Convergence behavior of central paths for convex homogeneous self-dual cones

This is a preliminary discussion note resulted among Nesterov, Todd and myself. I hope it would be interesting to some love-to-see-superlinear-convergence researchers.

We are interesting in solving the pair of primal and dual problems in K and K^*, which are self-scaled and convex homogeneous primal and dual cones in a finite-dimensional real vector space E, see Nesterov and Todd [1].

We further assume that there is a unique fix point, e, in $\operatorname{int} K$ such that

$$e = -F'(e), \quad \langle e, e\rangle = \nu, \qquad \text{and} \quad F''(e) = I$$

where I is the identical operator or matrix. Thus, we can view K^* as the dual of K with respect to the inner product on E defined by $\langle x, s\rangle := \langle F''(e)x, s\rangle$. Also, $x + s$ makes sense because it corresponds to $x + [F''(e)]^{-1}s$, which is certainly in E. One property for these cones is that for any w and x in $\operatorname{int} K$,

$$F''(w)x \in \operatorname{int} K^* = \operatorname{int} K, \tag{1}$$

where F is the ν self-scaled barrier for K.

Let us first give a definition.

Definition 1 A pair of primal and dual feasible solution are strictly complementary if

$$Ax = b, \quad s = c - A^T y, \quad x \in K, \quad s \in K^*,$$

$$x + s \in \operatorname{int} K, \quad \text{and} \quad \langle x, s\rangle = 0.$$

Lemma 1 *Let x^1 and x^2 be in $\operatorname{int} K$, and $s^1 = -F'(x^1)$ and $s^2 = -F'(x^2)$ in $\operatorname{int} K^*$. Then, if $x^1 - x^2 \in \operatorname{int} K$ then $s^2 - s^1 \in \operatorname{int} K^*$.*

Proof. For any $w \in K$ and $w \neq 0$,

$$\begin{aligned}
\langle s^2 - s^1, w\rangle &= \langle F'(x^1) - F'(x^2), w\rangle \\
&= \langle \int_0^1 F''(x^2 + t(x^1 - x^2))(x^1 - x^2)dt, w\rangle \\
&= \int_0^1 \langle F''(x^2 + t(x^1 - x^2))(x^1 - x^2), w\rangle dt \\
&> 0.
\end{aligned}$$

Here we have used fact (1) which implies that $F''(x^2 + t(x^1 - x^2))(x^1 - x^2) \in \operatorname{int} K^*$ and

$$F''(x^2 + t(x^1 - x^2))(x^1 - x^2), w\rangle > 0,$$

since $x^2 + t(x^1 - x^2) \in \operatorname{int} K$ for any $0 \leq t \leq 1$ and $x^1 - x^2 \in \operatorname{int} K$, and $0 \neq w \in K$.

□

Now we prove the following theorem.

[1] Yu. Nesterov and M. Todd, *Self-scaled barriers and interior-point methods for convex programming* (9462) and *Primal-dual interior-point methods for self-scaled cones* (9544), CORE, Louvain-la-Neuve, 1994-1995.

Theorem 2 *Let* (y^*, x^*, s^*) *be a strictly complementary solution such that*

$$x^* + s^* - \rho e \in K,$$

and let $(y(\mu), x(\mu), s(\mu))$*, for any* $\mu > 0$*, be on the central path, that is,*

$$Ax(\mu) = b, \quad s(\mu) = c^T - A^T y(\mu), \quad x(\mu) \in \text{int}\, K, \quad s(\mu) \in \text{int}\, K^*,$$

and

$$s(\mu) = -\mu F'(x(\mu))$$

or equivalently

$$x(\mu) = -\mu F'(s(\mu)).$$

Then,

$$x(\mu) + s(\mu) - \frac{3\rho}{4\nu} e \in K.$$

Proof. Since

$$\langle x(\mu) - x^*, s(\mu) - s^* \rangle = 0,$$

$$\begin{array}{rcl} \mu\nu = & = & \langle x(\mu), s^* \rangle + \langle s(\mu), x^* \rangle \\ & = & \langle -\mu F'(s(\mu)), s^* \rangle + \langle -\mu F'(x(\mu)), x^* \rangle \\ & \geq & \langle -\mu F'(s(\mu) + x(\mu)), s^* \rangle + \langle -\mu F'(x(\mu) + s(\mu)), x^* \rangle \\ & = & \langle -\mu F'(s(\mu) + x(\mu)), s^* + x^* \rangle \\ & \geq & \langle -\mu F'(s(\mu) + x(\mu)), \rho e \rangle. \end{array}$$

Thus,

$$\langle -F'(s(\mu) + x(\mu)), e \rangle \leq \nu/\rho.$$

From Theorem 5.2 and (5.6) of Nesterov and Todd's first paper and Lemma 3.3 of Nesterov and Todd's second paper, this implies that

$$\sigma_{x(\mu)+s(\mu)}(e) \leq \frac{4\nu}{3\rho}$$

or

$$x(\mu) + s(\mu) - \frac{3\rho}{4\nu} e \in K.$$

□

The result holds for any point in a neighborhood of the central path, based on the above result.

Corollary 3 *Let* (y^*, x^*, s^*) *be a strictly complementary solution such that*

$$x^* + s^* - \rho e \in K,$$

and let (y, x, s)*, for* $\mu = \langle x, s \rangle / \nu > 0$*, be in the neighborhood of the central path, that is,*

$$Ax = b, \quad s = c^T - A^T y, \quad x \in \text{int}\, K, \quad s \in \text{int}\, K^*,$$

and for some $0 < \beta < 1$

$$s - \beta(-\mu F'(x)) \in K^*$$

or equivalently (Lemma 3.2 of Nesterov and Todd's second paper)

$$x - \beta(-\mu F'(s)) \in K.$$

Then,

$$x + s - \frac{3\beta\rho}{4\nu} e \in K.$$