Orbital integrals, Hecke algebras and relative Langlands program: global rigidity and local conjectures

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Schedule

Principles for fundamental lemmas and so on

1. Study local objects by local geometry and counting: locally constancy, limits, parabolic descents, stratification and inductive structures.
2. It is very hard to locally prove some refined local conjectures of local objects, e.g. orbital integrals, local Whittaker functions.
3. With **global rigidity and purity**, we could "easily" prove local conjectures, without refined study of local objects, e.g. **local rigidity and purity** (hard to prove!). Only need to know in simple cases, conjectures are true by computations, or by induction.

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4. Local conjectures are formulated with global motivations, and based on local computations in simple cases. **Often global proofs appear first, then several years later we understand refined local geometry / analysis and probably find local proofs**. Examples: fundamental lemmas (Ngo), variants with ramifications (smooth transfers), **local Langlands correspondences for classical groups** (Arthur), local Jacquet-Langlands transfer for GL_n, stable base change (Clozel-Labesse).

What Langlands said in 1992

Remarks on Igusa Theory and Real Orbital Integrals, Robert P. Langlands, the zeta functions of Picard modular surfaces, Les Publications CRM Montreal (1992).

> *"The study of orbital integrals on p-adic groups has turned out to be singularly difficult, and even the most basic results in the simplest examples are surprisingly hard to come by ([H])."*
>
> *"For real groups there is a much more incisive technique that is based on the differential equations exploited by Harish-Chandra to study the behaviour of orbital integrals near singular elements ([HC I]), and that Shelstad used to obtain the first serious results on transfer and endoscopy."*

Global applications of local conjectures

Often such **local conjectures are last missing pieces to prove global conjectures with non-trivial arithmetic applications**.

1. The proof of global Gan-Gross-Prasad conjecture on L-functions and period integrals for $\mathrm{U}_n \times \mathrm{U}_{n+1}$ (2009-2020s, related to fundamental lemmas today). It applies to prove rank 0 Bloch-Kato conjectures for $\mathrm{GL}_n \times \mathrm{GL}_{n+1}$ motives via bipartite Euler systems ([LTXZZ], 2019).
2. The proof of Gross-Zagier formula on modular curves (1986) and arithmetic Gan-Gross-Prasad conjecture for $\mathrm{U}_n \times \mathrm{U}_{n+1}$ (related to arithmetic fundamental lemmas and my PHD thesis). It applies to prove one direction of rank 1 Bloch-Kato conjectures for $\mathrm{GL}_n \times \mathrm{GL}_{n+1}$ motives (Disegni-Zhang 2024) e.g. BSD conjecture ($n = 1$), and also to Gauss class number problem (Goldfeld, 1976).
3. (E. Frenkel, Gauge theory and Langlands duality) From A-branes (**Hitchin fibration**) to Hecke eigensheaves has an analogue as a passage from orbital integrals to Hecke eigenfunctions.

Global rigidity

1. Global rigidity over number fields: construct related modular forms on SL_2. Used in W. Zhang's proof of arithmetic fundamental lemma, no use of refined local structures (e.g. local modularity / Bruhat-Tits).
2. Toy model: the space of modular forms with given level K and weight k is finite dimensional.
3. For a summary of such a global method and applications, see my recent arxiv paper "Non-reductive special cycles and Twisted Arithmetic Fundamental Lemma" and an almost modularity method to handle ramification in my PHD thesis.

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4. Global rigidity over function fields: construct related perverse sheaves on the relative Hitchin base.
5. Toy model: given a finite map $f : X \to Y$, $f_*\mathbb{Q}_\ell$ is often determined on the locus of Y where f is fintie étale.
6. **(Today) use global rigidity to study relative orbital integrals for Hecke functions (joint work with Griffin Wang).**

Schedule

General orbital integrals

G acts on X over a local arithmetic field F (e.g. $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$).

$$\mathrm{Orb}(G.x, f) := \int_{y \in G.x} f(y)dy, \ \forall f \in \mathcal{S}(X), x \in X.$$

Classical: consider the G-conjugacy action on $X = G$.

Relative: $H_1 \to G \leftarrow H_2$. $H_1 \times H_2$-action on G, or H_1-action on G/H_2.

Example (Tate thesis)

$\mathbb{G}_m$ acts on $\mathbb{A}^1$. $f_p = 1_{\mathbb{Z}_p}$, $x = 1 \in \mathbb{A}^1$.

$$\zeta_p(s) = \int_{\mathbb{G}_m} f_p(t.1)|t|^s dt = (1 - p^{-s})^{-1}.$$

$$\zeta_\infty(s) = \int_{\mathbb{G}_m} f_\infty(t.1)|t|^s dt = \pi^{-s/2}\Gamma(s/2).$$

$$\zeta_\infty(s)\zeta(s) = \prod_p \zeta_p(s) = \int_0^\infty y^{s/2} \sum_{n=1}^\infty e^{-n^2\pi y} dy.$$

Geometrization: affine Springer fibers

Orbital integrals for G-actions could be related to $\mathbb{F}_q$-points of certain infinite type algebraic variety (**affine Springer fiber**) for G.
Let $K \leq G$ be a compact open subgroup. Let $f \in \mathcal{S}(X)^K$. Any $x \in X$ gives a function f_x on $L \in G/K$ (note $G.x \cong G_x \backslash G$)

$$f_x(L) = f(L^{-1}.x).$$

$$\mathrm{Orb}(G.x, f) = \int_{L \in G_x \backslash G/K} f_x(L) dL$$

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G/K is an affine Grassmanian where we could do algebraic geometry.

Example

$GL_n(\mathbb{Q}_p)/GL_n(\mathbb{Z}_p) = \{\text{lattices in } \mathbb{Q}_p^n\}$

However, f_x is complicated and these fibers are **not smooth, nor proper, and could not be deformed in an algebraic way.**
Example: there is no algebraic family whose generic fiber is a chain of projective lines and the special fiber is a single projective line.

Orb=local Hitchin densities (Frenkel-Langlands-Ngo)

the local Hitchin map $h_X : X/G \to X//G$ from stack to GIT quotient (assuming $X//G$ is a good quotient).
Think $a \in X//G(F_x)$ as the set of regular semi-simple G-orbits on X (assuming there is a rational Kostant type section $X//G \to X, a \to x_a$).
Let $f = 1_\Omega$ for a K-stable compact open subset Ω of X (thought as an integral model of X). Study $h_X : \Omega \to \Omega//K$ (over O_F).

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Proposition (why Orb appears in Langlands-Kottwitz method)

$$\mathrm{Orb}(a, f) = \#\{x \in \Omega | h_X(x) = h_X(x_a)\}.$$

$\#$ means counting volume ($F = \mathbb{Q}_p, F_p((t))$). See also Cho-Yamauchi.
We have so called **local evaluation map** of affine Springer fibers:

$$ev_x : \{L \in G/K | f_x(L) \neq 0\} \to X(F)/G(O), L \mapsto L^{-1}.x$$

which allow us to pullback Hecke functions on X to affine Springer fibers e.g. (for $H_1 \times H_2$-action on G, pullback Hecke algebra $S(K\backslash G/K)$).

Globalization and local-global product formula

Idea (after Ngo): construct a global **Hitchin fibration**

$$h_{X,C} : \mathcal{M} = Map(C, \text{``}[X/G]\text{''}) \to \mathcal{A} = Map(C, \text{``}[X//G]\text{''})$$

over a curve C, s.t. local geometry of C at a given point remembers above local geometry. Fibers are called Hitchin fibers.

Proposition

$\int_{h_{X,C}{}^{-1}(a)} f$ *remembers all local orbital integrals* $Orb(a_x, f_x), x \in C$.

Here we use **a global deformed version** of "$[X/G]$" and "$[X//G]$" (after choosing a line bundle D on C).

- It is very easy to deform global Hitchin fibers, as you can deform spectral curves easily (hence a defomation of compactified Jacobian).
- Moreover, global sheaf theory is well-defined and we even have the Weil conjecture (global purity)!

Deformation and global version on C

$D = K_C$ is interesting for physics but hard. Let $degD >> 0$ (avoid derived stacks). For M with $\mathbb{G}_m$-action, get a bundle $D \times_{\mathbb{G}_m} M$ on C.

Proposition (classical Lie algebra case $(D \times_{\mathbb{G}_m} \mathfrak{g})//G$, Ngo)

Good deformation exists globally as fibers of Hitchin fibrations.

Example: deformation of curves and compactified Jacobians.

Proposition (classical group case $(D \times_{\mathbb{G}_m} M)//G$, Griffin Wang 2024)

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Good deformation exists globally as multiplicative Hitchin fibrations.

Use $G \to M$ to monoids (Higgs fields with poles), where M behaves like Lie algebra. Example: $GL_n \to \mathrm{Mat}_{n\times n}$. Note $G//G \cong \mathfrak{g}//G$ when G is semisimple and simply-connected (not true generally e.g. torus).

Proposition (relative GGP cases $(D \times_{\mathbb{G}_m} M)//H$, Wang-Z. 2024)

Good deformation exists globally as relative multiplicative Hitchin fibrations. Kostant section exists (with obstruction for rationality).

Set up: relative invariant theory on two sides

Let F/F_0 be an unramified quadratic extension of local fields. Let V be a n-dimensional hermitian space over F. Let $V^\sharp = V \oplus Fe$, $(e,e)=1$.

$$H = \mathrm{U}(V) \to G_{tot} = \mathrm{U}(V) \times \mathrm{U}(V^\sharp) \leftarrow H = \mathrm{U}(V).$$

Proposition (Unitary side)

The $H \times H$-action on G_{tot} could be reduced to **the conjugacy action of** H **on** $G = \mathrm{U}(V^\sharp)$.

$$H' = \mathrm{GL}(V) \to G'_{tot} = \mathrm{GL}(V) \times \mathrm{GL}(V^\sharp) \overset{\eta}{\leftarrow} H'_2 = \mathrm{GL}_n(F_0) \times \mathrm{GL}_{n+1}(F_0).$$

Proposition (Symmetric side)

The $H' \times H'_2$-action on G'_{tot} could be reduced to **the conjugacy action of** $H'_0 = \mathrm{GL}_n(F_0)$ **on**
$\mathrm{GL}_{n+1}(F)/\mathrm{GL}_{n+1}(F_0) \cong S_{n+1} = \{\gamma \in GL_{n+1}(F) | \gamma\overline{\gamma} = 1\}$.

Orbits and orbital integrals

Proposition

There is an natural isomorphism of GIT quotient (preserving character polynomials $G//G \cong S_{n+1}//GL_{n+1}$)

$$G//H \cong S_{n+1}//H_0'$$

hence an embedding of regular semi-simple orbits $[\mathrm{U}(V^\sharp)//\mathrm{U}(V)]_{\mathrm{rs}}$ (together with inner forms) to $[S_n//\mathrm{GL}_n(F_0)]_{\mathrm{rs}}$.

Classical version: Lie algebra. We use monoid M_G (resp. $M_{S_{n+1}}$)"containing" G (resp. S_{n+1}), and extend above relative invariant theory to monoids. A universal example: Vinberg monoid.
The advantage of monoid is that it looks like "Lie algebra" (with scaling actions) and encode information for Satake functions on reductive group G (rather than Lie algebras).

Combinatorial matching of Hecke functions

Assume there a self-dual lattice L in V, i.e. V is quasi-split.

$$S_{n+1} \to \mathrm{GL}(V^\sharp) \leftarrow \mathrm{U}(V^\sharp)$$

Given $C_\lambda = \mathrm{GL}_{n+1}(O_F)\varpi^\lambda \mathrm{GL}_{n+1}(O_F) \leq \mathrm{GL}(V^\#)$ (Cartan). Let

$$C_\lambda^U = C_\lambda \cap U(V^\sharp),\, C_\lambda^S = C_\lambda \cap S_{n+1}.$$

1. GL_{n+1}-spherical variety $S_{n+1} = \coprod_\lambda C_\lambda^S$ as $\mathrm{GL}_{n+1}(O_F)$-orbits.
2. $\mathrm{U}(V^\sharp) \times \mathrm{U}(V^\sharp)$-spherical variety $\mathrm{U}(V^\sharp) = \coprod_\lambda C_\lambda^U$ as $\mathrm{U}(L^\sharp) \times \mathrm{U}(L^\sharp)$-orbits.

Matching of functions $\mathcal{S}(S_{n+1})^{\mathrm{GL}_{n+1}(O_F)} \to \mathcal{S}(\mathrm{U}(V^\sharp))^{\mathrm{U}(L^\sharp)\times \mathrm{U}(L^\sharp)}$.

Example

$1_{C_\lambda^S} \leftrightarrow 1_{C_\lambda^U},\ \ 1_{Sat_\lambda^S} \leftrightarrow 1_{Sat_\lambda^U}.$

In general, for G-spherical variety X, decompose $K\backslash X$ via $K.ax_0$, $a \in A/Stab(x_0) = A_X$ (A maximal torus of G) from a coweight $\mathbb{G}_m \to A_X$.

Orbital integrals

Definition (Orbital integral: symmetric side)

Consider $f' \in \mathcal{S}(S_{n+1})$, and a regular semi-simple orbit $\gamma \in S_{n+1}(F_0)$. Define

$$\mathrm{Orb}(\gamma, f') = \int_{h \in \mathrm{GL}_n(F_0)} f'(h^{-1}\gamma h)\eta(\det h)dh.$$

$\eta : F_0^\times \to \{\pm 1\}$ be the quadratic character associated to F/F_0 by class field theory. If F/F_0 is inert (resp. split), we have $\eta(x) = (-1)^{\mathrm{val}(x)}$ (resp. $\eta = 1$).

Definition (Orbital integrals: unitary side)

Consider $f \in \mathcal{S}(\mathrm{U}(V^\sharp))$, and a regular semi-simple orbit $g \in \mathrm{U}(V^\sharp)(F_0)$. Define

$$\mathrm{Orb}(g, f) = \int_{h \in \mathrm{U}(V)} f(h^{-1}gh)dh.$$

Here we normalize the Haar measure such that $\mathrm{GL}_n(O_{F_0})$ has volume 1.

Fundamental Lemmas and smooth transfers for Hecke algebras

Relative fundamental lemmas improve our understandings of relative Langlands program ([BZSV]), numerically L-functions and period integrals.

Theorem (Wang-Zhang 2024, fundamental Lemmas and smooth transfers for Hecke algebras)

Assume $p > 2n$ and $F_0 = \mathbb{F}_q((t))$. For any strongly regular semi-simple orbits $\gamma \leftrightarrow g$ and spherical $f' \leftrightarrow f$, we have

$$\omega(\gamma)\mathrm{Orb}(\gamma, f') = \mathrm{Orb}(g, f).$$

And $\omega(\gamma) \in \{\pm 1\}$ is a transfer factor such that left hand side only depends on orbits of γ. Via model theory, our result may be used to obtain a new global proof of (explicit) smooth transfers for spherical Hecke algebras over p-adic fields. Methods are general (**apply to other cases without extra symmetry**).

Why Hecke algebras?

Above theorem over p-adic local fields is proved by Spencer Leslie (2022), via **existence of smooth transfers**, **refined local haramonic analysis** and **global comparison of relative trace formulas** built on works of Beuzart-Plessis (2020), which is not known in equal characteristic.

Why Hecke algebras?

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For global automorphic representation π of G, π is built from local representation π_v of G_v.
For almost every place v, π_v is unramified and determined by the action of Hecke algebra $S(K_v \backslash G_v / K_v)$ (Satake isomorphism), which gives a collection of Hecke eigenvalues.
In good cases, multiplicity one theorem determines π from these Hecke eigenvalues at v for almost every place v.
Toy model: a Hecke eigenform of positive weight $f = \sum_n a_n q^n$ is determined by a_p for almost all p.
Therefore, **fundamental lemmas for spherical Hecke algebras are enough** in many cases to understand automorphic forms and L-functions.

Hecke correspondence on modular curves

Compactified modular curve
$X_0(N) = \{N$-cyclic isogeny of generalized elliptic curves $f : E_1 \to E_2\}$
For $m \geq 1, (m, N) = 1$, the Hecke correspondence T_m on $X_0(N)$ given by

$$T_m x = \sum_C x_C$$

where the sum is taken over all subgroups C of order m in E_1 with $C \cap Ker f = 1$, where x_C is the point $f : E/C \to E'/f(C)$.

$$\deg T_m = \sum_{d|m, d>0} d.$$

Reconstruct functions from relative orbital integrals

Proposition (Proposition 8.1.1, spherical AFL, Li-Rapoport-Zhang)

Assume $F_0 = \mathbb{Q}_p$. For f_1', f_2' in $\mathcal{S}(S_{n+1})^{\mathrm{GL}_n(O_F)}$, if $\mathrm{Orb}(\gamma, f_1') = \mathrm{Orb}(\gamma, f_2')$ for all regular semi-simple γ, then $f_1' = f_2'$.

Proof uses results on local characters on tempered representations (local Rankin–Selberg periods and the local Flicker–Rallis periods) by Beuzart-Plessis.

Is the same thing true in char p?

Then spherical FL implies the same is true for

$$Orb : \mathcal{S}(\mathrm{U}_{n+1})^{\mathrm{U}_n \times \mathrm{U}_n} \mapsto C^\infty(\mathrm{U}_{n+1}(F_0)_{rs}).$$

Base change from U_n to GL_n=restriction of Satake parameters from $\mathbb{C}[T(F)/F(O)]^{W_{\mathrm{GL}_n}}$ to $\mathbb{C}[A(F)/A(O)]^{W_{\mathrm{U}_n}}$ (along $A^\vee \to T^\vee$).

Generalization to singular orbits and orbital integrals

Theorem (Deligne-Ranga Rao (1972))

Classical adjoint orbital integral is convergent for any element $\gamma \in G(F)$.

For relative orbital integrals on singular obits, regularization are needed. For transfers e.g.on regular non-semisimple orbits, see

1. Section 7 of "geometric side of the Jacquet-Rallis relative trace formula" (Weixiao Lu, 2024).
2. 'Endoscopic transfer for unitary Lie algebras' (Jingwei Xiao, 2018).

An extension of our fundamental lemmas for Hecke functions to singular orbits, may be obtained by studying degenerations / truncations of Hitchin fibers.

Let $f = 1_{\Omega}$, $f' = 1_{\Omega'}$. Then U_1 acts trivially on U_2, we have $\mathrm{Orb}(g, f) = f(g) \in \{0, 1\}$.

Need to show same thing happens on symmetric side. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

$$\mathrm{Orb}(\gamma, f') = \int_{t \in GL_1} f'(\begin{pmatrix} a & t^{-1}c \\ tb & d \end{pmatrix})\eta(t)dt = \sum_{m \in \mathbb{Z}} \eta(m) f'(\begin{pmatrix} a & \varpi^{-m}c \\ \varpi^m b & d \end{pmatrix}).$$

$= \sum_m \eta(m) = \pm(1 - 1 + 1 - 1 + 1 - 1....) \in \{-1, 0, 1\}$.
Everything is computable e.g. take $f'_m = 1_{GL_2(O_F).(0,\omega^m//\omega^{-m},0)}$ you get $\eta(m)$ or 0.
See Proposition 7.2.1., 7.3.1. and Remark 7.3.3. of Spherical AFL, Li-Rapoport-Zhang.

Schedule

Monoids for GL_n and its Galois twists

We first detwist everything. Set $G = \mathrm{GL}_n$, with the standard rep of G.

- $T \subseteq G$ maximal torus, $Z \subseteq G$ center.
- Consider the universal monoid M of $G^{sc} = \mathrm{SL}_n$. The unit group of M is $(T^{sc} \times G^{sc})/Z^{sc}$, with center $Z_M = T^{sc}$.
- Construction of universal monoid (advantage: deformations remember all directions given by Satake functions):

$$\rho_* : M^\times \to \mathbb{A}^{n-1} \oplus \oplus_{i=1}^{n-1} \mathrm{End}(\wedge^i V), \ (z, g) \mapsto (\alpha_i(z), \omega_i(z) \wedge^i g)$$

Here α_i are simple roots and ω_i are fundamental weights of G. M will be the normalization of the closure of $\rho_*(M^\times)$.

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  Here α_i are simple roots and ω_i are fundamental weights of G. M will be the normalization of the closure of $\rho_*(M^\times)$.
- $H = \mathrm{GL}_{n-1} \subseteq G$, which is the stabilizer of $\mathbf{e} = (0, \ldots, 0, 1)^t \in V$ and $\mathbf{e}^* = (0, \ldots, 1)^t \in V^*$.
- Fact: invariant theory $M \to M//H \to M//G$ (adjoint action) could be described (affine spaces with Kostant sections). Ex: $\dim M//H = (3n - 3)$.

Global moduli spaces for deformations

Choose an etale double cover $\theta : X' \to X$ of curve over k. Consider

$$M/H/Z_M \to M//H/Z_M$$

We get the multiplicative Hitchin fibration for M from three steps:

- Replace $[M/H] \to M//H$ by its global deformed version $\underline{M} \to \underline{C}$
- Apply $\mathrm{Map}(X, -)$ to $\underline{M}/(Z_M \times \mathbb{G}_m) \to \underline{C}/(Z_M \times \mathbb{G}_m)$.
- Restrict to the open substack of boundary divisors.

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We need Galois twistings of M, which produces $Z_{\mathfrak{M}'} \leq \mathfrak{M}'$ (symmetric side) and $Z_{\mathfrak{M}} \leq \mathfrak{M}$ (unitary side).

Definition

$h : \mathcal{M} \to \mathcal{A}$ (resp. $h' : \mathcal{M}' \to \mathcal{A}$): the Jacquet–Rallis mH-fibration on the symmetric (resp. unitary) side.

Additional $\mathbb{G}_m$ occurs due to global deformation. Note modulo $Z_M \times \mathbb{G}_m$ amounts to choose a Z_M-torsor L, and a line bundle D on X.

Moduli stack of Boundary divisor B

Z_M on M plays the role of $\mathbb{G}_m$-action on Lie algebra.
By general theory, the monoid M comes with an abelianization map (remember $\mathbb{A}^{n-1}$-part of ρ)

$$\alpha_M : M \to A_M = M//(G^{sc} \times G^{sc}) \cong \mathbb{A}^{n-1}$$

We have $\alpha_M(A_M^\times) = M^\times$, $\alpha_M(1) = G^{sc}$. Recall center $Z_M = T^{sc}$.

Example

$n = 2$, $M = Mat_{2\times 2}$, $\alpha_M : M \to \mathbb{A}^1$ is the determinant map. $Z_M = \mathbb{G}_m$.

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Mapping stack construction naturally lies over $B^+ := Map(X, A_M/Z_M)$ (via $(M//H)/Z_M \to (M//G)/Z_M \to A_M/Z_M$). Not seen in Lie algebra version: $[\mathfrak{g}/(G \times \mathbb{G}_m)] \to [g//G/\mathbb{G}_m] \to [*/\mathbb{G}_m]$.
There is a maximal open Deligne-Mumford substack B, called the moduli stack of boundary divisors. Pull back to B to make things well-behaved.

Deformed quotient stack over $k = \mathbb{F}_q$

Let S be a k-scheme. Let $\underline{M}(S)$ be the groupoid of $(\mathcal{E}, x, e, e^\vee)$ where

- $\mathcal{E}$ a vector bundle on $X \times S$ of rank n,
- x a section in $M \times^G \mathcal{E}(S)$,
- $e : \mathcal{O}_S \to \mathcal{E}$.
- $e^\vee : \mathcal{E} \to \mathcal{O}_S$.

Get $\underline{M} \to \mathbb{A}^1$, $(\mathcal{E}, x, e, e^\vee) \to e^\vee \circ e \in \mathcal{O}_S$.
The preimage of 1 is $\underline{M}_1 \cong [M/H]$.
Obtain the map $\underline{M} \to \underline{C} := M//H \times \mathbb{A}^1$ as global deformation of $[M/H] \to M//H$.
There is an addition action of $\mathbb{G}_m$ on $\underline{M}$ by scaling e and $e^\vee$, preserving $e^\vee \circ e$.

Affine Jacquet–Rallis fibers

$$\mathcal{M} \xrightarrow{h} \mathcal{A} \xleftarrow{h'} \mathcal{M}'$$

The strongly regular semi-simple locus $\mathcal{A}^{\heartsuit} \subseteq \mathcal{A}$ is the complement of a relative discriminant divisor $Disc_{M,H}$.

For any local place v of X (split or inert), local Hitchin map gives

$$\mathcal{M}_v \xrightarrow{\chi} \mathcal{C}_v \xleftarrow{\chi} \mathcal{M}'_v$$

Definition

Choose strongly regular semi-simple $a \in \mathcal{C}_v(\mathcal{O}_v)^{\heartsuit}$, consider $\mathcal{M}_v(a) := \chi^{-1}(a) \cap \mathcal{M}(O_v)$ (resp. $\mathcal{M}'_v(a) := \chi^{-1}(a) \cap \mathcal{M}'(O_v)$) as affine Jacquet-Rallis fiber for symmetric side (resp. for unitary side).

At a split place v, take (γ, e, e^*) with invariant a, we have
$\mathcal{M}_v(a)(k) = \mathcal{M}'_v(a)(k) = \{\Lambda \subseteq V_v | \gamma_i \Lambda_i \subseteq \Lambda_i, e \in \Lambda, e^* \in \Lambda\}$.

Matching orbits and Hecke functions as Satake sheaves

Matching of functions $\mathcal{S}(S_n)^{\mathrm{GL}_n(O')} \to \mathcal{S}(\mathrm{U}_n)^{\mathrm{U}_n(O)\times \mathrm{U}_n(O)}$. Affine Grassmanian

$$S_n/\mathrm{GL}_n(O') \to \mathrm{GL}_n(O')\backslash \mathrm{GL}_n(F')/\mathrm{GL}_n(O') \leftarrow \mathrm{U}_n(O)\backslash \mathrm{U}_n(F)/\mathrm{U}_n(O)$$

From geometric Satake, we have the $\mathrm{GL}_n(O')$-equivariant perverse sheaves Sat_λ on affine Grassmanian for $\mathrm{GL}_n(F')/\mathrm{GL}_n(O')$. Pull back we get

Definition

IC^S_λ (resp. IC^U_λ): the pullback of local Satake sheaf to $S_n/\mathrm{GL}_n(O')$ (resp. $\mathrm{U}_n(F)/\mathrm{U}_n(O)$) with highest coweight λ.

We may further pullback these sheaves to affine Jacquet-Rallis fibers (via local evaluation map to affine Grassmanian). Via function-sheaf dictionary $\mathcal{F} \to \#_{\mathcal{F}}(-)$, fundamental lemma is equivalent to

Proposition (Local identity, L_η on $\mathcal{M}_v^\heartsuit$, Wang-Z. 2024)

$\#_{IC^S_\lambda \otimes L_\eta} \mathcal{M}_v(a)(k_v) = \#_{IC^S_\lambda} \mathcal{M}'_v(a)(k_v)$.

Comparison of perverse sheaves and global equality

Proposition (Global identity)

On a dense open subset of $\mathcal{A}^{\heartsuit}$, we have $Rh_*(IC^S_\lambda \otimes L_\eta) \cong Rh_*(IC^U_\lambda)$ (up to semi-simplification) as perverse sheaves.

Proved via stratified smallness of Hitchin fibrations and some simple computations. Similar to Yun's proof of Jacquet-Rallis fundamental lemma, but no spectral curves are used.
By local-global product formulas, local model of singularity, we get local equality from global identities.

why it is semi-small

Schedule

Relative Langlands duality and orbital integrals

Moment map $\mu : M = T^*X \to \mathfrak{g}^*$ (symplectic cotangent stack).
$\mu//G : T^*X//G \cong \mathfrak{t}_X^*//W_X \to \mathfrak{t}^*//W \cong \mathfrak{g}^*//G$.
Future hopes: a story for general M (with interesting microlocal geometry). Then Orb could be defined for functions on Lagrangians of M but only depends on M (Fourier transforms commute with transfers).

$$Orb : S(X)^K \to Func(X//G)$$

$Orb_{RTF} : S(X \times X)^{K \times K} \to Func(X \times X//G)$.

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Example ($S(V^r)$ = Weil rep ω of $G_r = \mathrm{U}(r,r)$, V hermitian space)

$\mathrm{Herm}_r \cong N_r \leq G_r$ radical of Siegel parabolic. For $\varphi \in \mathcal{S}(V^r), g \in G_r$,

$$W_T(g, \varphi) := \int_{b \in N_r(F)} (\omega(w_r^{-1})\omega(b)\omega(g).\varphi)(0)\psi(-\mathrm{Tr}(Tb))db.$$

When $r = \dim V$ and $k \geq 0$, get $W_T(s = k, g = 1, \varphi)$ = local density of hermitian lattices (from T to $L_\varphi \oplus \langle 1 \rangle^{2k}$).

Structures of polarized M and Hecke modules

Question: a uniform way to prove conjectures for all good examples?
A side (G, M): periods / orbital integrals. ("automorphic quantizations").
B side $(G^\vee, M^\vee)$: L-functions ("spectral quantizations").
$(G, M) \rightsquigarrow (G^\vee, M^\vee = \mathrm{WhitInd}_\iota^{G^\vee}(S_X))$ (symplectic rep S_X via even spherical colors of X, $\iota : G_X^\vee \times \mathrm{SL}_2 \to G^\vee$).
Example: $(\mathrm{PGL}_2, T^*(\mathrm{PGL}_2/\mathbb{G}_m)) \rightsquigarrow (\mathrm{SL}_2, T^*\mathbb{A}^2)$.

Conjecture (BZSV, conjecture 7.5.1, which assumes $M = T^*X$.)

$$Sh(X_F/G_O) = QCoh^{shear}(M^\vee/G^\vee(-2\rho))$$

compatible with pointings (conjecturally $M^\vee//G^\vee = \mathfrak{h}^{\vee,*}//H^\vee$ is an affine space, L-algebra=$O(M^\vee//G^\vee)$), Hecke actions, Galois actions and Poisson structures.

Version over $\mathbb{F}_p$ implies a description of $G(O)\backslash G(F)/G(O)$-action on $S(X)^K$. Note $H(F)\backslash G(F)/G(O) = \Lambda_X^\vee/W_X$ (Cartan decomposition) only depends on M. **And $M^\vee = S_X$ is linear if M is strongly tempered.**

Kostant sections and gauge fixing in physics

Theorem (Kostant 1963)

Let G be a complex semi-simple group, $\mathfrak{g} = \mathrm{Lie}\, G$.

1. *There exists a global slice $S \subseteq \mathfrak{g}^*_{reg}$ for the coadjoint action.*
2. *The stabilizer G_x is abelian for $x \in \mathfrak{g}^*_{reg}$.*
3. *(Codimension / Hartogs) holmorphic functions on $\mathfrak{g}^*_{reg}$ extends to $\mathfrak{g}^*$.*
4. *(Kirillov-Kostant-Souriau) each coadjoint orbit $G.x$ is symplectic.*

Proof uses $\mathfrak{sl}_2$-triples. Crucially used in proving fundamental lemma via regular centralizers [Ngo10, Section 2], proving derived Satake equivalence [BF08], Whittaker reduction of T^*G and $H^*_{G(O)}(Gr_G)$ [BFM05].

Conjecture

For a $G^\vee$-variety $M^\vee$, find Kostant section for $M^\vee/G \to M^\vee//G^\vee$?

Relative Satake via taking cohomology (Devalapurkar, 2024)
$M^\vee//G^\vee \times_{\kappa, M^\vee/G^\vee, \kappa} M^\vee//G^\vee = J^\vee = H^*_{G(O)}(V(F))$.

Examples

1. Examples where $G_X^\vee$ is small (rank 1 or 2, highly non-tempered)

G	GL_n	SO_n	Sp_{2n}	SL_3	G_2	Spin_7	F_4	E_6
H	GL_{n-1}	SO_{n-1}	Sp_{2n-2}	SO_3	SL_3	G_2	Spin_9	F_4

   Theta correspondence are useful to study them (both locally and globally). See also generalized Whittaker models (Gan-Jun, 2024).

2. Examples where $G_X^\vee = G^\vee$ (iff strongly tempered iff all root type T, related to today's talk):

G	$\mathrm{GL}_{n-1} \times \mathrm{GL}_n$	$\mathrm{SO}_{n-1} \times \mathrm{SO}_n$
H	GL_{n-1}	SO_{n-1}

3. A Levi L is spherical in a split classical group G (Kramer's classification):

$$\mathrm{GL}_j \times \mathrm{GL}_{n-j} \subseteq \mathrm{GL}_n,\ \mathrm{SO}_2 \times \mathrm{SO}_{2n-1} \subseteq \mathrm{SO}_{2n+1},\ \mathrm{GL}_n \subseteq \mathrm{SO}_{2n+1},$$

$$\mathbb{G}_m \times \mathrm{Sp}_{2n-2} \subseteq \mathrm{Sp}_{2n}, \mathrm{GL}_n \subseteq \mathrm{Sp}_{2n}, \mathrm{SO}_2 \times \mathrm{SO}_{2n-2} \subseteq SO_{2n}, \mathrm{GL}_n \subseteq \mathrm{SO}_{2n}$$

4. Can also consider non-smooth or non-homogeneous examples.
5. (Whittaker induction) $(H, M) \rightsquigarrow (G, (M \times \mathfrak{n}/\mathfrak{n}^+) \times^{HN}_{\mathfrak{h}^* \oplus \mathfrak{n}^*} (T^*G))$.

Thank you for Listening