Z}(u))_{\mathcal{N}_{GL'_n}}$ *is an explicit derived orbital integral, $p > 2$.*

Application: arithmetic of twisted triple product of elliptic curves.

One core idea for the method: global modularity and induction.

Motivation: arithmetic fundamental lemmas (AFL)

Theorem (AFL, W. Zhang, Mihatsch-Zhang, Z.)

$(g\mathcal{N}_{U_n}, \mathcal{N}_{U_n})_{\mathcal{N}_{U_n}\times\mathcal{N}_{U_{n+\mathbf{1}}}}$ *is an explicit derived orbital integral,* $p > 2$.

Application: p-adic Gross-Zagier for $\rho_n \otimes \rho_{n+1}$ (Disegni-Zhang).

Theorem (Twisted AFL, Z.)

$(g\mathcal{N}_{U_n}, \mathcal{N}_{U_n}, \mathcal{Z}(u))_{\mathcal{N}_{GL'_n}}$ *is an explicit derived orbital integral,* $p > 2$.

Application: arithmetic of twisted triple product of elliptic curves.

One core idea for the method: global modularity and induction.

- TAFL is different and complicated (Wadspurger v.s. Hecke). New ideas: $GL_n(E)\backslash GL_n(L)/GL_n(K)$ (orbits); mirabolic cycles (otherwise no induction); twisted CM cycles on Hilbert Shimura varieties (mysterious); relations of non-classical cycles (modularity).
- Other important but complicated open questions: Hecke AFL, linear AFL.. This core idea does not work directly. New ideas?

From char p to char 0?

Question

Is there a new uniform proof without induction and global modularity?

Theorem (Wang-Yun-Z, in progress)

$(g\mathcal{N}_{U_n}, \mathcal{N}_{U_n})_{\mathcal{N}_{U_n} \times \mathcal{N}_{U_{n+1}}}$ *is an explicit derived orbital integral,* $p > 2n + 2$.
We follow the road (after Ngo)

powerful function field → local function field → p-adic field → theorems over $\mathbb{Q}$

We may transfer back to $\mathbb{Q}_p$ (for minuscule g) to provide a new proof of AFL without induction. General g: model theory?
This method of globalization may also apply to Hecke AFL.

Convolution algebra

Let $f : X \to Y$ be a stratified small map ($\dim f^{-1}(y) < 2\mathrm{codim}\, y$) where Y is irreducible and X has mild singularity. So f is proper and generically finite. We consider $H^{top}(X \times_Y X)$, an algebra under convolution.

Proposition (Convolution algebra)

$$\overline{\mathbb{Q}_\ell}[f^{-1}(y) \times f^{-1}(y)]^\Gamma \cong H^{top}(X \times_Y X) \cong \mathrm{End}_Y(f_* IC_X)$$

induces natural gradings of $f_* IC_X$. Here y a generic geometric point of Y, and fiber $f^{-1}(y)$ with a monodromy action of Γ.

Enhanced relative set ups

Theorem (Wang-Yun-Z.)

- *On a dense open subset of* $\mathcal{A}^{\heartsuit}$, $Rh_*(IC_\lambda^S \otimes L_\eta) \cong Rh'_*(IC_\lambda^U)$.
- ($\lambda = 0$) *For the irreducible component passing a, convolution algebra*

$$Conv_{2d} = \overline{\mathbb{Q}_\ell}[(\mathbb{Z}/2\mathbb{Z})^{2d} \times (\mathbb{Z}/2\mathbb{Z})^{2d}]^\Gamma, \quad \Gamma = (\mathbb{Z}/2d)^{2d} \rtimes S_{2d}.$$

$Conv_{2d}$ = the space of S_{2d}-invariant functions on $(\mathbb{Z}/2\mathbb{Z})^{2d}$. For $0 \leq i \leq d$, $T_i \in Conv_{2d}$, the characteristic function of (i bits equal to 1). $Conv_{2d} \cong \mathbb{Q}_\ell^{\oplus(2d+1)}$ is commutative with characters χ_i, $-d \leq i \leq d$, where $\chi_i(T_1) = -2i$. So we have a grading

$$Rh_*(IC_{\mathcal{M}^U}) = \oplus_{i=-d}^{d} K_d$$

which also matches the grading on $Rh_*(IC_{\mathcal{M}^S} \otimes L_\eta)$ by L_η.

Global theorem: *Sht*

Hecke stack $Hk^r \to Bun_{U_n} \times Bun_{U_n} \times (X')^r$ (elementary modifications)

$$\mathcal{E}_0 \overset{x_1', \sigma(x_1')}{--} \mathcal{E}_1 \overset{x_2', \sigma(x_2')}{--} .. \overset{x_r', \sigma(x_r')}{--} \mathcal{E}_r.$$

This produces moduli of hermitian Shtukas $Sht^r_{U_n} \to (X')^r$. Similarly, we have the Hecke stack $Hk^r_{\mathcal{M}^U} \to \mathcal{M}^U \times_{\mathcal{A}} \mathcal{M}^U \to \mathcal{A}$. Derived Shtuka space

$$\mathrm{RSht}^{<r>}_{M^U} = (id, Fr)\mathcal{M}^U \times_{(\mathcal{M}^U \times \mathcal{M}^U} Hk^r_{\mathcal{M}^U} \to \mathcal{A}.$$

Global theorem: *Sht*

Hecke stack $Hk^r \to Bun_{U_n} \times Bun_{U_n} \times (X')^r$ (elementary modifications)

$$\mathcal{E}_0 \overset{x_1',\sigma(x_1')}{--} \mathcal{E}_1 \overset{x_2',\sigma(x_2')}{--} \ldots \overset{x_r',\sigma(x_r')}{--} \mathcal{E}_r.$$

This produces moduli of hermitian Shtukas $Sht^r_{U_n} \to (X')^r$. Similarly, we have the Hecke stack $Hk^r_{\mathcal{M}^U} \to \mathcal{M}^U \times_{\mathcal{A}} \mathcal{M}^U \to \mathcal{A}$. Derived Shtuka space

$$\mathrm{RSht}^{<r>}_{M^U} = (id, Fr)\mathcal{M}^U \times_{(\mathcal{M}^U \times \mathcal{M}^U} Hk^r_{\mathcal{M}^U} \to \mathcal{A}.$$

For nice $a \in \mathcal{A}^{srs}(k)$, $\deg(\mathrm{RSht}^{<r>}_{M^U,a})$ encodes derived intersection numbers of special cycles with invariants a on $Sht^r_{U_n}$ (global analog).

Proposition

$Hk^r_{M^U} = T_1 \in H^{top}(\mathcal{M}^U \times_{\mathcal{A}} \mathcal{M}^U) = Conv_{2d}$.

Global to local

Theorem

$$\deg(\mathrm{RSht}^{<r>}_{M^U,a}) = \sum_{i\in\mathbb{Z}}(-1)^i(-2i)^r\#\mathcal{M}^S_{i,a}(k).$$

We may assume both sides are non-zero. The natural map (image has finitely many points)

$$\mathrm{RSht}^{<r>}_{M^U,a} \to X^r.$$

So $\deg(\mathrm{RSht}^{<r>}_{M^U,a})$ is a sum of terms indexed by closed points of X^r (not r closed points of X, unless $r = 1$).

Global to local

Theorem

$$\deg(\mathrm{RSht}^{<r>}_{M^U,a}) = \sum_{i\in\mathbb{Z}}(-1)^i(-2i)^r\#\mathcal{M}^S_{i,a}(k).$$

We may assume both sides are non-zero. The natural map (image has finitely many points)

$$\mathrm{RSht}^{<r>}_{M^U,a} \to X^r.$$

So $\deg(\mathrm{RSht}^{<r>}_{M^U,a})$ is a sum of terms indexed by closed points of X^r (not r closed points of X, unless $r=1$).
We specialize in $r=1$ for uniformization. Product formula ($r=1$)

$$\deg(\mathrm{RSht}^{<r>}_{M^U,a}) = \sum_x \mathrm{Int}_{|x|}(a)\prod_{v\neq|x|} Orb_v(a).$$

RHS$=\sum_x \partial Orb_{|x|}(a)\prod_{v\neq|x|} Orb_v(a)$.

Epilogue: questions and future goals

Thank you for your attention!