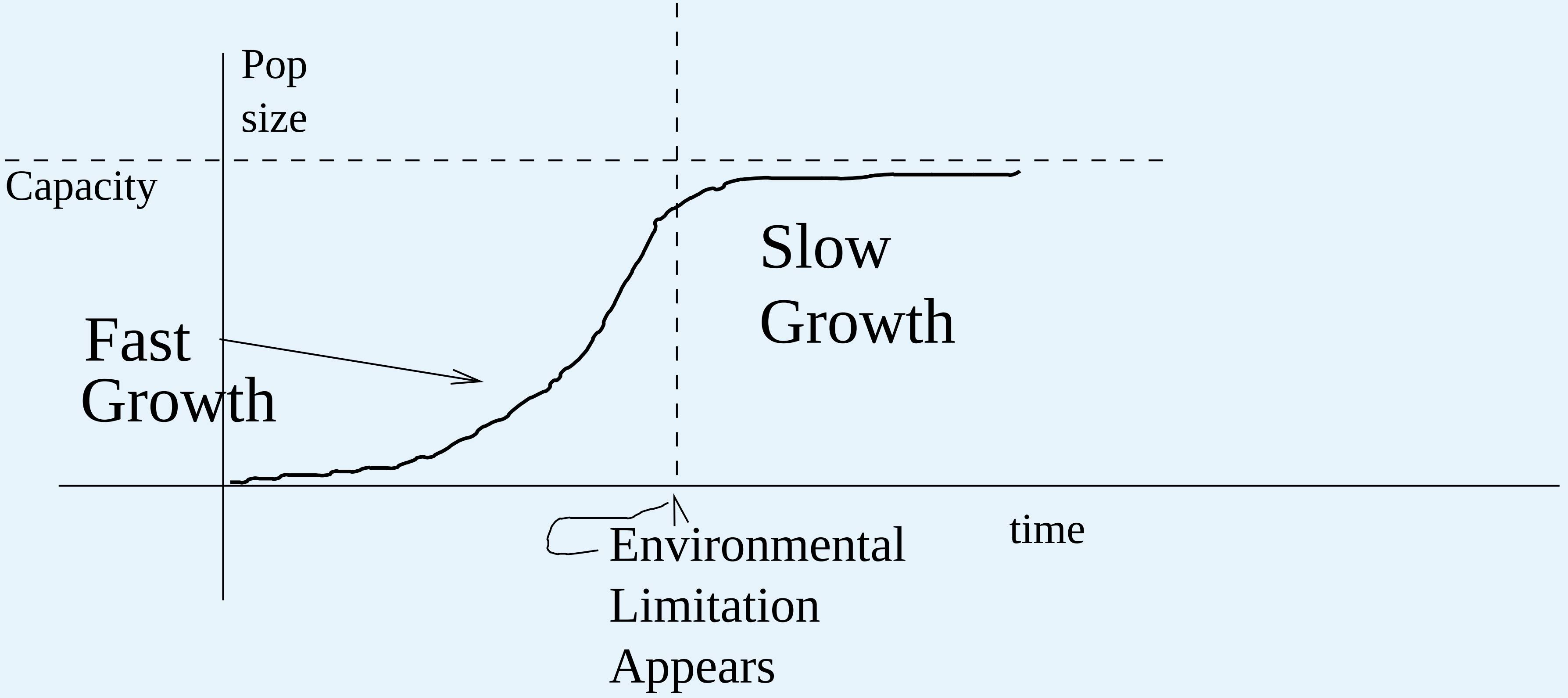
Lecture 3: Non-Linear Systems



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Philosophy: Reality demands non-linear terms, generating effects impossible to model with purely linear systems; there's a fidelity/analyzability tradeoff that can often (but not always) be avoided by linearization analysis.

Caveat: I don't know too much about non-linear systems.

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$N(t) = e^{rt}N(0)$.

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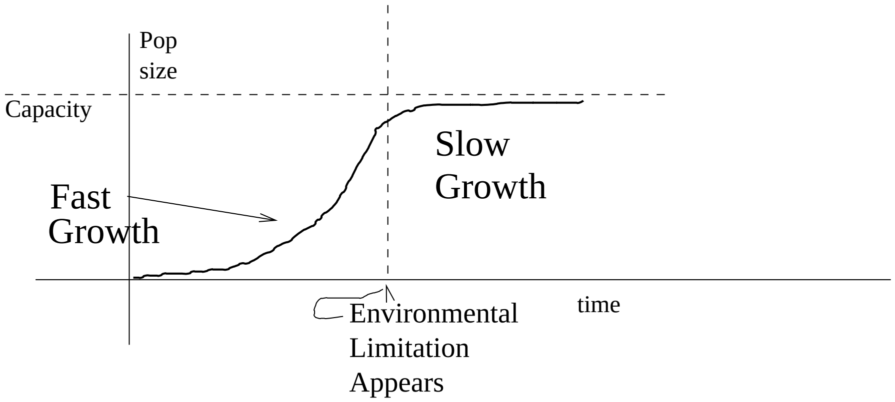

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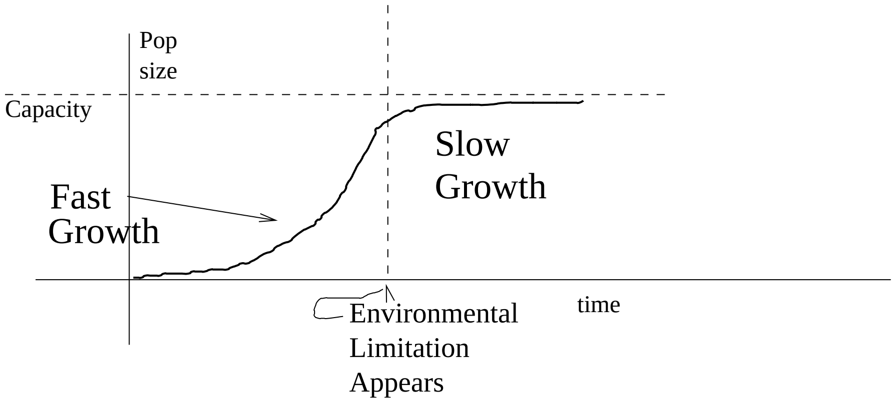

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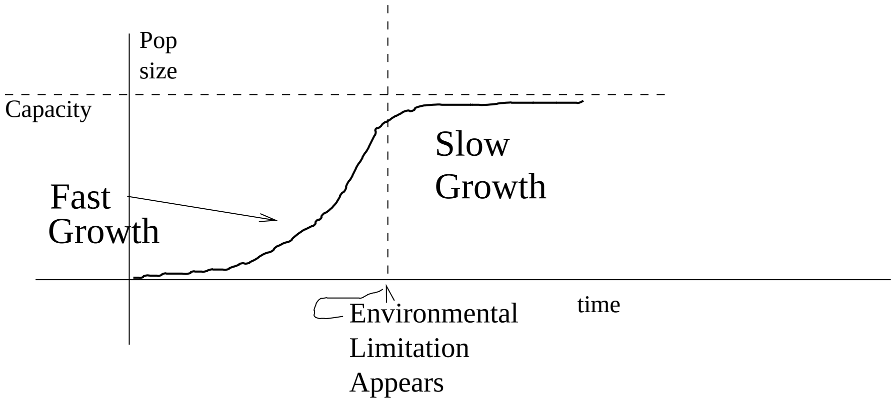

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Answer: Lecture 2 analysis $\Rightarrow$ NO linear system can model resource response.

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- If $N \ll K$, $\dot{N} \sim rN$, with fast growth as we wanted.

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Problem 2 What is the easiest strategy to evaluate the integral on the LHS?

Answer: Partial fractions.

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Problem 3 Solve for C as a function of $N(0)$.

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“Integrating factors" are multipliers you add in to get it to form 1, and remove after.

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Illustrates another method: substitution.

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Strategy: solve ODE, find asymptotes, fixed points, stability from trajectory behavior. Why is this a bad strategy?

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- Even when ODE is solvable, the answer can be opaque.
- Often ODE is unsolvable.

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Answer: $\dot{N} = 0 \;\Rightarrow\; rN\,(1 - N/K) = 0 \;\Rightarrow N = 0, N = K.$

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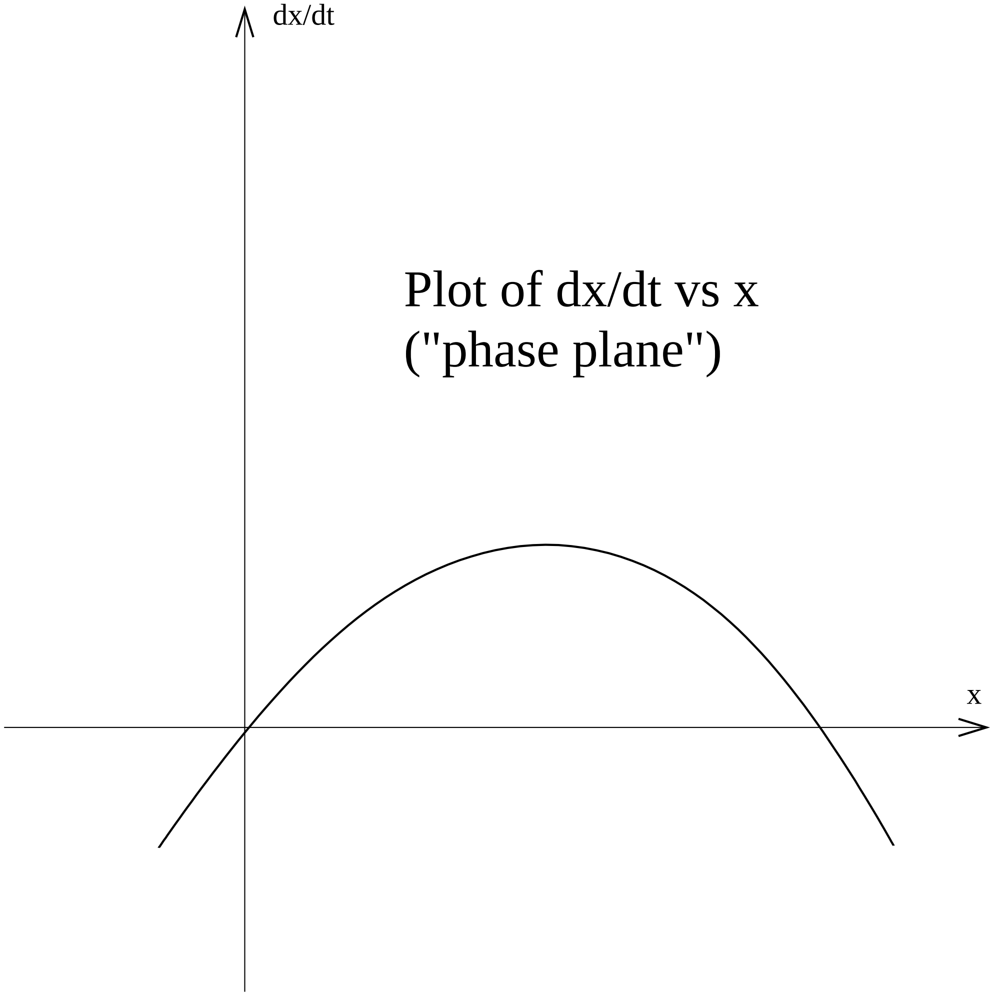

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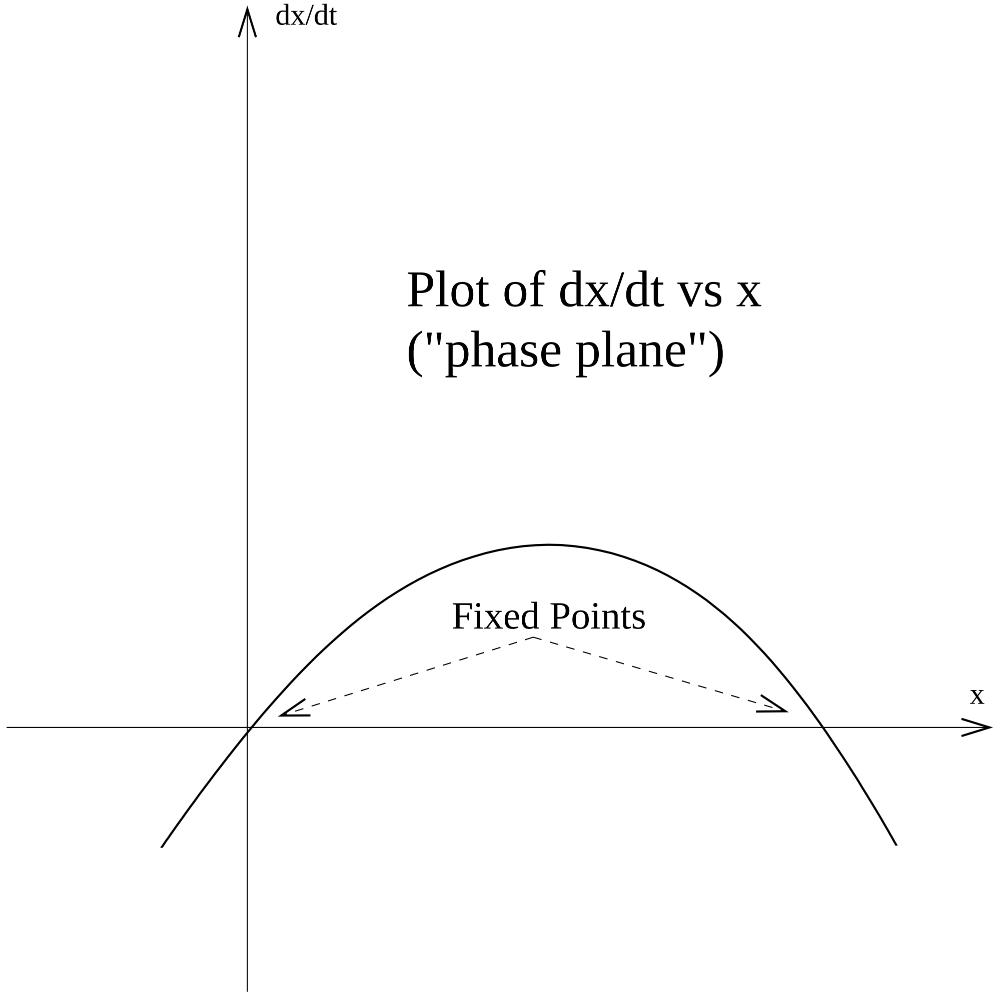

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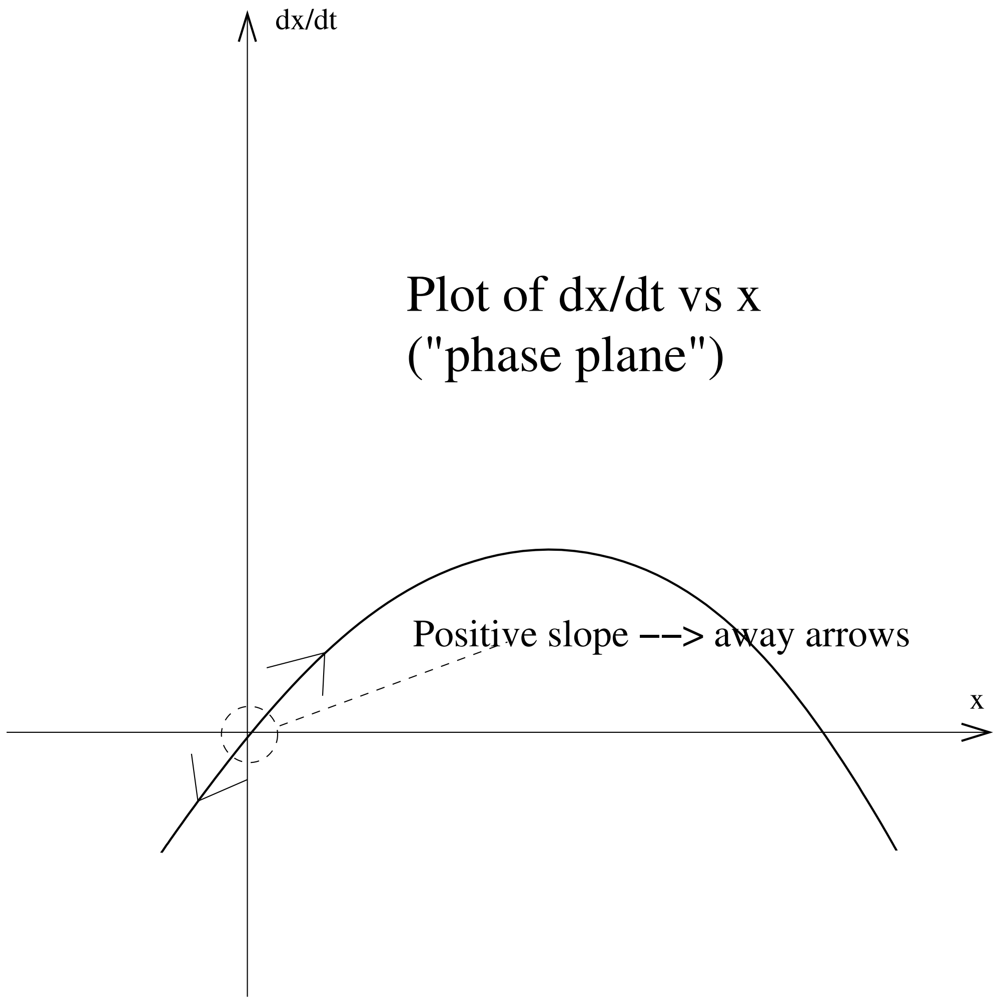
Positive slope around fixed point $\Rightarrow$



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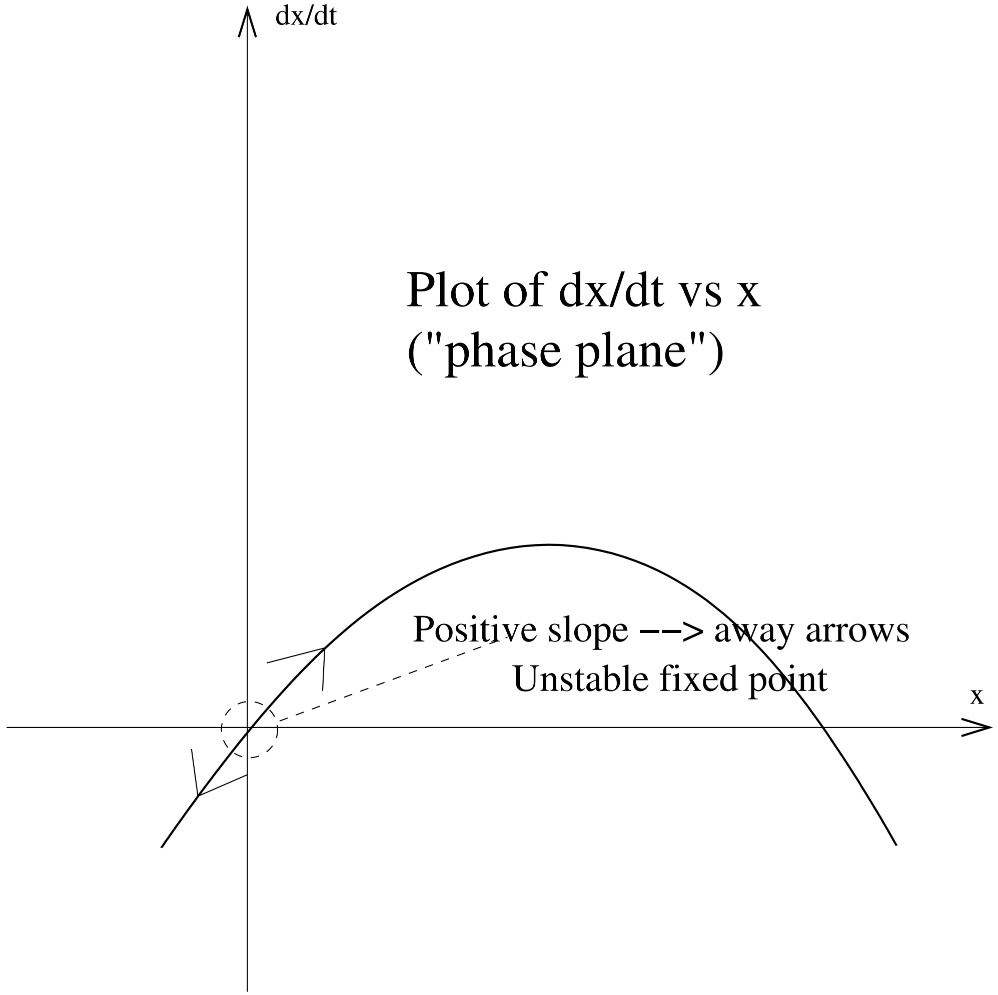

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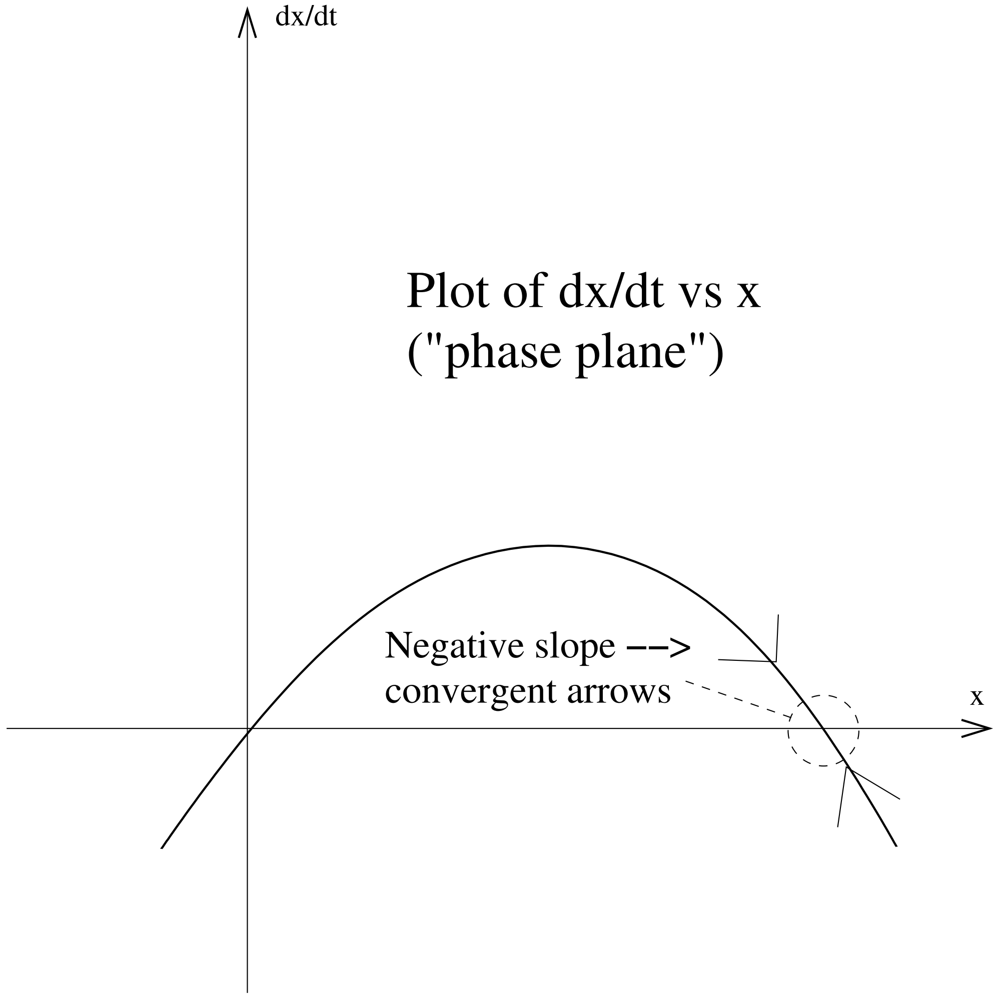

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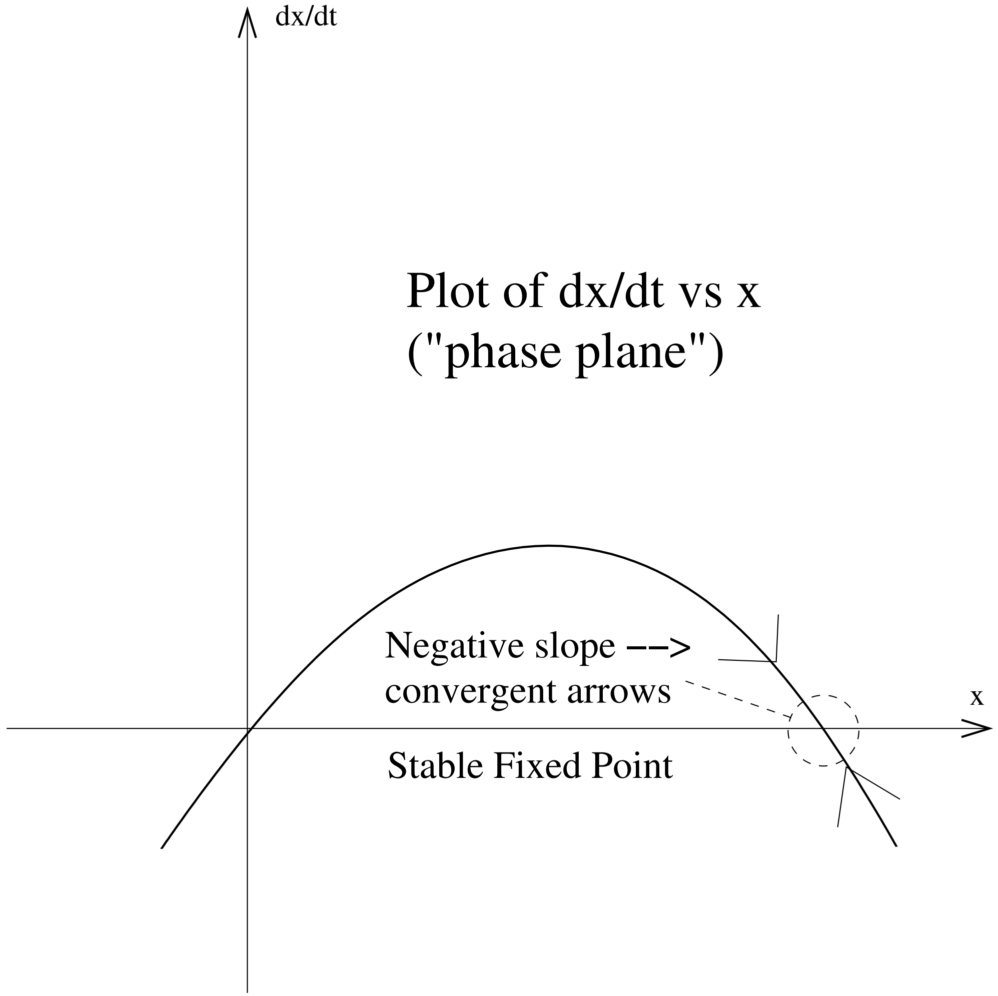

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Thus, if $f'(x_{fp}) < 0$, displacement from x_{fp} shrinks (at least locally). If $f'(x_{fp}) > 0$, displacement grows.

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It’s both easy (or easier) to do and gives the insight we wanted anyway.

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This info, along with the stability calculations, allows us to qualitatively map out trajectories.

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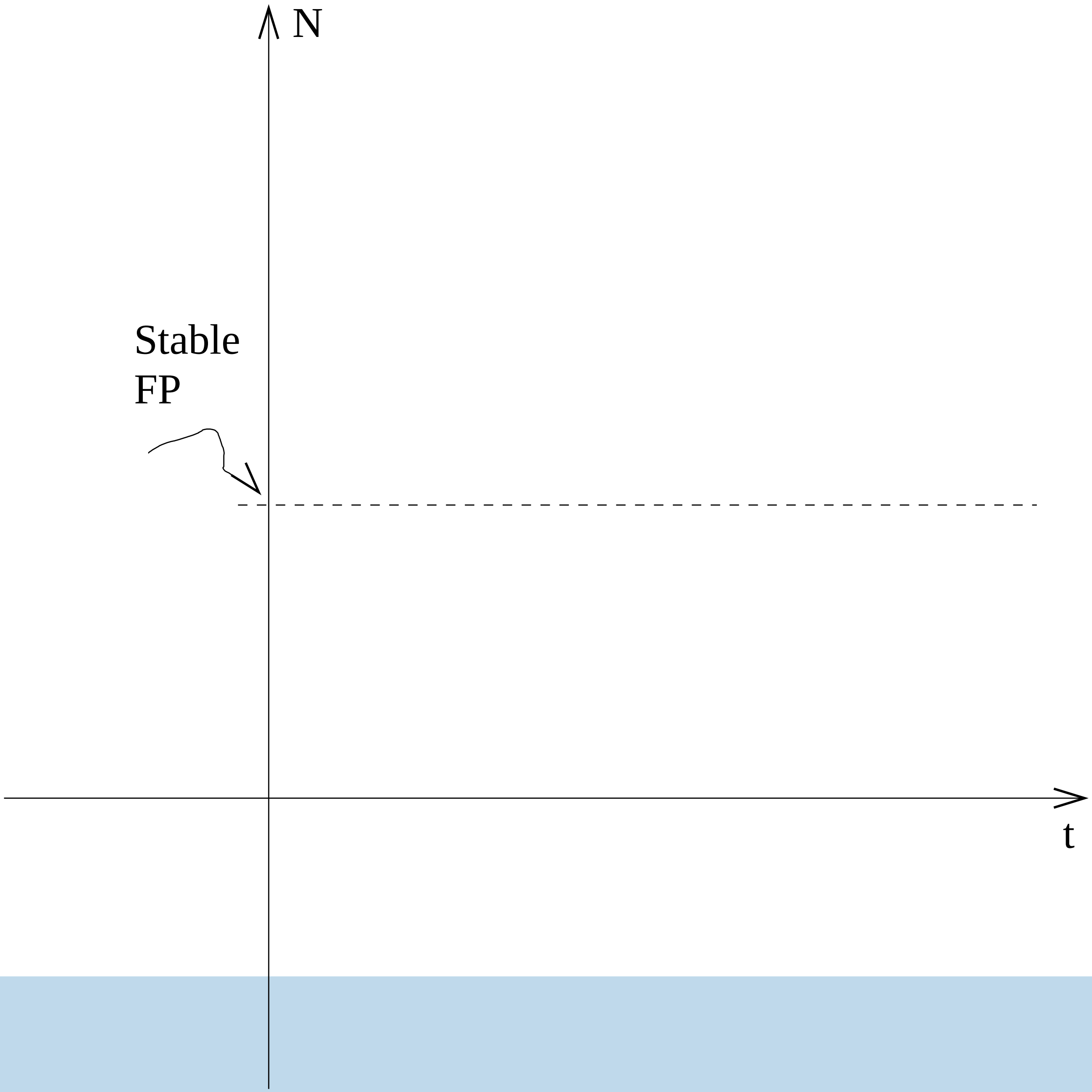

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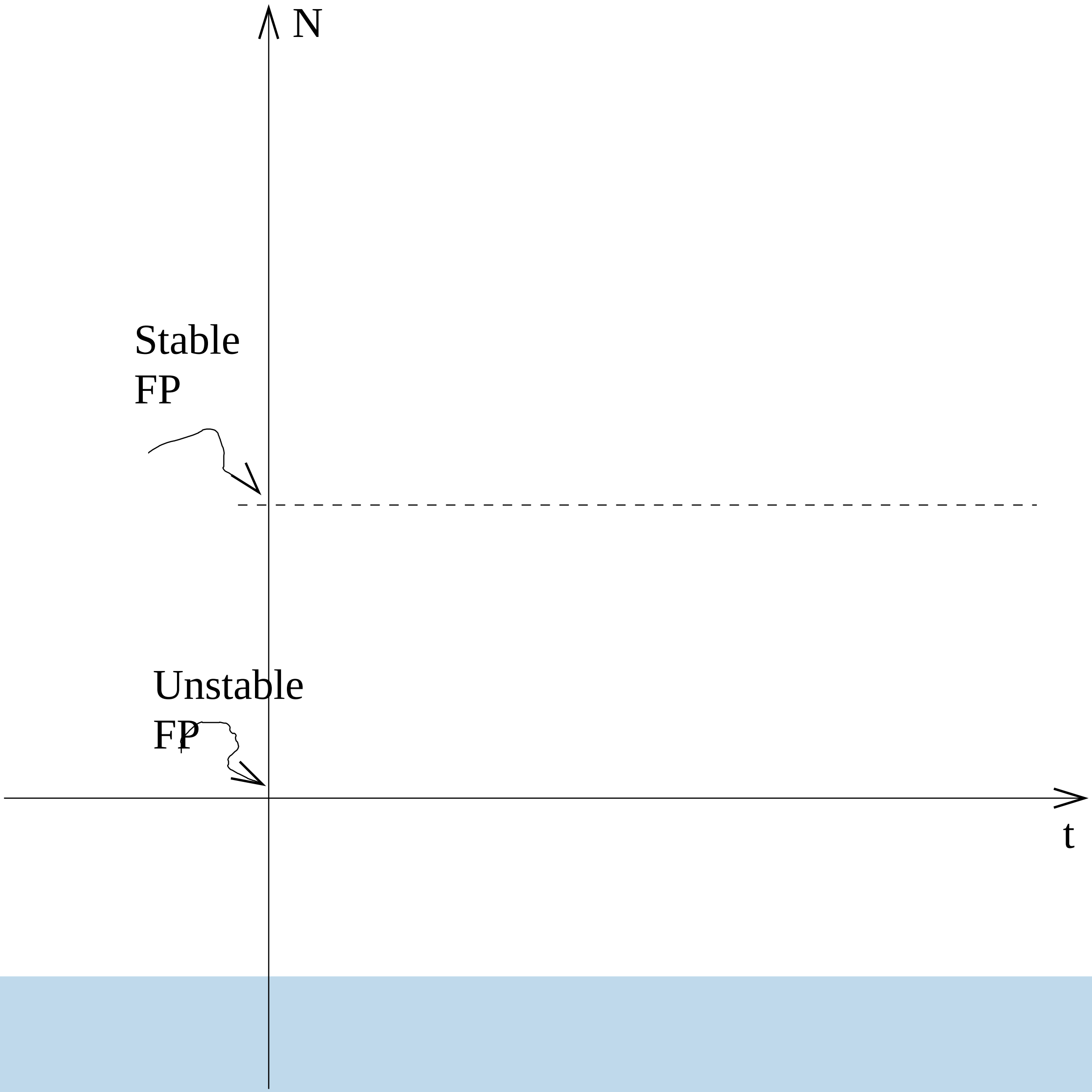

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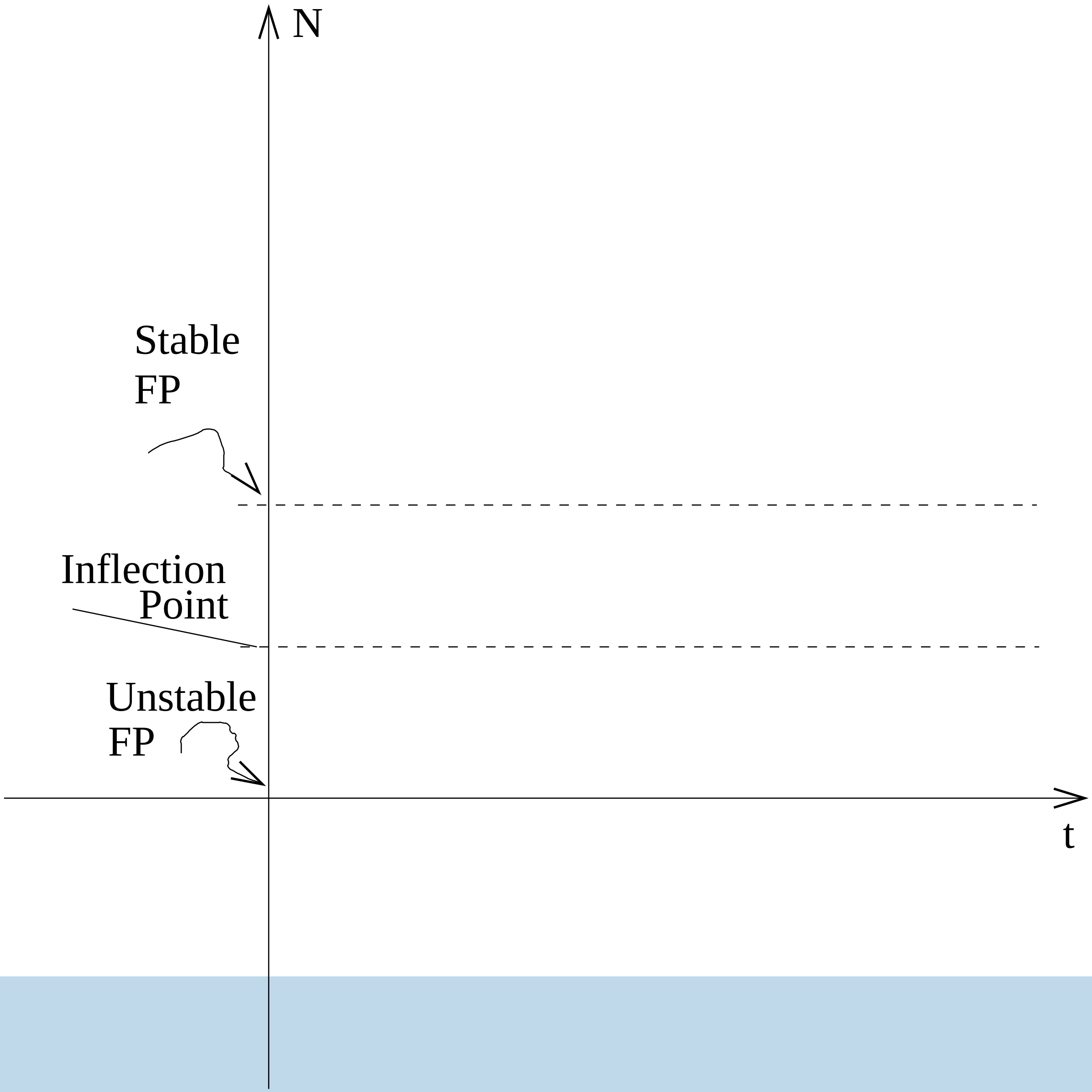

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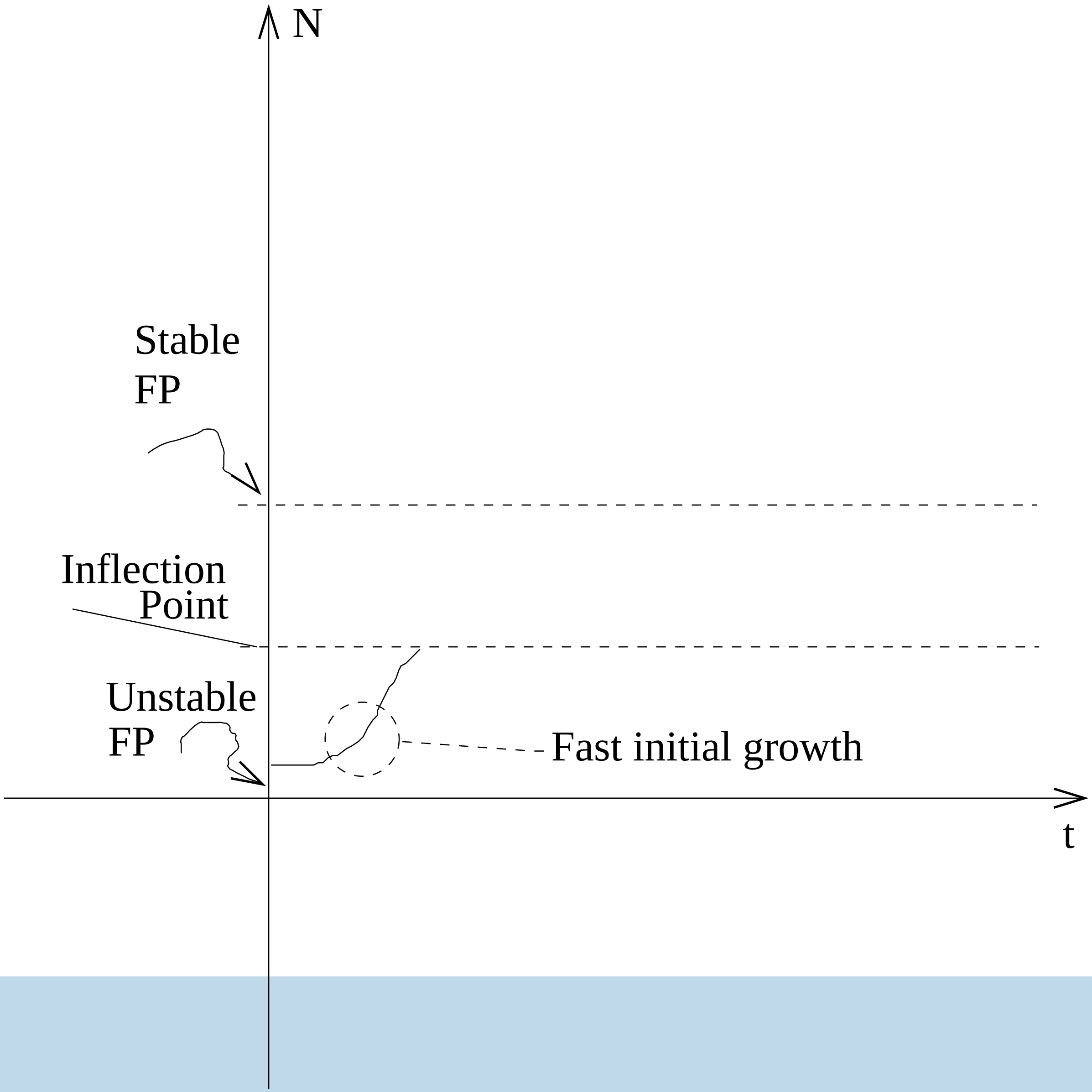

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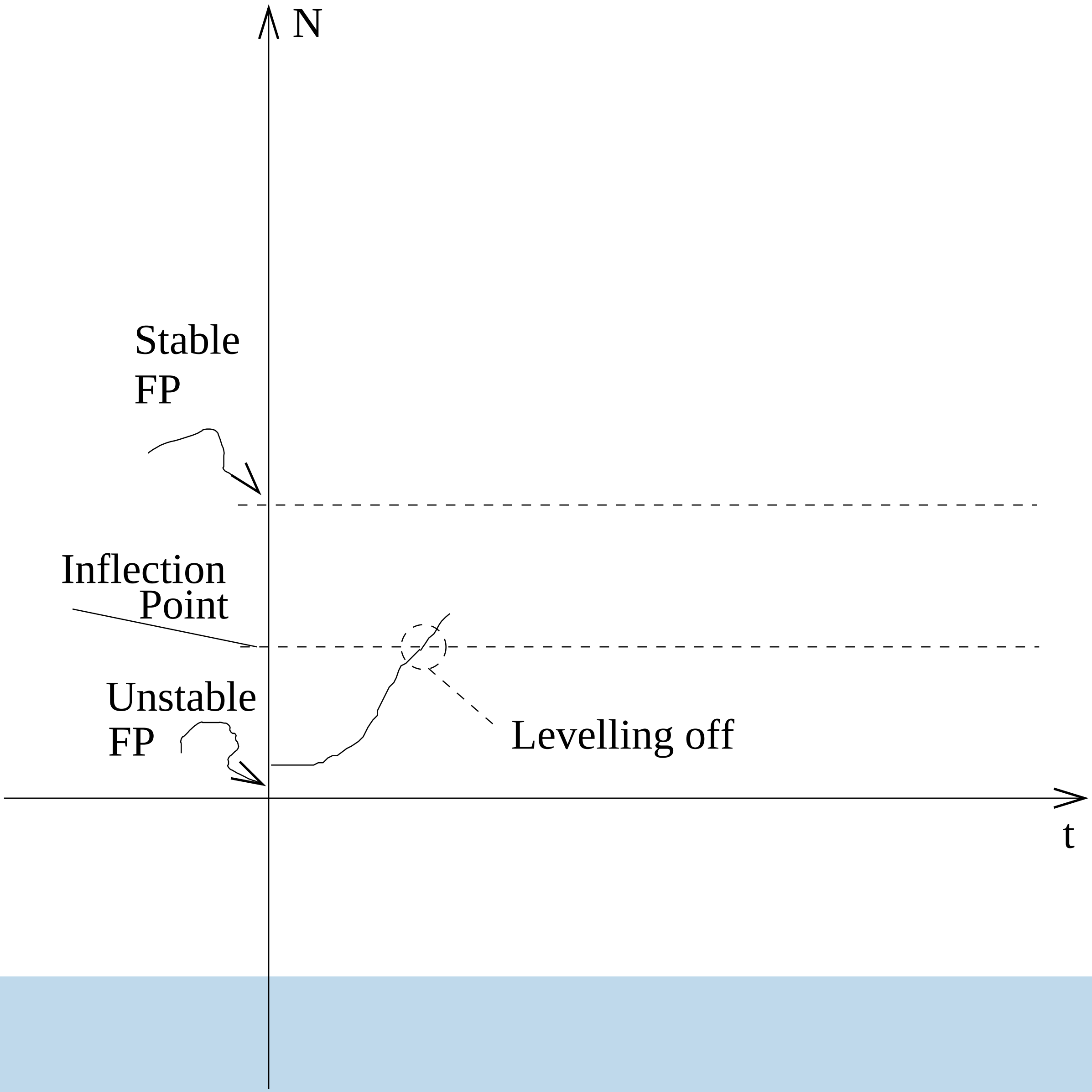

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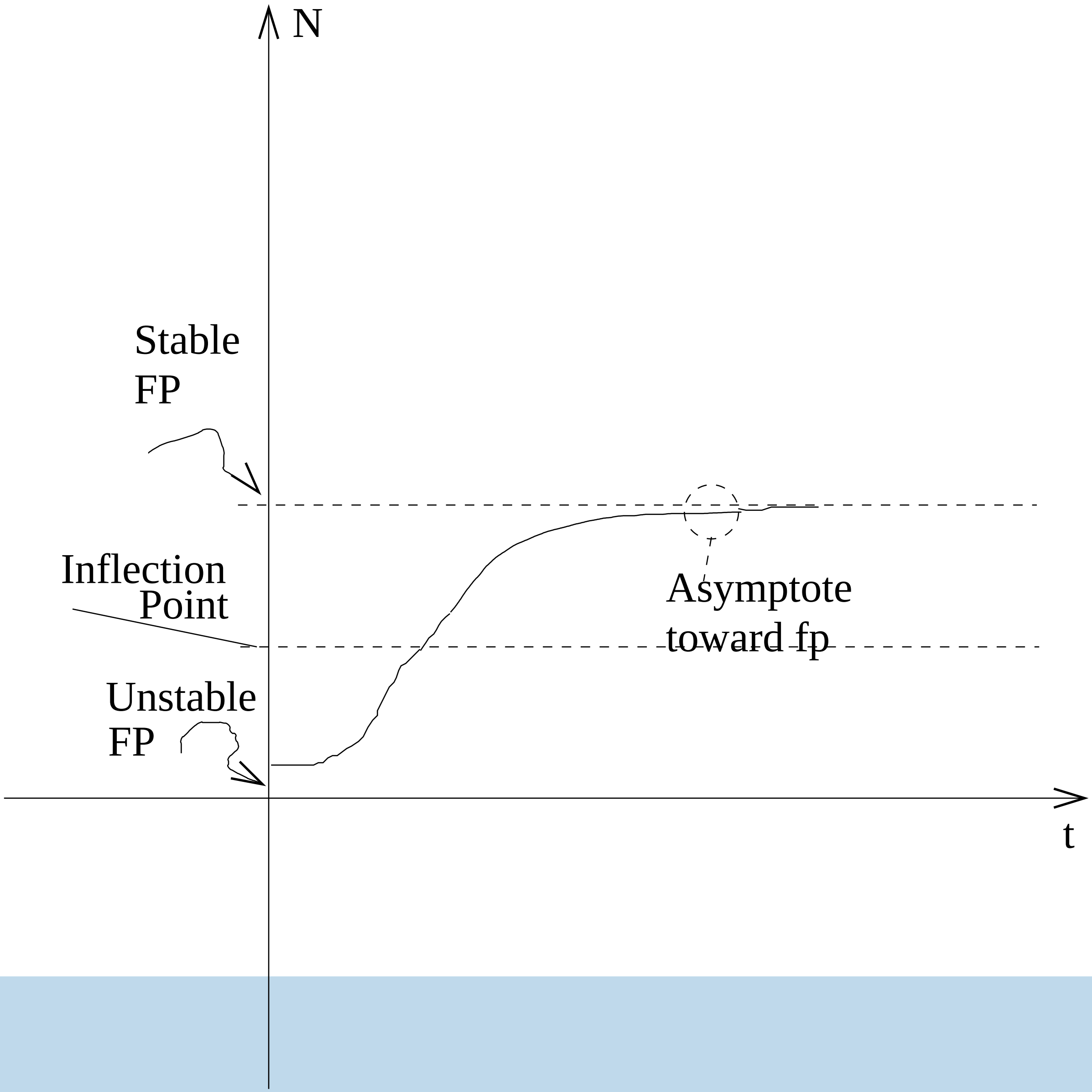

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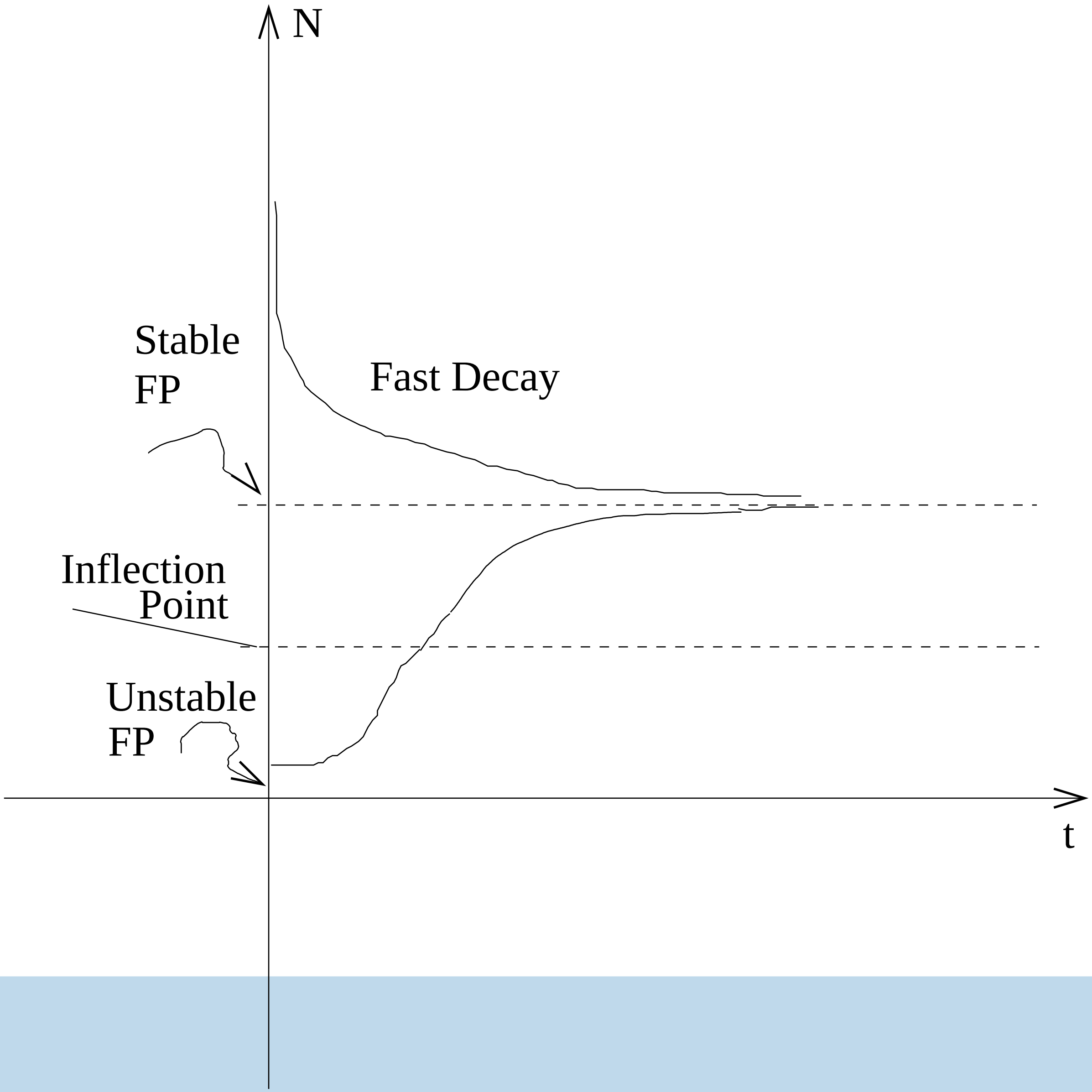

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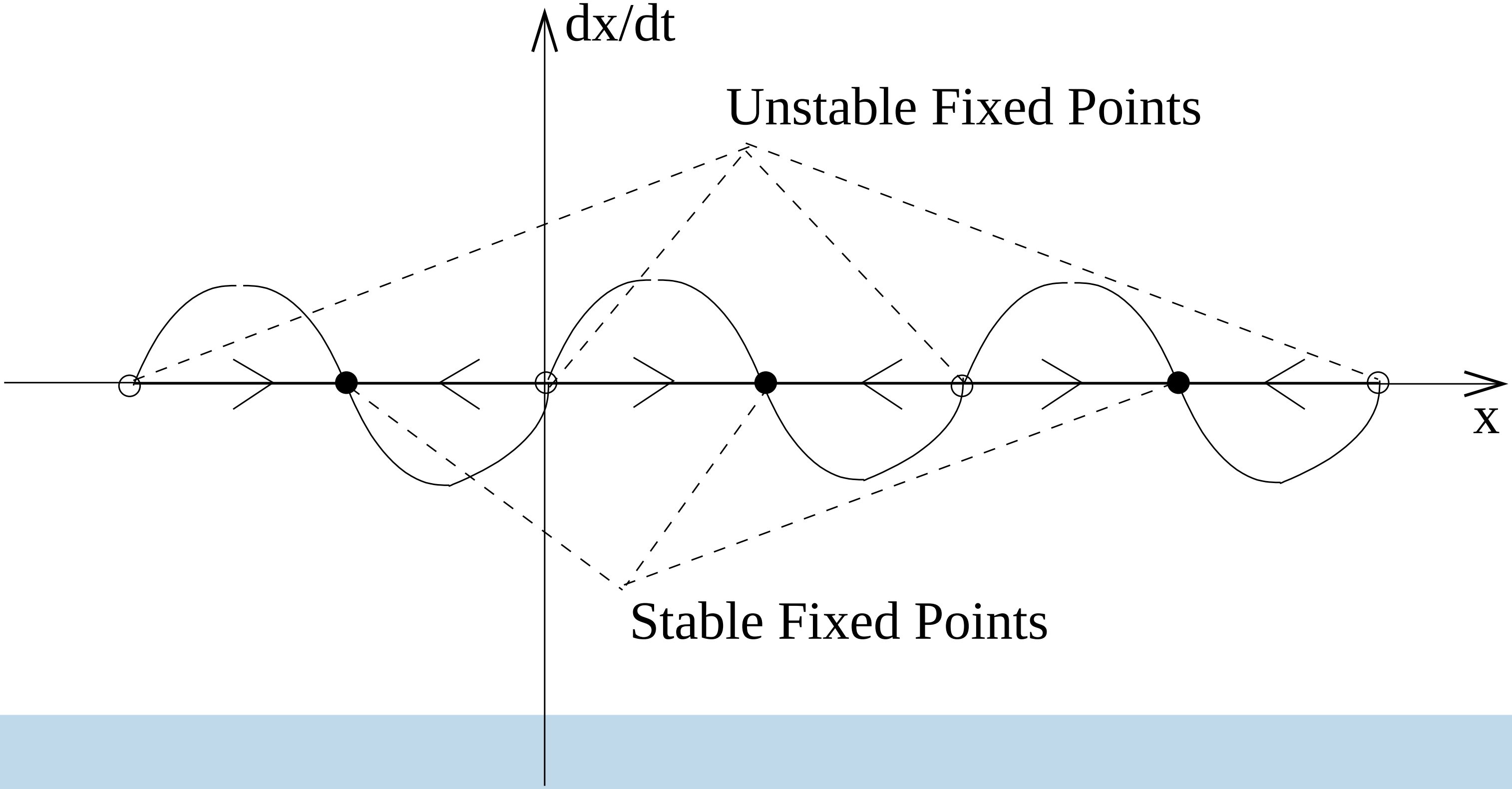

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The 2-variable version of Taylor expansion is:

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I.e., zeroth-order + first-order + higher order terms.

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Problem 11 Compute the Taylor expansion to second order for $f(x,y) = sin(xy)$ about $(0,1)$.

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We want to generalize the linearization process from 1-D to 2-D.

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Suppose (x_{fp}, y_{fp}) is a fixed point of the system, i.e.

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Now, let's use Taylor series as we did before; then on g

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But we know all about these!

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A model that formalizes these assumptions is:

$$\dot{x} = x(r_1 - x - c_1 y); \ \dot{y} = y(r_2 - c_2 x - y)$$

where x is rabbits, y is albatross, $r_1 > r_2$, $c_1 > c_2$, $c_1 c_2 > 1$, $r_1 < c_1 r_2$, and $r_2 < c_2 r_1$.

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This is the well-known *Lotka-Volterra* model; the constant relationships have meaning we’ll understand.

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Answer: $(x, y) = (0,0), (0, r_2), (r_1, 0)$, and

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So now let's do the fixed point analysis one by one.

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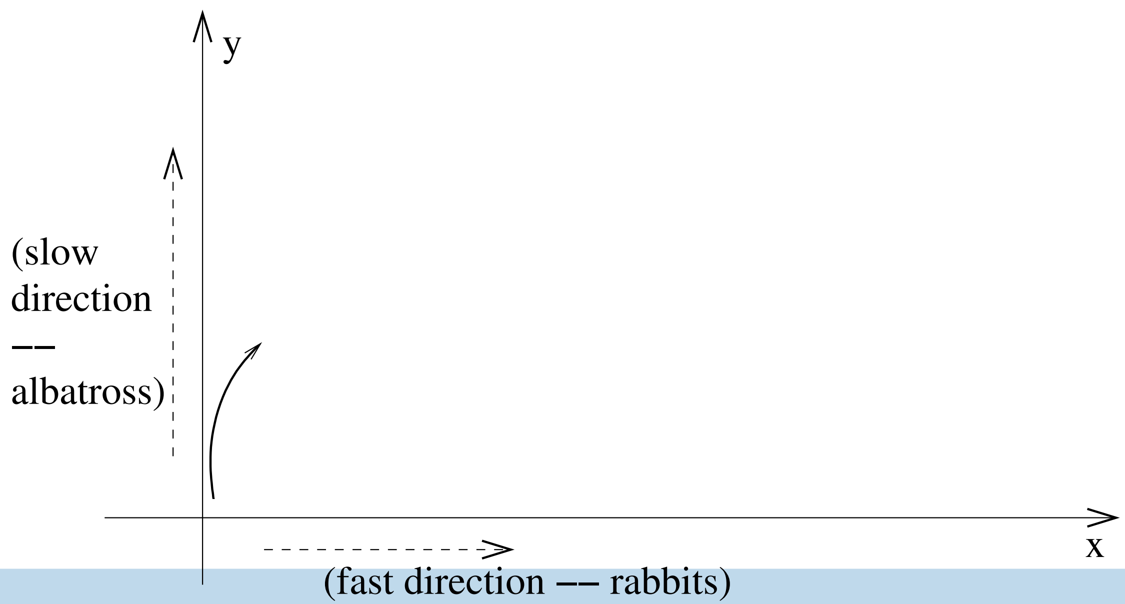

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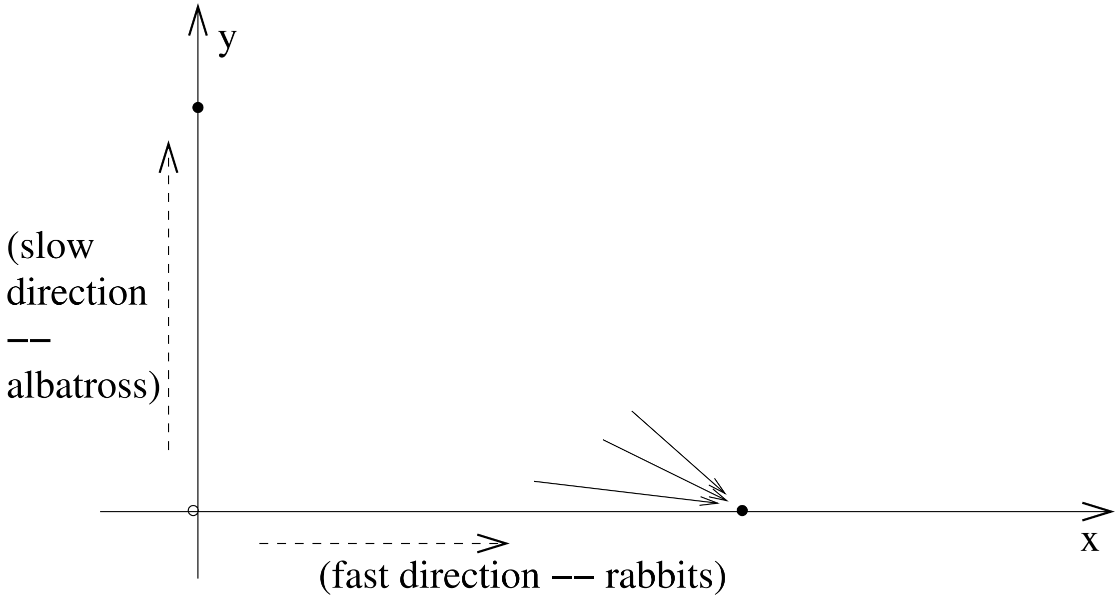

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Trace is negative; determinant is negative; hence it's a ...

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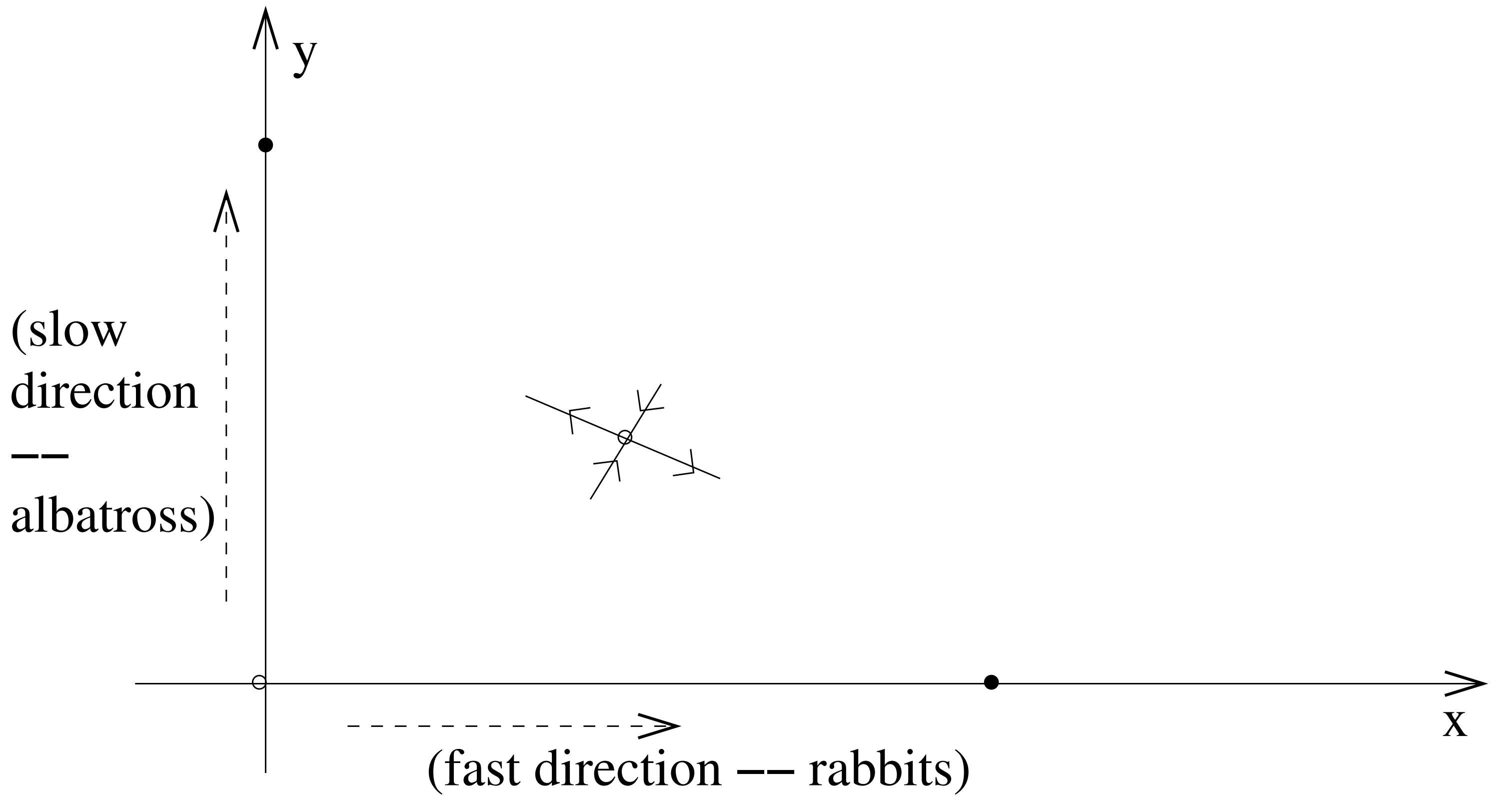

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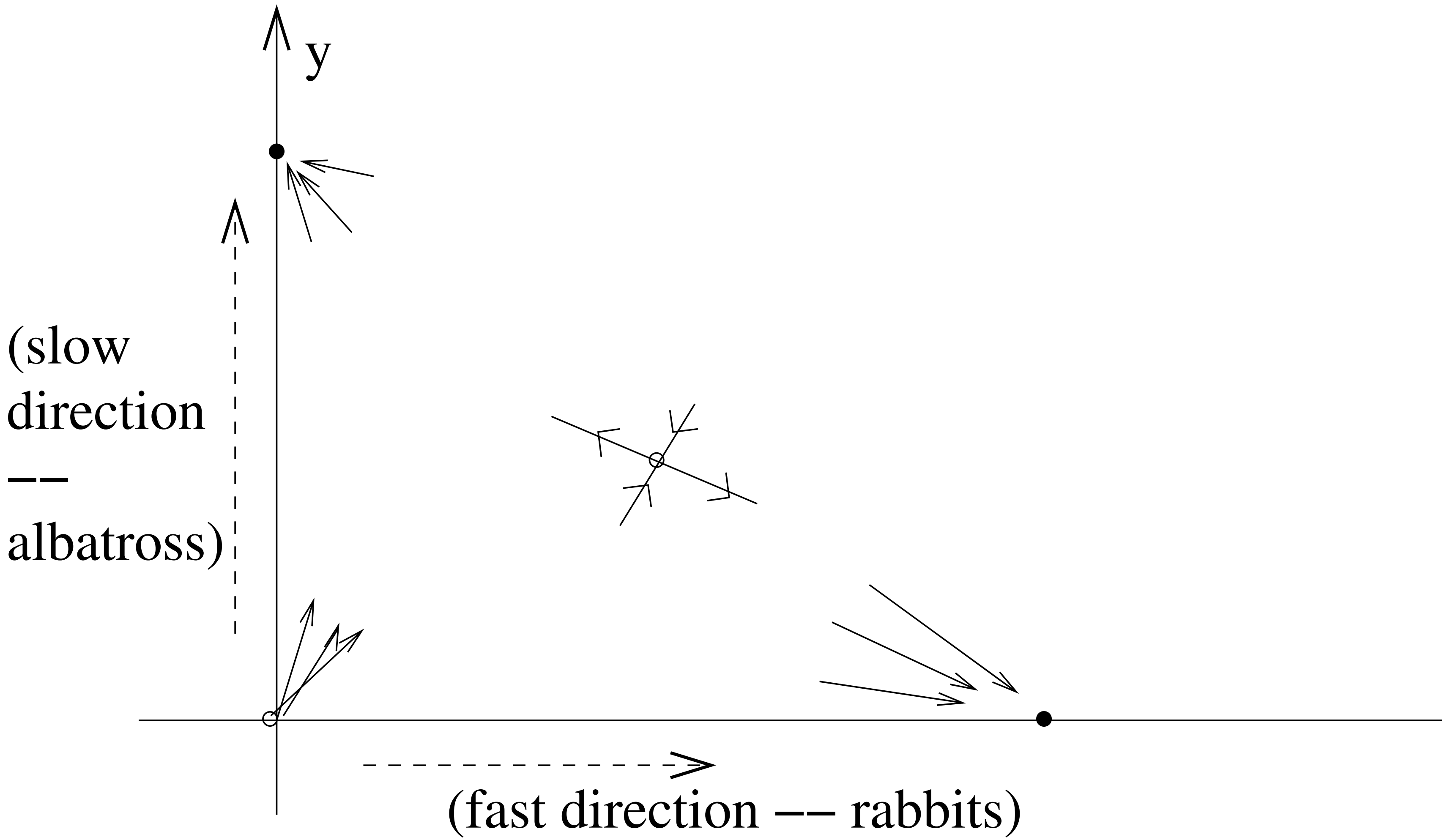

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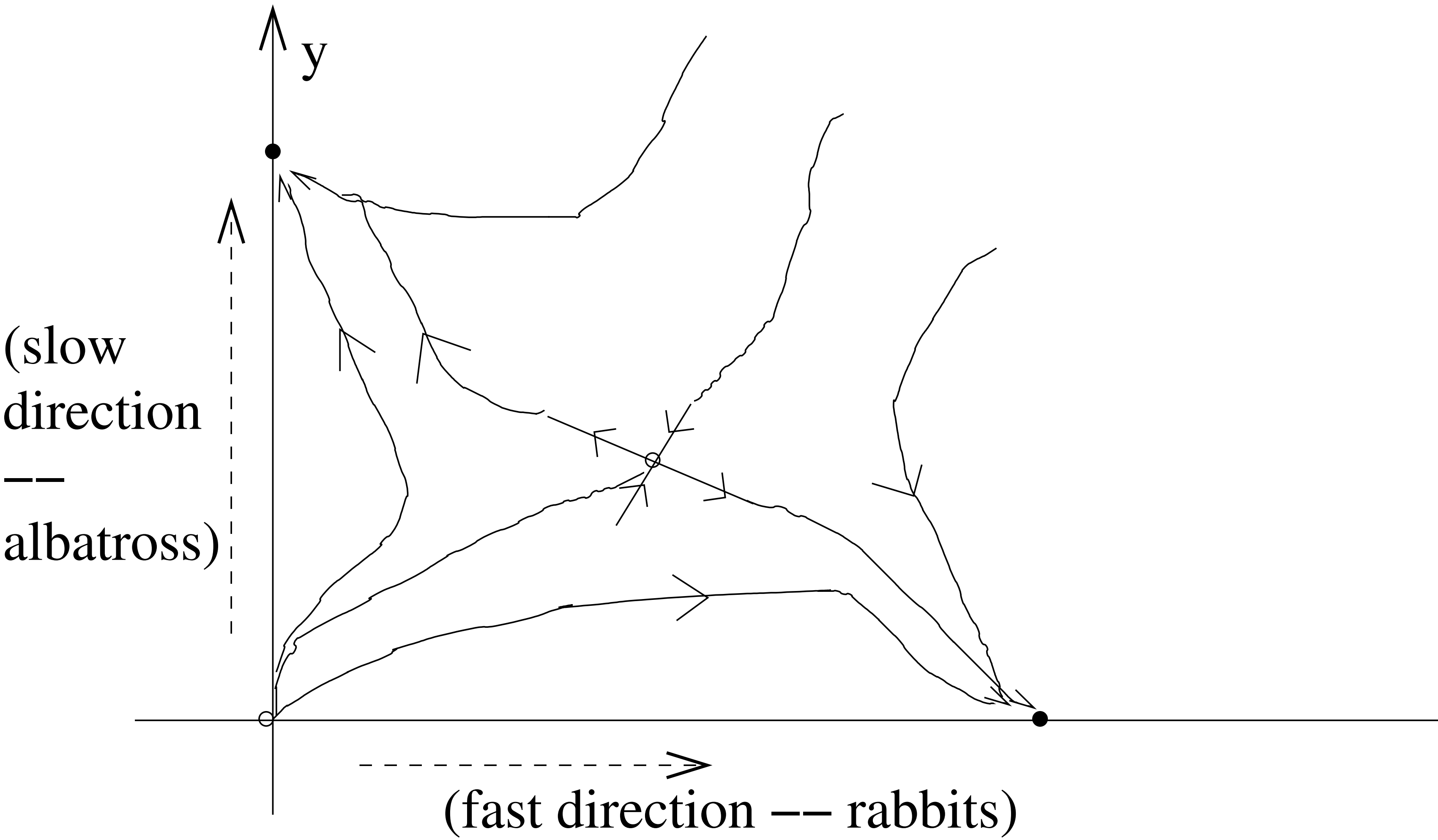

Limitations of Linearization

Consider the system (Strogatz p. 153)

$$\dot{x} = -y + ax(x^2 + y^2); \quad \dot{y} = x + ay(x^2 + y^2)$$

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Answer: Ignoring non-linear terms (since we're at $(0, 0)$) gives

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predicting that the system will rotate around the center for all values of a.
However ...

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This is a *decoupled* system and can be analytically solved.

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What are the bad (sensitive) cases?

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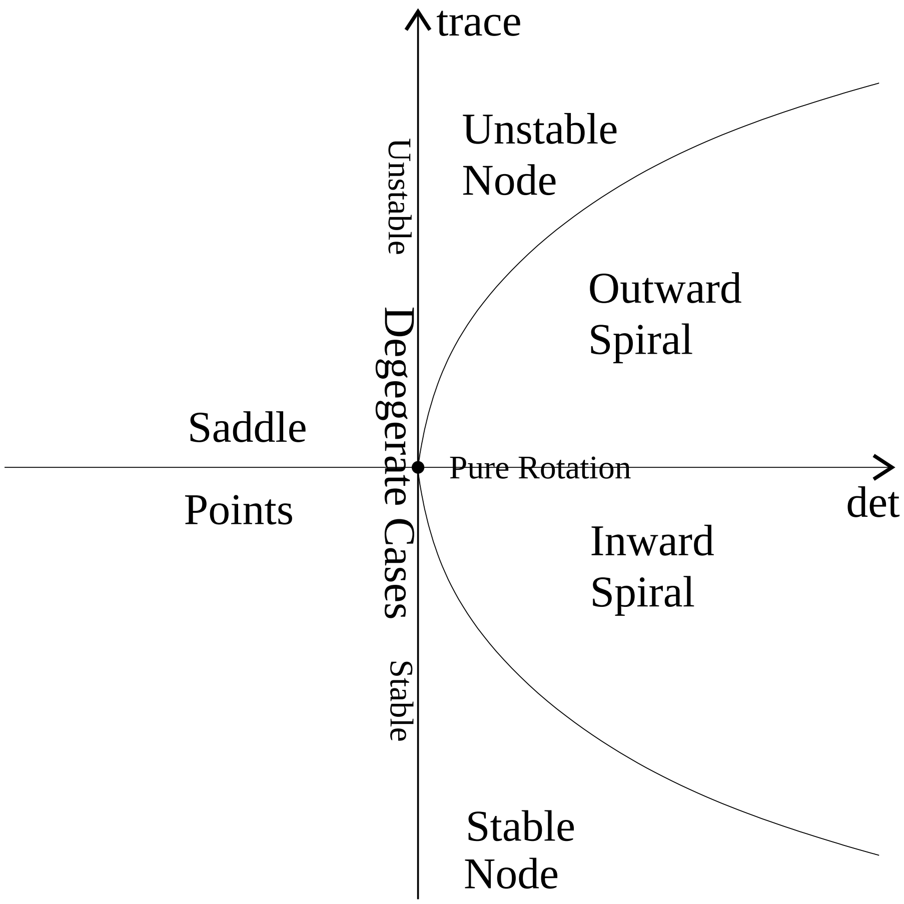
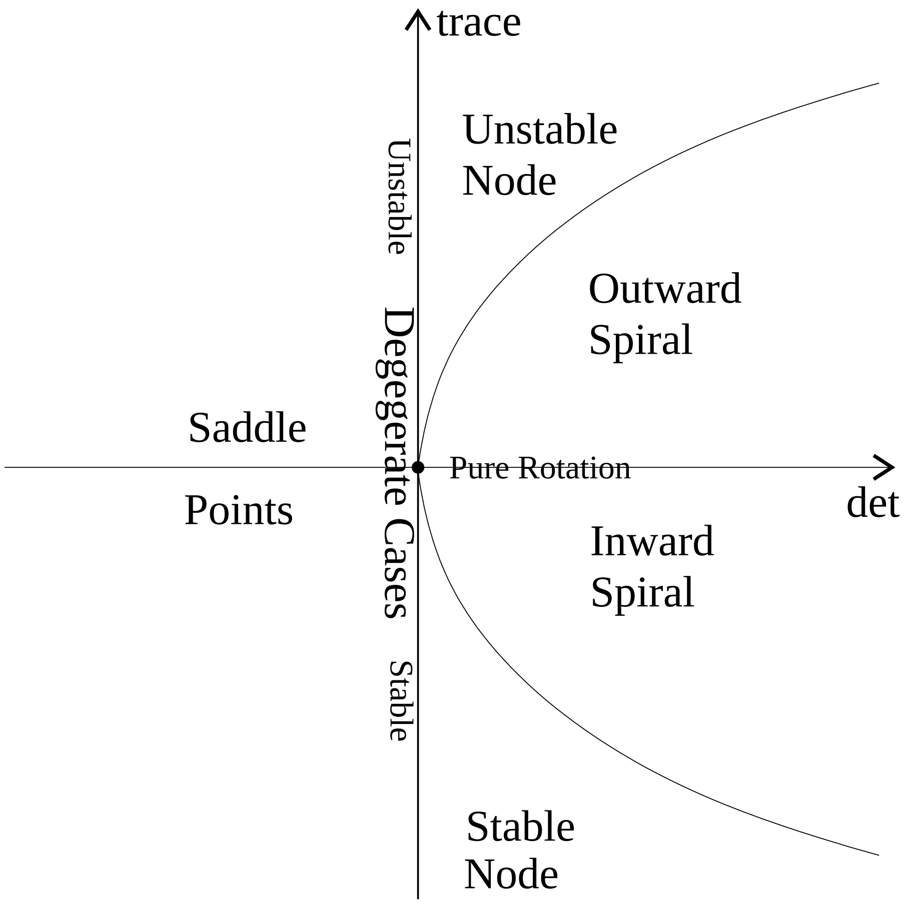
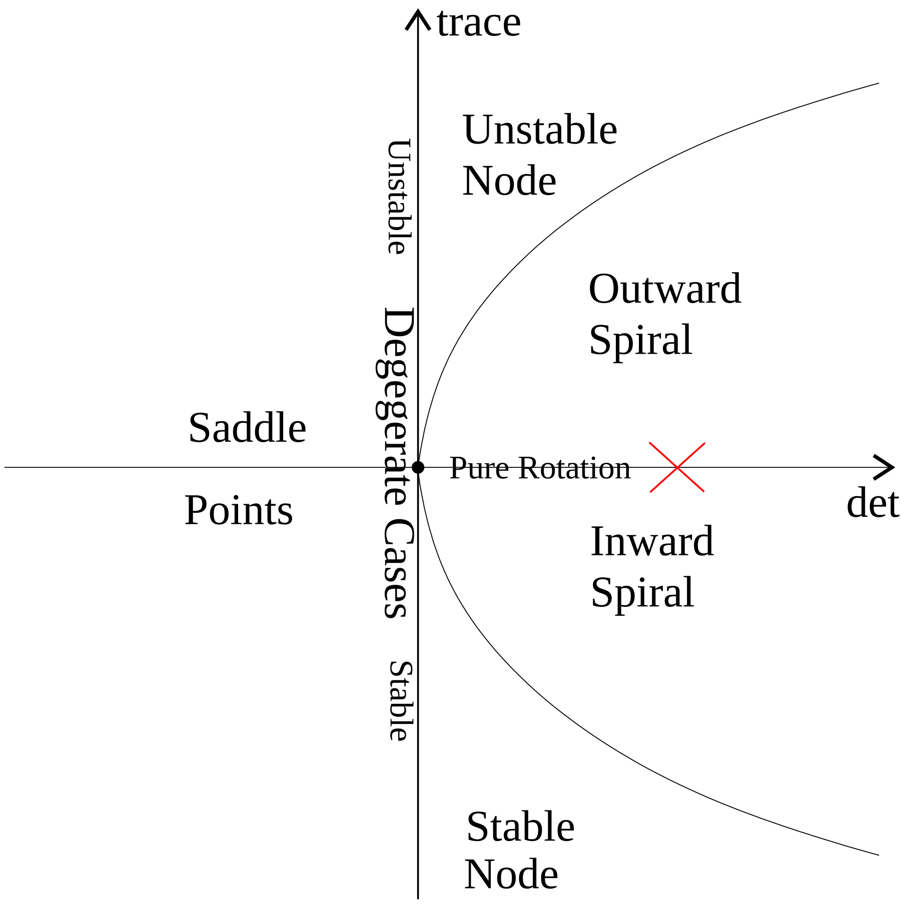


Limitations of Linearization



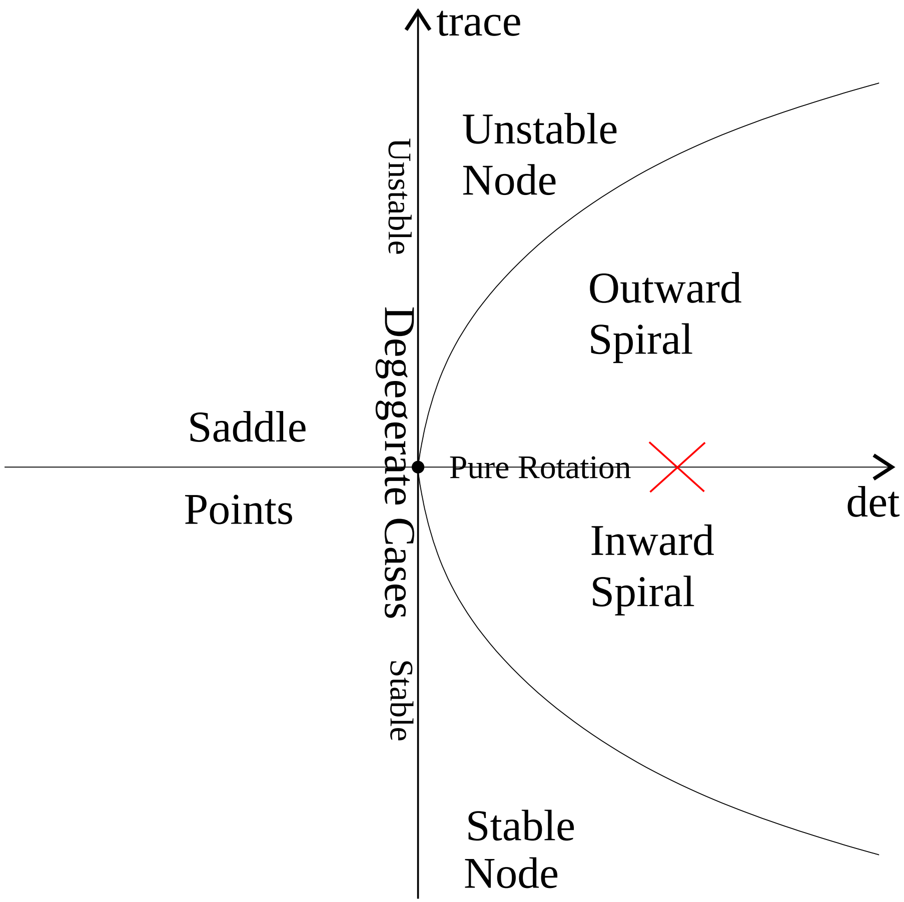
Question: Where on this picture was the bad example we just saw?

Limitations of Linearization



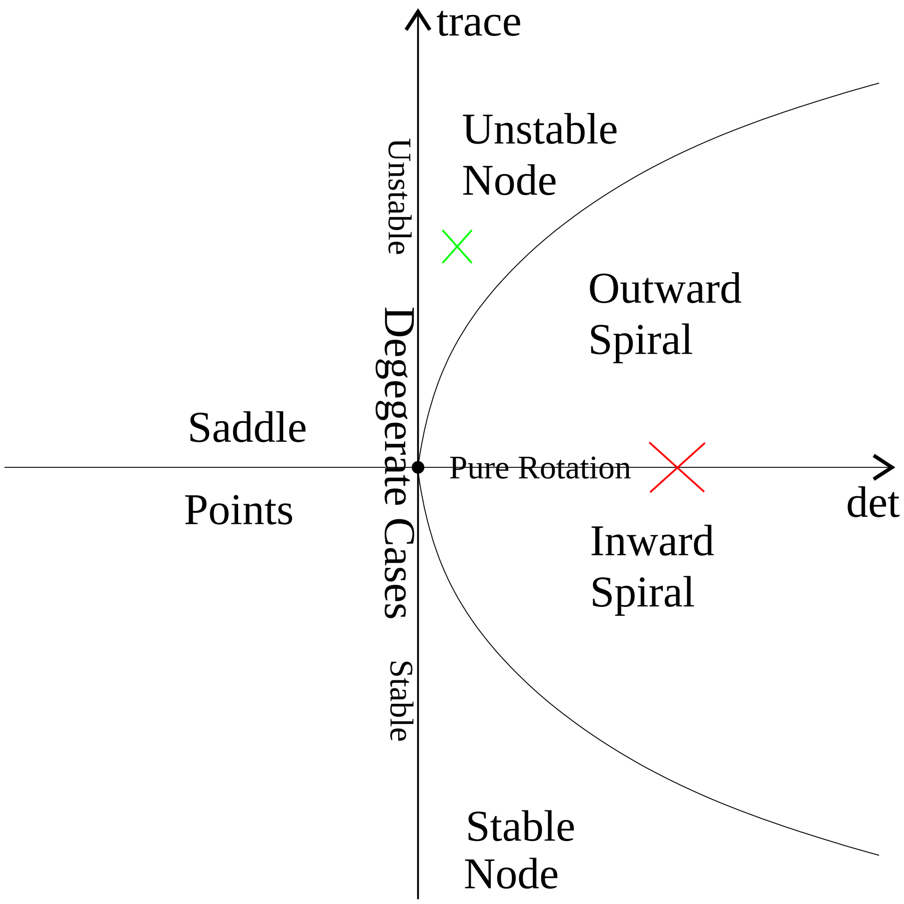
Answer: A pure rotation, on the stable/unstable boundary.

Limitations of Linearization



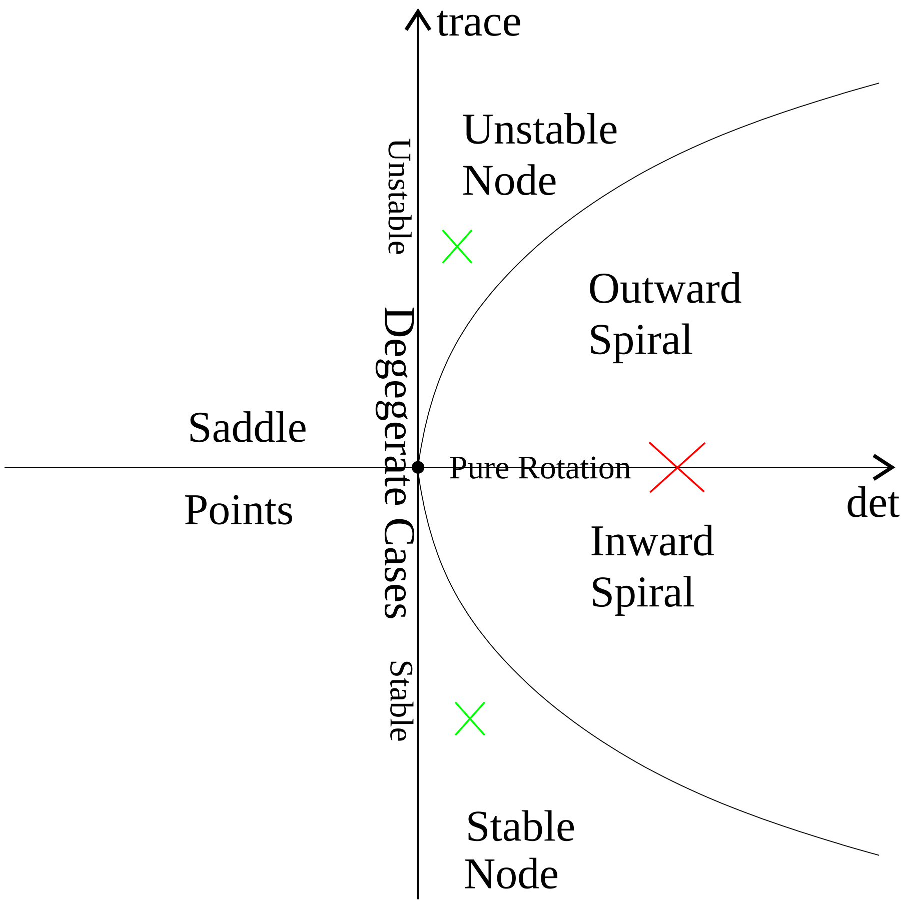
Question: Where were the correct examples, from the population model?

Limitations of Linearization



Answer: One was an unstable node.

Limitations of Linearization



Answer: Another was a stable node.

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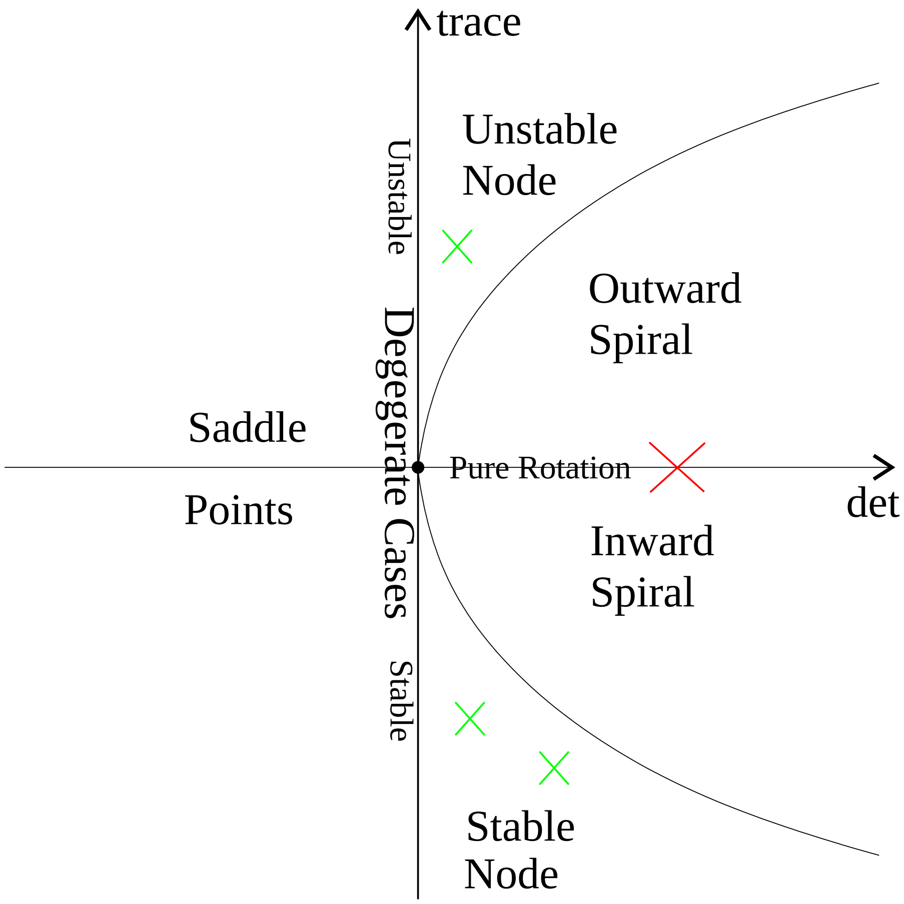
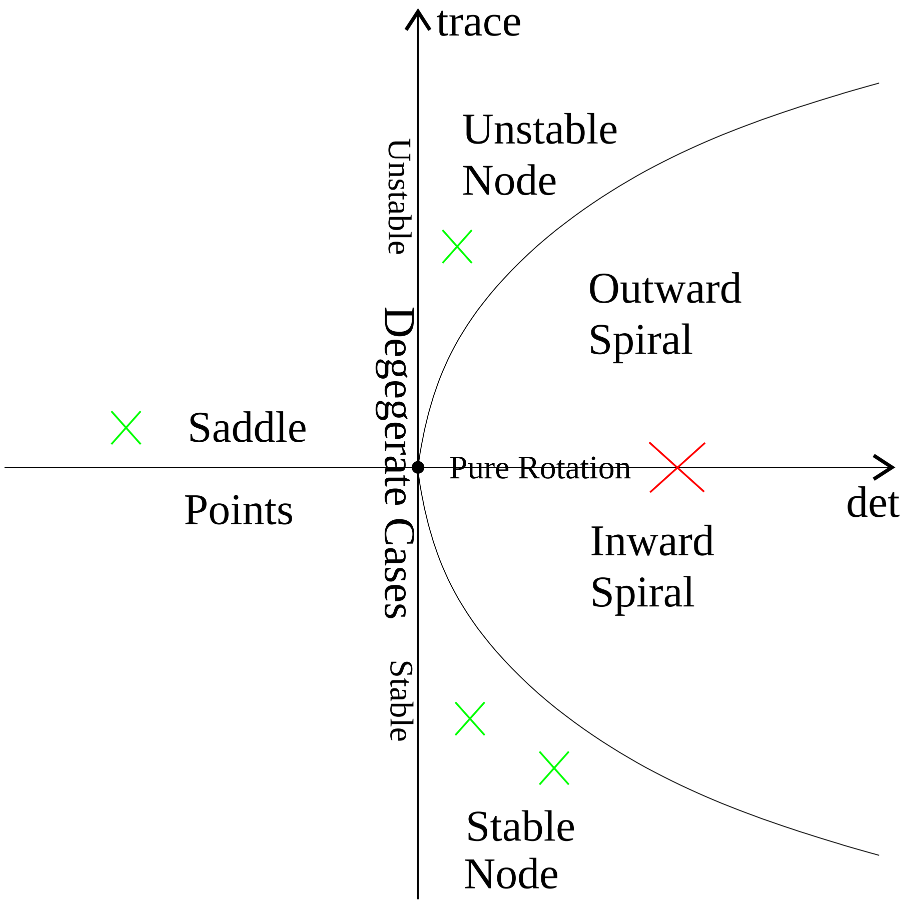


Answer: As was the third.

Limitations of Linearization

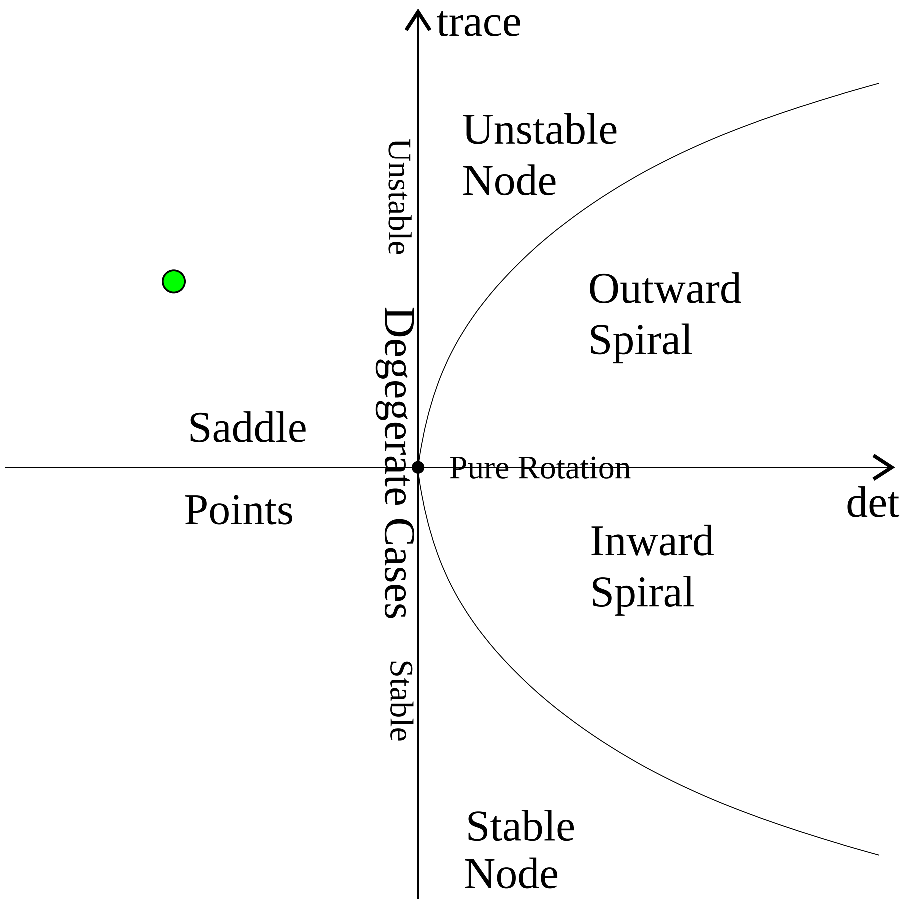


Answer: And the fourth was a saddle.

Limits of Linearization

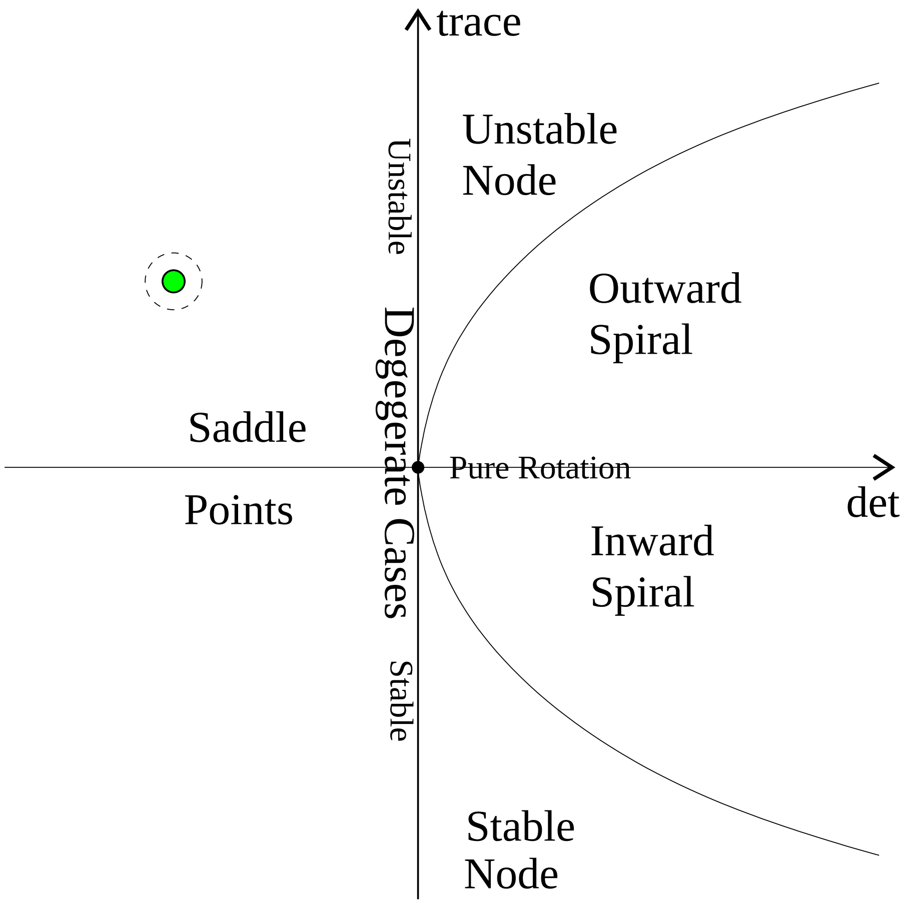
Theorem 1 *(Hartman-Grobman, etc...) Linearization is accurate in 2D if – and only if – you can draw a small circle around the point and still be in the same region in the 2-D classification diagram. That is, if you're not on the border. If you are on the border, small non-linear perturbations can qualitatively change the behavior.*

Limits of Linearization



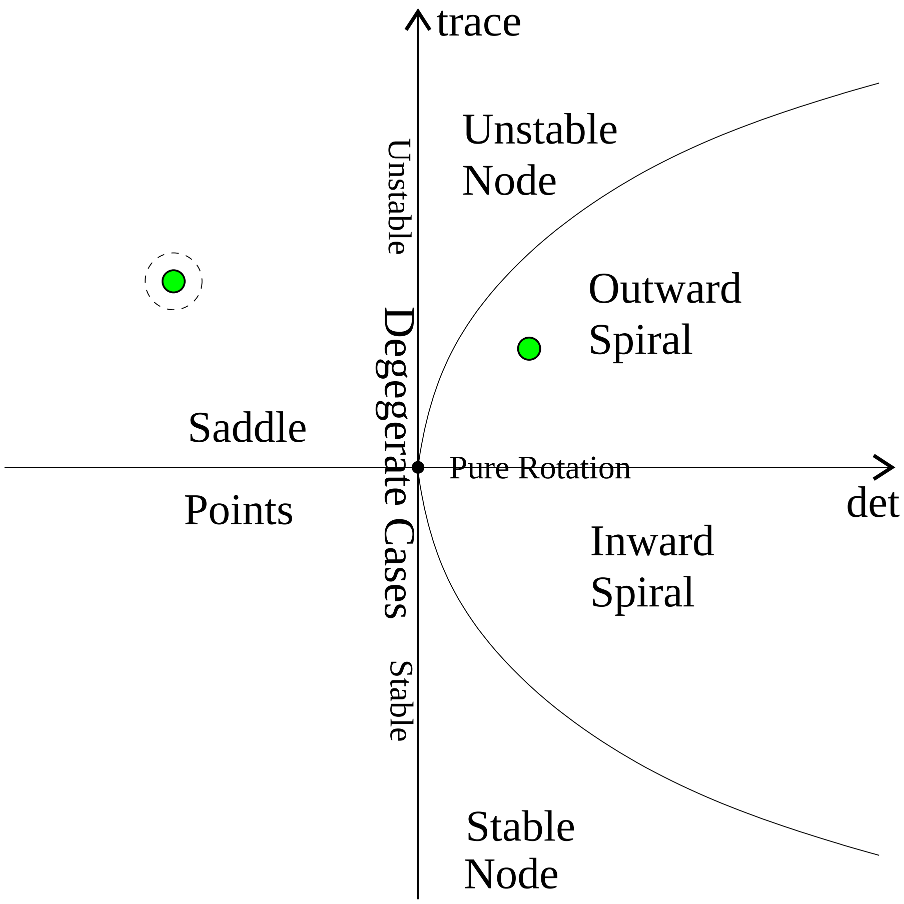
This is in the middle of a region.

Limits of Linearization



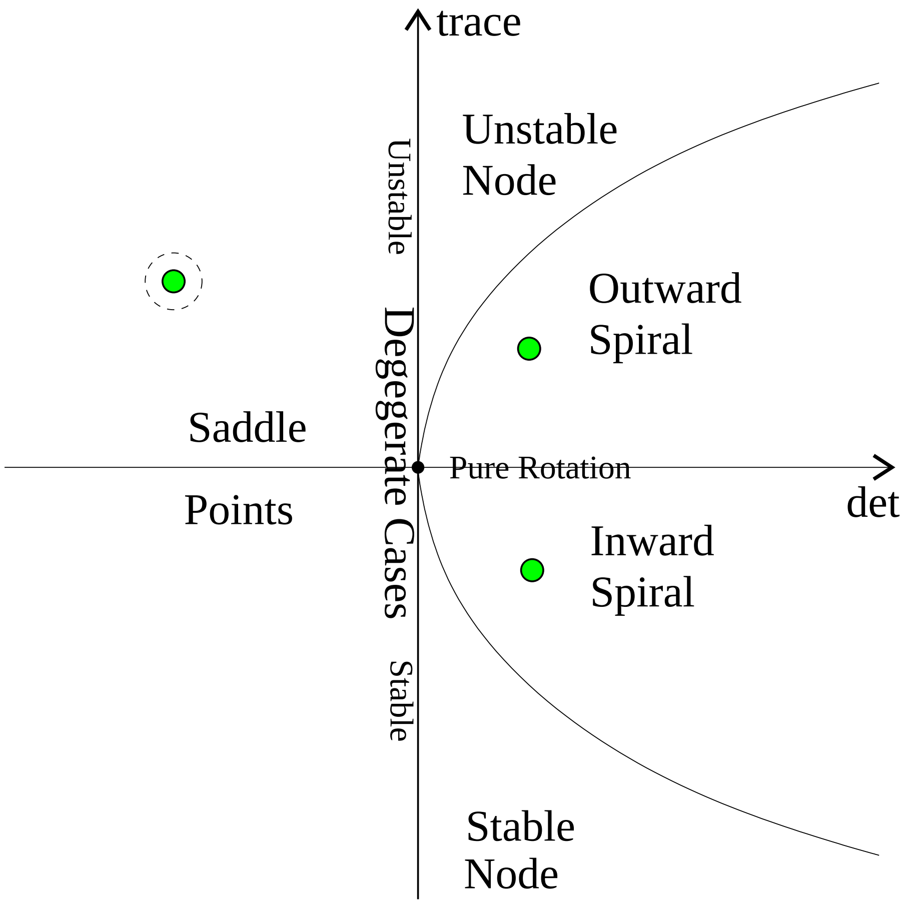
So will accurately predict dynamics.

Limits of Linearization



This one,

Limits of Linearization



and this one,

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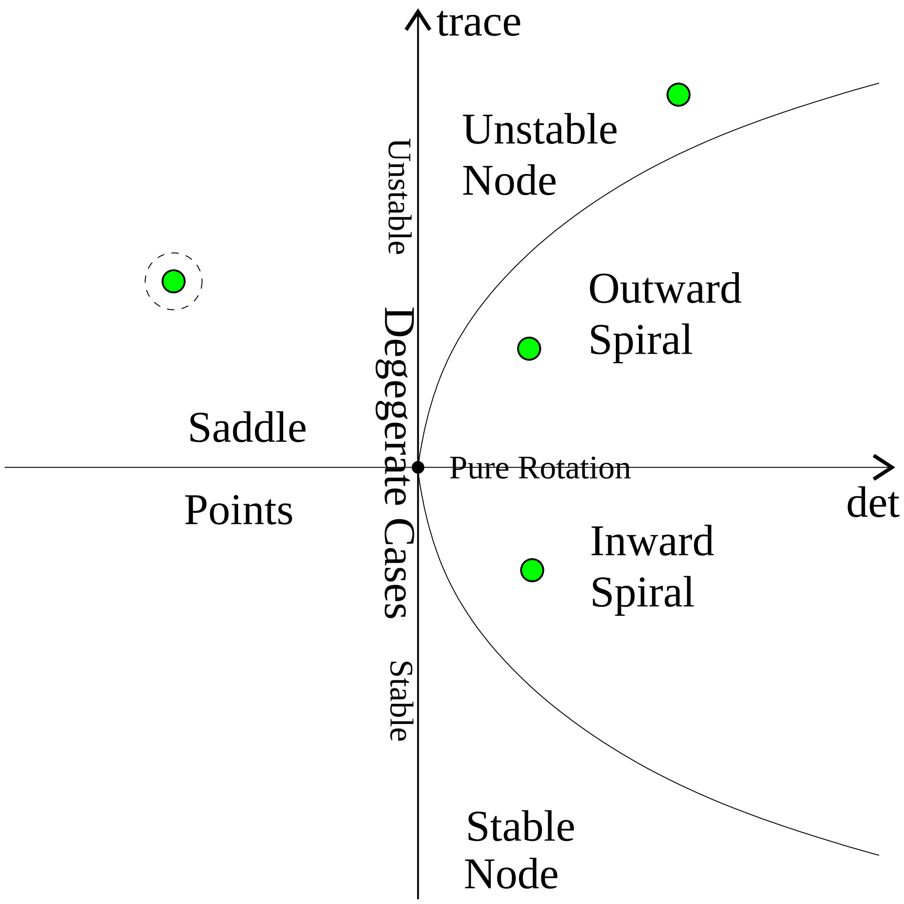

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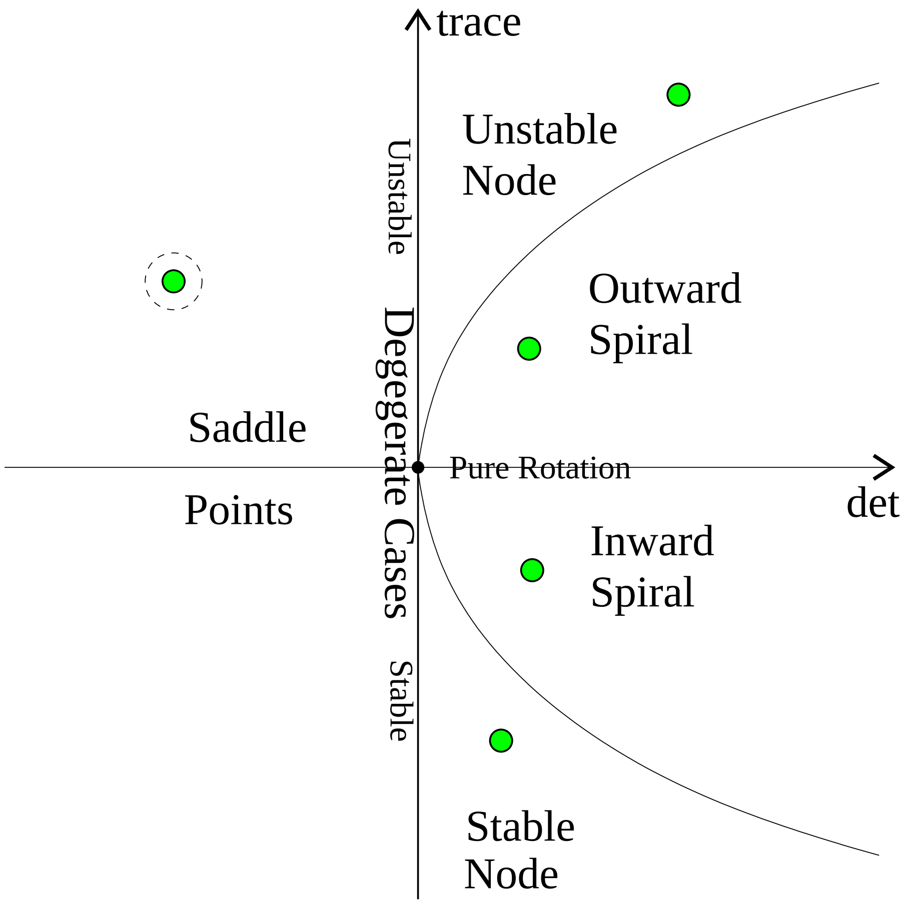

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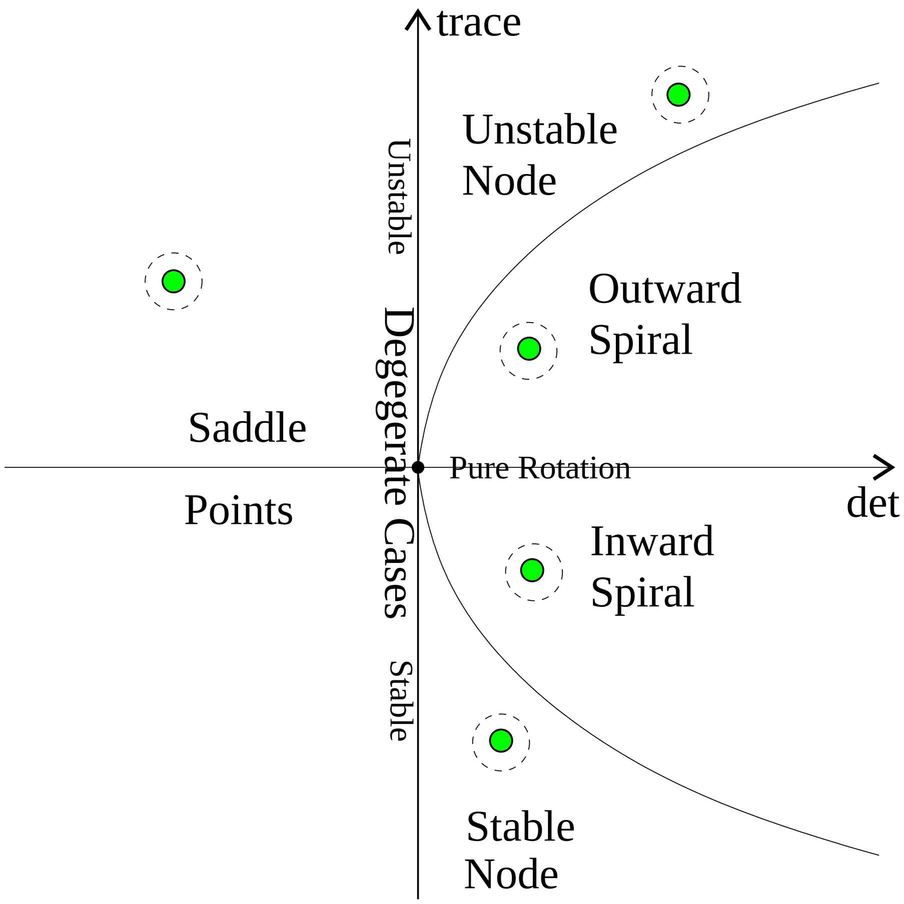
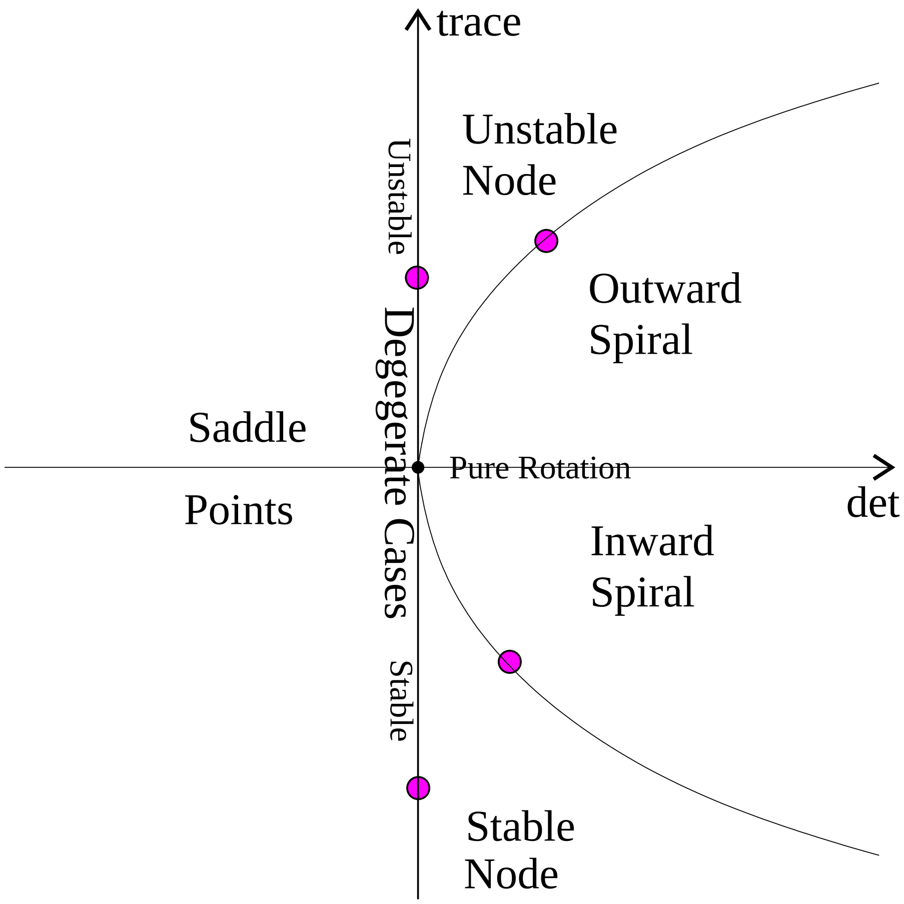


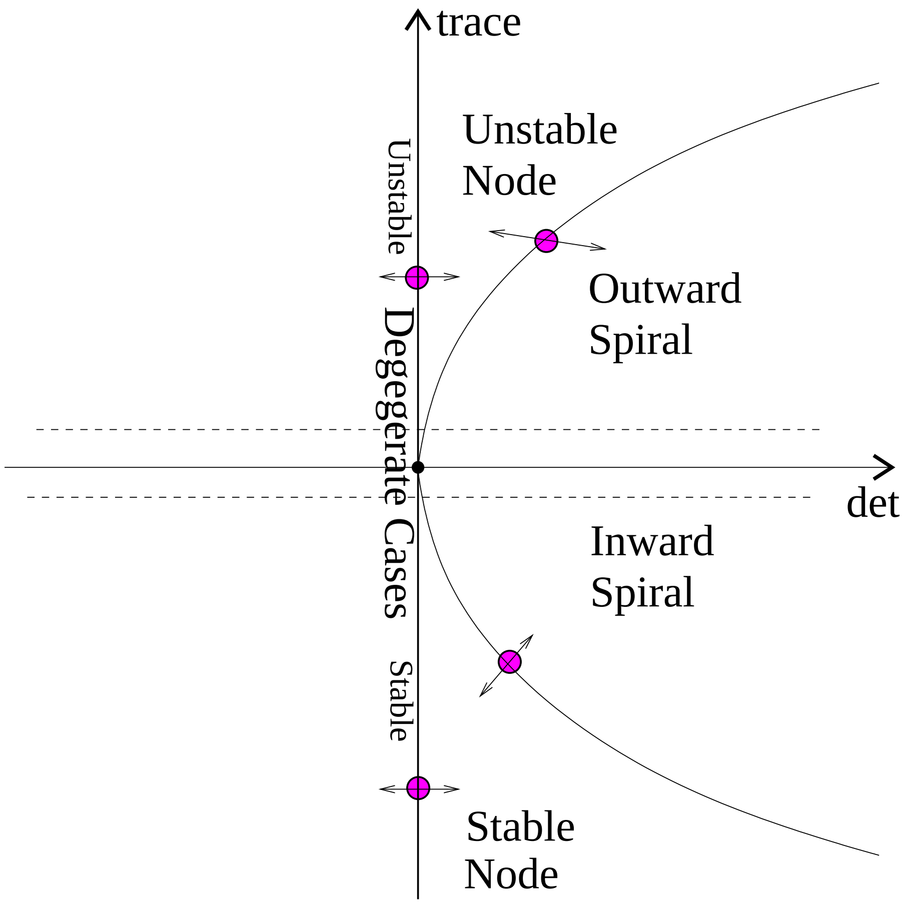
are also all accurate.

Limits of Linearization



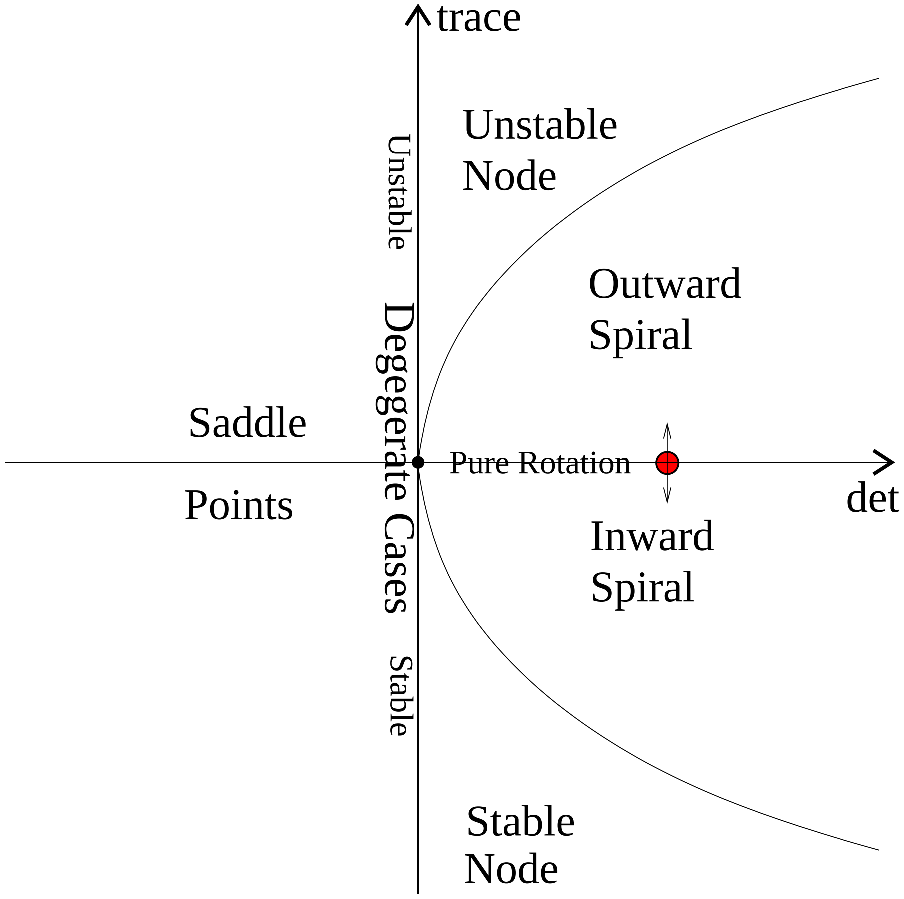
These border cases may be wrong about shape (i.e. spiral vs. saddle vs. node) ...

Limits of Linearization



... but not about stability, since they're isolated from the stability dividing line.

Limits of Linearization



This border case (pure rotation) is the worst ... here, linearization may mispredict shape *and* stability.

Limitations of Linearization

Most cases are not on the border,

Limitations of Linearization

Most cases are not on the border, So linearization is “usually" close enough ...

Limitations of Linearization

> “Linearization may not be perfect, but it sure is close enough for government work."

– Tom, United Technologies aerospace engineer (Pratt & Whitney), retired.

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Philosophy: eigenvalues/vectors are (almost) everything.