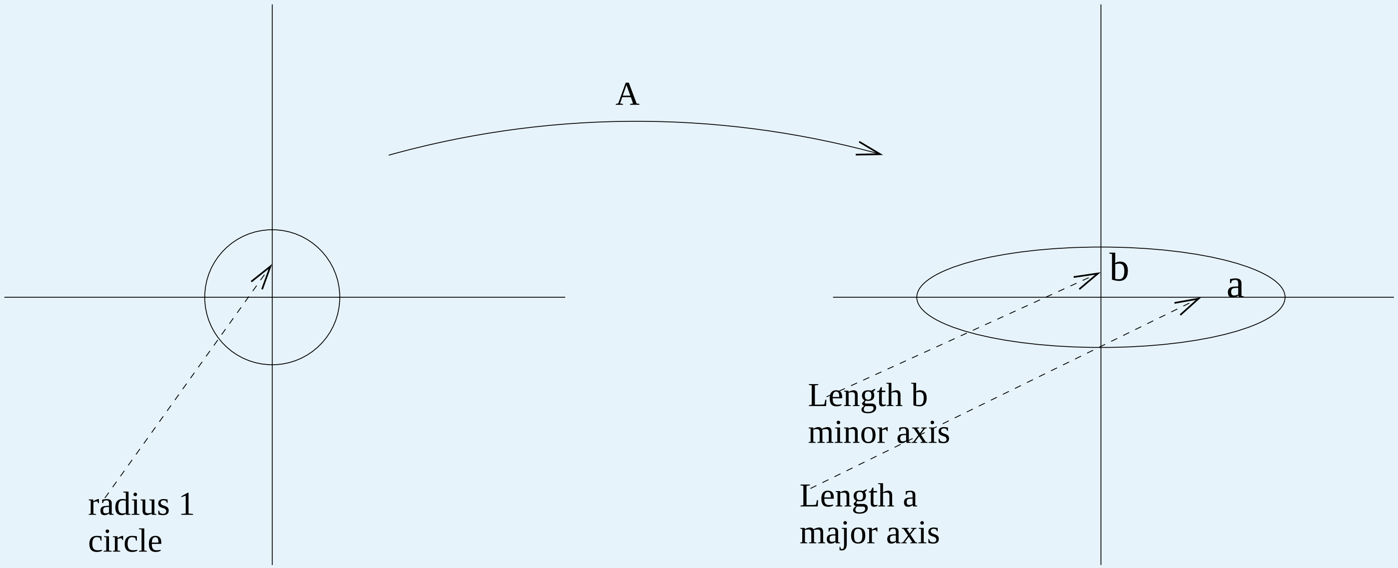
Lecture 1: More Linear Algebra



Overview

The topics will be:

Overview

The topics will be:

- More Linear Algebra (Day 1)

Overview

The topics will be:

- More Linear Algebra (Day 1)
- Analyzing Linear ODEs (Day 1 & 2)

Overview

The topics will be:

- More Linear Algebra (Day 1)
- Analyzing Linear ODEs (Day 1 & 2)
- Light Intro to Non-linear Systems (Day 2)

Overview

The topics will be:

- More Linear Algebra (Day 1)
- Analyzing Linear ODEs (Day 1 & 2)
- Light Intro to Non-linear Systems (Day 2)

The philosophy:

Overview

The topics will be:

- More Linear Algebra (Day 1)
- Analyzing Linear ODEs (Day 1 & 2)
- Light Intro to Non-linear Systems (Day 2)

The philosophy: get as comfortable as possible with qualitative behavior of linear systems(topic 2, requiring topic 1);

Overview

The topics will be:

- More Linear Algebra (Day 1)
- Analyzing Linear ODEs (Day 1 & 2)
- Light Intro to Non-linear Systems (Day 2)

The philosophy: get as comfortable as possible with qualitative behavior of linear systems(topic 2, requiring topic 1); then understand how non-linear systems can quickly differ (topic 3).

More Linear Algebra

Matrices Represent ALOT

Matrices:

Matrices Represent ALOT

Matrices:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & \dots & & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & \dots & & a_{nn} \end{bmatrix}$$

Matrices Represent ALOT

Matrices:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \cdots & & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & \cdots & & a_{nn} \end{bmatrix}$$

represent a lot of things.

Matrices Represent ALOT

Matrices:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \cdots & & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & \cdots & & a_{nn} \end{bmatrix}$$

represent a lot of things.
Simultaneous linear algebra.

Matrices Represent ALOT

Matrices:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & \dots & & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & \dots & & a_{nn} \end{bmatrix}$$

represent a lot of things.
Simultaneous linear algebra.

Operators that map vectors to vectors:

$$L : \mathbb{R}^n \longrightarrow \mathbb{R}^n; \text{ given by } x \mapsto Ax$$

Matrices Represent ALOT

Matrices:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \cdots & & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & \cdots & & a_{nn} \end{bmatrix}$$

represent a lot of things.
Simultaneous linear algebra.

Operators that map vectors to vectors:

$$L : \mathbb{R}^n \longrightarrow \mathbb{R}^n; \text{ given by } x \mapsto Ax$$

Linear ODEs:

$$\frac{dx}{dt} = Ax; x(0) = x_0$$

Matrices Represent ALOT

Matrices:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & \dots & & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & \dots & & a_{nn} \end{bmatrix}$$

represent a lot of things.
Simultaneous linear algebra.

Operators that map vectors to vectors:

$$L : \mathbb{R}^n \longrightarrow \mathbb{R}^n; \text{ given by } x \mapsto Ax$$

Linear ODEs:

$$\frac{dx}{dt} = Ax; x(0) = x_0$$

And, as you'll see in SB200, probabilistic processes.

The Main Point

But the Main Point is that:

The Main Point

But the Main Point is that:

Matrices are COMPLETELY classifiable.

The Main Point

But the Main Point is that:

Matrices are COMPLETELY classifiable.

Meaning, there is a standard “view" that every matrix can be put into that renders all of its properties, like:

The Main Point

But the Main Point is that:

Matrices are COMPLETELY classifiable.

Meaning, there is a standard “view" that every matrix can be put into that renders all of its properties, like:

- The existence and uniqueness of solutions to $Ax = b$

The Main Point

But the Main Point is that:

Matrices are COMPLETELY classifiable.

Meaning, there is a standard “view" that every matrix can be put into that renders all of its properties, like:

- The existence and uniqueness of solutions to $Ax = b$
- The range and behavior of the linear operator L

The Main Point

But the Main Point is that:

Matrices are COMPLETELY classifiable.

Meaning, there is a standard “view" that every matrix can be put into that renders all of its properties, like:

- The existence and uniqueness of solutions to $Ax = b$
- The range and behavior of the linear operator L
- the dynamic and steady-state behavior of $\dot{x} = Ax$

The Main Point

But the Main Point is that:

Matrices are COMPLETELY classifiable.

Meaning, there is a standard “view" that every matrix can be put into that renders all of its properties, like:

- The existence and uniqueness of solutions to $Ax = b$
- The range and behavior of the linear operator L
- the dynamic and steady-state behavior of $\dot{x} = Ax$
- and a great many other things,

The Main Point

But the Main Point is that:

Matrices are COMPLETELY classifiable.

Meaning, there is a standard “view" that every matrix can be put into that renders all of its properties, like:

- The existence and uniqueness of solutions to $Ax = b$
- The range and behavior of the linear operator L
- the dynamic and steady-state behavior of $\dot{x} = Ax$
- and a great many other things,

COMPLETELY obvious.

The Main Point

But the Main Point is that:

Matrices are COMPLETELY classifiable.

Meaning, there is a standard “view" that every matrix can be put into that renders all of its properties, like:

- The existence and uniqueness of solutions to $Ax = b$
- The range and behavior of the linear operator L
- the dynamic and steady-state behavior of $\dot{x} = Ax$
- and a great many other things,

COMPLETELY obvious.

Goal of this lecture: give you intuition for how this works.

A Practical Problem

The most important practical problem in basic linear algebra is:

A Practical Problem

The most important practical problem in basic linear algebra is:

Figuring out how to calculate

A Practical Problem

The most important practical problem in basic linear algebra is:

Figuring out how to calculate

$$e^A$$

A Practical Problem

The most important practical problem in basic linear algebra is:

Figuring out how to calculate

$$e^A$$

efficiently.

A Practical Problem

The most important practical problem in basic linear algebra is:

Figuring out how to calculate

$$e^A$$

efficiently.

This problem is inspired by ODEs.

A Practical Problem

The most important practical problem in basic linear algebra is:

Figuring out how to calculate

$$e^A$$

efficiently.

This problem is inspired by ODEs.

It drives all (or really, most) of the theory.

Commutativity

For any two real (or complex) numbers a and b,

Commutativity

For any two real (or complex) numbers a and b,

$$a \cdot b = b \cdot a.$$

Commutativity

For any two real (or complex) numbers a and b,

$$a \cdot b = b \cdot a.$$

This is the *commutativity* property.

Commutativity

For any two real (or complex) numbers a and b,

$$a \cdot b = b \cdot a.$$

This is the *commutativity* property. But matices are *not always commutative*.

Commutativity

For any two real (or complex) numbers a and b,

$$a \cdot b = b \cdot a.$$

This is the *commutativity* property. But matices are *not always commutative*.

Problem 1 Find two 2x2 matrices A and B such that

$$AB \neq BA; \text{ that is, } [A, B] = AB - BA \neq 0.$$

$[A, B]$ is called the “commutator" of A and B.

Commutativity

For any two real (or complex) numbers a and b,

$$a \cdot b = b \cdot a.$$

This is the *commutativity* property. But matices are *not always commutative*.

Problem 1 Find two 2x2 matrices A and B such that

$$AB \neq BA; \text{ that is, } [A, B] = AB - BA \neq 0.$$

$[A, B]$ is called the "commutator" of A and B.

Notice that A always commutes with e^A, because

$$Ae^A = A\left(\sum_{n=0}^{\infty} \frac{A^n}{n!}\right) = \sum_{n=0}^{\infty} \frac{A^{n+1}}{n!} = \left(\sum_{i=0}^{\infty} \frac{A^n}{n!}\right) A.$$

Commutativity

Why do we care about commutativity?

Commutativity

Why do we care about commutativity?

Remember the motivating problem (exponentiation).

Commutativity

Why do we care about commutativity?

Remember the motivating problem (exponentiation).

If $[A, B] = AB - BA = 0$, then

Commutativity

Why do we care about commutativity?

Remember the motivating problem (exponentiation).

If $[A, B] = AB - BA = 0$, then

$$e^{A+B} = e^A e^B = e^B e^A$$

Commutativity

Why do we care about commutativity?

Remember the motivating problem (exponentiation).

If $[A, B] = AB - BA = 0$, then

$$e^{A+B} = e^{A}e^{B} = e^{B}e^{A}$$

which might make the computation easier.

Commutativity

Why do we care about commutativity?

Remember the motivating problem (exponentiation).

If $[A, B] = AB - BA = 0$, then

$$e^{A+B} = e^A e^B = e^B e^A$$

which might make the computation easier.

Problem 2 Compute

$$exp\left(\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} t\right).$$

Commutativity

Why do we care about commutativity?

Remember the motivating problem (exponentiation).

If $[A, B] = AB - BA = 0$, then

$$e^{A+B} = e^A e^B = e^B e^A$$

which might make the computation easier.

Problem 2 Compute

$$exp\left(\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} t\right).$$

Answer:

$$\begin{bmatrix} e^t & 0 \\ 2te^t & e^t \end{bmatrix}.$$

Commutativity

But there’s a deeper interpretation.

Commutativity

But there’s a deeper interpretation.

Problem 3 Show the following Little Fact 1: if A and B are diagonal matrices, then they commute.

Commutativity

But there's a deeper interpretation.

Problem 3 Show the following Little Fact 1: if A and B are diagonal matrices, then they commute.

Reason: because diagonal matrix multiplication is just like a parallel version of regular number multiplication, separately on each diagonal.

Commutativity

But there's a deeper interpretation.

Problem 3 Show the following Little Fact 1: if A and B are diagonal matrices, then they commute.

Reason: because diagonal matrix multiplication is just like a parallel version of regular number multiplication, separately on each diagonal.

Problem 4 Show the following Little Fact 2: diagonal matrices with all the diagonal numbers being the same commute with *all* matrices.

Commutativity

But there's a deeper interpretation.

Problem 3 Show the following Little Fact 1: if A and B are diagonal matrices, then they commute.

Reason: because diagonal matrix multiplication is just like a parallel version of regular number multiplication, separately on each diagonal.

Problem 4 Show the following Little Fact 2: diagonal matrices with all the diagonal numbers being the same commute with *all* matrices.

Reason: $A(bI) = b(AI) = bA = (bI)A$; i.e. the identity matrix (obviously) commutes with everything.

Rotation Matrices

Before we go on, a little aside.

Rotation Matrices

Before we go on, a little aside.

Problem 5 What does the matrix

$$R_\theta = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix}$$

do?

Rotation Matrices

Before we go on, a little aside.

Problem 5 What does the matrix

$$R_\theta = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix}$$

do?

Answer: it rotates the plane through angle θ.

Rotation Matrices

Before we go on, a little aside.

Problem 5 What does the matrix

$$R_\theta = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix}$$

do?

Answer: it rotates the plane through angle θ.
What is its inverse? Well,

Rotation Matrices

Before we go on, a little aside.

Problem 5 What does the matrix

$$R_\theta = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix}$$

do?

Answer: it rotates the plane through angle θ.
What is its inverse? Well,

$$(R_\theta)^{-1} = R_{-\theta} = \begin{bmatrix} cos(-\theta) & -sin(-\theta) \\ sin(-\theta) & cos(-\theta) \end{bmatrix}.$$

But remember $cos(-\theta) = cos(\theta)$ and $sin(-\theta) = -sin(\theta)$, so

$$R_\theta^{-1} = \begin{bmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{bmatrix}.$$

Rotation Matrices

To rotate in three dimensions, we need three different rotations:

Rotation Matrices

To rotate in three dimensions, we need three different rotations:

$$R_\theta^{x,y} = \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix};\ R_\theta^{y,z} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\theta) & -sin(\theta) \\ 0 & sin(\theta) & cos(\theta) \end{bmatrix}$$

Rotation Matrices

To rotate in three dimensions, we need three different rotations:

$$R_\theta^{x,y} = \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} ; \; R_\theta^{y,z} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\theta) & -sin(\theta) \\ 0 & sin(\theta) & cos(\theta) \end{bmatrix}$$

and

Rotation Matrices

To rotate in three dimensions, we need three different rotations:

$$R_\theta^{x,y} = \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}; \; R_\theta^{y,z} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\theta) & -sin(\theta) \\ 0 & sin(\theta) & cos(\theta) \end{bmatrix}$$

and

$$R_\theta^{x,z} = \begin{bmatrix} cos(\theta) & 0 & -sin(\theta) \\ 0 & 1 & 0 \\ sin(\theta) & 0 & cos(\theta) \end{bmatrix}.$$

Rotation Matrices

To rotate in three dimensions, we need three different rotations:

$$R_\theta^{x,y} = \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} ; \; R_\theta^{y,z} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\theta) & -sin(\theta) \\ 0 & sin(\theta) & cos(\theta) \end{bmatrix}$$

and

$$R_\theta^{x,z} = \begin{bmatrix} cos(\theta) & 0 & -sin(\theta) \\ 0 & 1 & 0 \\ sin(\theta) & 0 & cos(\theta) \end{bmatrix}.$$

There are higher-dimensional versions for each n.

Commutativity

Back to commutativity. Let’s consider two 2x2 matrices.

Commutativity

Back to commutativity. Let's consider two 2x2 matrices. First,

$$A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

where a and b are real numbers.

Commutativity

Back to commutativity. Let's consider two 2x2 matrices. First,

$$A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

where a and b are real numbers.
A stretches the x axis by factor a and the y axis by factor b, making a circle into an ellipse.

Commutativity

Back to commutativity. Let's consider two 2x2 matrices. First,

$$A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

where a and b are real numbers.
For example, if $a < 1$ and $b > 1$, then the picture is

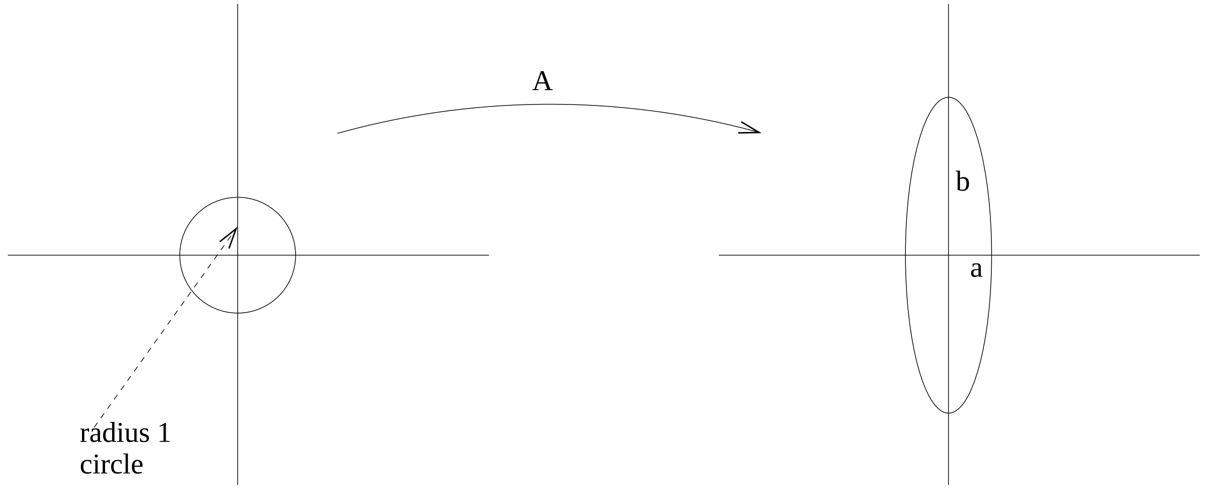


Figure 1: Stretching action of a 2x2 diagonal matrix

Commutativity

Back to commutativity. Let's consider two 2x2 matrices. First,

$$A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

where a and b are real numbers.

Conversely, if $a > 1$ and $b < 1$, then the picture is

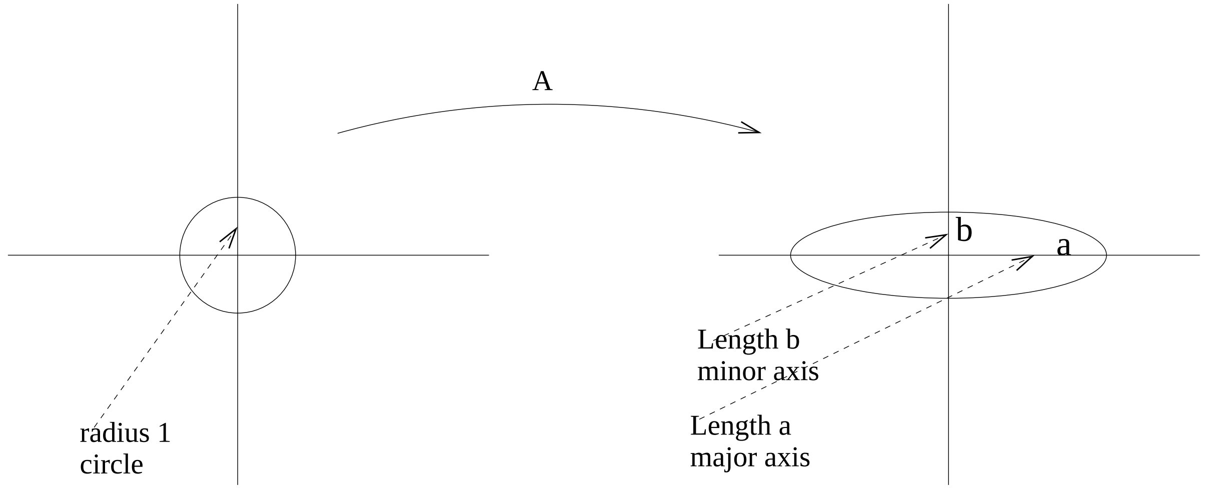


Figure 1: Stretching action of a 2x2 diagonal matrix

Commutativity

Back to commutativity. Let's consider two 2x2 matrices. First,

$$A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

where a and b are real numbers.

Problem 6 What are the eigenvalues and eigenvectors of A?

Commutativity

Back to commutativity. Let's consider two 2x2 matrices. First,

$$A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

where a and b are real numbers.

Problem 6 What are the eigenvalues and eigenvectors of A?

Answer: $([1\ 0], a)$ and $([0\ 1], b)$.

Commutativity

Ok, so we have one matrix, A. Now for the second matrix, B.

Commutativity

Ok, so we have one matrix, A. Now for the second matrix, B.

Problem 7 Find a 2x2 matrix B with eigenvalues α and β, and whose eigenvectors are rotated from x and y axis basis vectors by angle θ. Hint: use R_θ as a change-of-basis.

Commutativity

Ok, so we have one matrix, A. Now for the second matrix, B.

Problem 7 Find a 2x2 matrix B with eigenvalues α and β, and whose eigenvectors are rotated from x and y axis basis vectors by angle θ. Hint: use R_θ as a change-of-basis.

Answer:

$$B = R_\theta \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} (R_\theta)^{-1} = \begin{bmatrix} \alpha cos^2(\theta) + \beta sin^2(\theta) & (\alpha - \beta) sin(\theta) cos(\theta) \\ (\alpha - \beta) sin(\theta) cos(\theta) & \alpha sin^2(\theta) + \beta cos^2(\theta) \end{bmatrix}$$

Commutativity

Ok, so we have one matrix, A. Now for the second matrix, B.

Problem 7 Find a 2x2 matrix B with eigenvalues α and β, and whose eigenvectors are rotated from x and y axis basis vectors by angle θ. Hint: use R_θ as a change-of-basis.

Answer:

$$B = R_\theta \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} (R_\theta)^{-1} = \begin{bmatrix} \alpha cos^2(\theta) + \beta sin^2(\theta) & (\alpha - \beta) sin(\theta) cos(\theta) \\ (\alpha - \beta) sin(\theta) cos(\theta) & \alpha sin^2(\theta) + \beta cos^2(\theta) \end{bmatrix}$$

By construction, the eigenvectors and eigenvalues of B are ($[cos(\theta), sin(\theta)]$ with value α) and $[cos(\theta + \pi/2), sin(\theta + \pi/2)]$ with value β).

Commutativity

Ok, so we have one matrix, A. Now for the second matrix, B.

Problem 7 Find a 2x2 matrix B with eigenvalues α and β, and whose eigenvectors are rotated from x and y axis basis vectors by angle θ. Hint: use R_θ as a change-of-basis.

Answer:

$$B = R_\theta \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} (R_\theta)^{-1} = \begin{bmatrix} \alpha cos^2(\theta) + \beta sin^2(\theta) & (\alpha - \beta) sin(\theta) cos(\theta) \\ (\alpha - \beta) sin(\theta) cos(\theta) & \alpha sin^2(\theta) + \beta cos^2(\theta) \end{bmatrix}$$

By construction, the eigenvectors and eigenvalues of B are ($[cos(\theta), sin(\theta)]$ with value α) and $[cos(\theta + \pi/2), sin(\theta + \pi/2)]$ with value β).

Question: why is the $\pi/2$ there in the second eigenvector?

Commutativity

Now let's figure out when A and B commute.

Commutativity

Now let's figure out when A and B commute. On the one hand,

$$AB = \begin{bmatrix} a\alpha cos^2(\theta) + a\beta sin^2(\theta) & a(\alpha - \beta)sin(\theta)cos(\theta) \\ b(\alpha - \beta)sin(\theta)cos(\theta) & b\alpha sin^2(th) + b\beta cos^2(\theta) \end{bmatrix}.$$

Commutativity

Now let's figure out when A and B commute. On the one hand,

$$AB = \begin{bmatrix} a\alpha cos^2(\theta) + a\beta sin^2(\theta) & a(\alpha-\beta)sin(\theta)cos(\theta) \\ b(\alpha-\beta)sin(\theta)cos(\theta) & b\alpha sin^2(th) + b\beta cos^2(\theta) \end{bmatrix}.$$

On the other hand,

$$BA = \begin{bmatrix} a\alpha cos^2(\theta) + a\beta sin^2(\theta) & b(\alpha-\beta)sin(\theta)cos(\theta) \\ a(\alpha-\beta)sin(\theta)cos(\theta) & b\alpha sin^2(th) + b\beta cos^2(\theta) \end{bmatrix}.$$

Commutativity

Now let's figure out when A and B commute. On the one hand,

$$AB = \begin{bmatrix} a\alpha cos^2(\theta) + a\beta sin^2(\theta) & a(\alpha - \beta)sin(\theta)cos(\theta) \\ b(\alpha - \beta)sin(\theta)cos(\theta) & b\alpha sin^2(th) + b\beta cos^2(\theta) \end{bmatrix}.$$

On the other hand,

$$BA = \begin{bmatrix} a\alpha cos^2(\theta) + a\beta sin^2(\theta) & b(\alpha - \beta)sin(\theta)cos(\theta) \\ a(\alpha - \beta)sin(\theta)cos(\theta) & b\alpha sin^2(th) + b\beta cos^2(\theta) \end{bmatrix}.$$

Problem 8 When are AB and BA equal?

Commutativity

Now let's figure out when A and B commute. On the one hand,

$$AB = \begin{bmatrix} a\alpha cos^2(\theta) + a\beta sin^2(\theta) & a(\alpha - \beta) sin(\theta) cos(\theta) \\ b(\alpha - \beta) sin(\theta) cos(\theta) & b\alpha sin^2(th) + b\beta cos^2(\theta) \end{bmatrix}.$$

On the other hand,

$$BA = \begin{bmatrix} a\alpha cos^2(\theta) + a\beta sin^2(\theta) & b(\alpha - \beta) sin(\theta) cos(\theta) \\ a(\alpha - \beta) sin(\theta) cos(\theta) & b\alpha sin^2(th) + b\beta cos^2(\theta) \end{bmatrix}.$$

Problem 8 When are AB and BA equal?

Answer: When 1) $a = b$, or 2) $\alpha = \beta$ or 3) $\theta = 0, \pi/2, \pi, 3\pi/2$.

Commutativity

But now, let’s look at the cases one by one.

Commutativity

Case 1: $a = b$. In that case,

Commutativity

Case 1: $a = b$. In that case, $A = aI_2$, where I_2 is the 2x2 identity matrix.

Commutativity

Case 1: $a = b$. In that case, $A = aI_2$, where I_2 is the 2x2 identity matrix.

Claim:

$[cos(\theta), sin(\theta)]$ and $[cos(\theta + \pi/2), sin(\theta + \pi/2)]$

are just as good eigenvectors for A as the original ones.

Commutativity

Case 1: $a = b$. In that case, $A = aI_2$, where I_2 is the 2x2 identity matrix.

Claim:

$$[cos(\theta), sin(\theta)] \text{ and } [cos(\theta + \pi/2), sin(\theta + \pi/2)]$$

are just as good eigenvectors for A as the original ones.

Problem 9 Why? (Don't give a computational proof.)

Commutativity

Case 1: $a = b$. In that case, $A = aI_2$, where I_2 is the 2x2 identity matrix.

Claim:

$$[cos(\theta), sin(\theta)] \text{ and } [cos(\theta + \pi/2), sin(\theta + \pi/2)]$$

are just as good eigenvectors for A as the original ones.

Problem 9 Why? (Don't give a computational proof.)

Answer: because the two original eigenvectors $[0, 1]$ and $[1, 0]$ have the *same eigenvalue*, so linear combinations are also eigenvectors.

Commutativity

Case 1: $a = b$. In that case, $A = aI_2$, where I_2 is the 2x2 identity matrix.

Claim:

$$[cos(\theta), sin(\theta)] \text{ and } [cos(\theta + \pi/2), sin(\theta + \pi/2)]$$

are just as good eigenvectors for A as the original ones.

Problem 9 Why? (Don't give a computational proof.)

Answer: because the two original eigenvectors $[0, 1]$ and $[1, 0]$ have the *same eigenvalue*, so linear combinations are also eigenvectors.

Hence: Case 1) $\Rightarrow$ A and B have a common set of eigenvectors.

Commutativity

Now, Case 2: $\alpha = \beta$.

Commutativity

Now, Case 2: $\alpha = \beta$.
In this case,

$$B = R_\theta \alpha \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (R_\theta)^{-1} = \alpha R_\theta (R_\theta)^{-1} = \alpha I_2.$$

Commutativity

Now, Case 2: $\alpha = \beta$.
In this case,

$$B = R_\theta \alpha \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (R_\theta)^{-1} = \alpha R_\theta (R_\theta)^{-1} = \alpha I_2.$$

But that means we're in the same situation as Case 1. Again, common eigenvectors.

Commutativity

Now, Case 2: $\alpha = \beta$.
In this case,

$$B = R_\theta \alpha \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (R_\theta)^{-1} = \alpha R_\theta (R_\theta)^{-1} = \alpha I_2.$$

But that means we're in the same situation as Case 1. Again, common eigenvectors.

Finally, Case 3: $\theta = 0, \pi/2, \pi,$ or $3\pi/2$.

Commutativity

Now, Case 2: $\alpha = \beta$.
In this case,

$$B = R_\theta \alpha \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (R_\theta)^{-1} = \alpha R_\theta (R_\theta)^{-1} = \alpha I_2.$$

But that means we're in the same situation as Case 1. Again, common eigenvectors.

Finally, Case 3: $\theta = 0, \pi/2, \pi,$ or $3\pi/2$.

Problem 10 What are the eigenvectors of B in this case?

Commutativity

Now, Case 2: $\alpha = \beta$.
In this case,

$$B = R_\theta \alpha \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (R_\theta)^{-1} = \alpha R_\theta (R_\theta)^{-1} = \alpha I_2.$$

But that means we're in the same situation as Case 1. Again, common eigenvectors.

Finally, Case 3: $\theta = 0, \pi/2, \pi,$ or $3\pi/2$.

Problem 10 What are the eigenvectors of B in this case?

Answer: $[0\ 1]$ and $[1\ 0]$.

Commutativity

Now, Case 2: $\alpha = \beta$.
In this case,

$$B = R_\theta \alpha \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (R_\theta)^{-1} = \alpha R_\theta (R_\theta)^{-1} = \alpha I_2.$$

But that means we're in the same situation as Case 1. Again, common eigenvectors.

Finally, Case 3: $\theta = 0, \pi/2, \pi,$ or $3\pi/2$.

Problem 10 What are the eigenvectors of B in this case?

Answer: $[0\ 1]$ and $[1\ 0]$.

Again! A and B have a common set of eigenvectors.

Commutativity

Little Facts 1 and 2 above *also say the same thing*.

Commutativity

Little Facts 1 and 2 above *also say the same thing*.

Problem 11 Show: if two matrices A and B can be diagonalized by the same matrix S, then A and B commute.

Commutativity

Little Facts 1 and 2 above *also say the same thing*.

Problem 11 Show: if two matrices A and B can be diagonalized by the same matrix S, then A and B commute.

Reason:

Commutativity

Little Facts 1 and 2 above *also say the same thing*.

Problem 11 Show: if two matrices A and B can be diagonalized by the same matrix S, then A and B commute.

Reason:

- $AB = (XD_AX^{-1})(XD_BX^{-1})$,

Commutativity

Little Facts 1 and 2 above *also say the same thing*.

Problem 11 Show: if two matrices A and B can be diagonalized by the same matrix S, then A and B commute.

Reason:

- $AB = (XD_AX^{-1})(XD_BX^{-1})$,
- $= XD_A(X^{-1}X)D_BX^{-1}$ by associativity

Commutativity

Little Facts 1 and 2 above *also say the same thing*.

Problem 11 Show: if two matrices A and B can be diagonalized by the same matrix S, then A and B commute.

Reason:
- $AB = (XD_AX^{-1})(XD_BX^{-1})$,
- $= XD_A(X^{-1}X)D_BX^{-1}$ by associativity
- $= XD_AD_BX^{-1}$ since $X^{-1}X = I$

Commutativity

Little Facts 1 and 2 above *also say the same thing*.

Problem 11 Show: if two matrices A and B can be diagonalized by the same matrix S, then A and B commute.

Reason:
- $AB = (XD_AX^{-1})(XD_BX^{-1})$,
- $= XD_A(X^{-1}X)D_BX^{-1}$ by associativity
- $= XD_AD_BX^{-1}$ since $X^{-1}X = I$
- $= XD_BD_AX^{-1}$ by Litte Fact 1

Commutativity

Little Facts 1 and 2 above *also say the same thing*.

Problem 11 Show: if two matrices A and B can be diagonalized by the same matrix S, then A and B commute.

Reason:

- $AB = (XD_AX^{-1})(XD_BX^{-1})$,
- $= XD_A(X^{-1}X)D_BX^{-1}$ by associativity
- $= XD_AD_BX^{-1}$ since $X^{-1}X = I$
- $= XD_BD_AX^{-1}$ by Litte Fact 1
- $= XD_B(X^{-1}X)D_AX^{-1}$ – we've inserted $X^{-1}X = I$

Commutativity

Little Facts 1 and 2 above *also say the same thing*.

Problem 11 Show: if two matrices A and B can be diagonalized by the same matrix S, then A and B commute.

Reason:

- $AB = (XD_AX^{-1})(XD_BX^{-1})$,
- $= XD_A(X^{-1}X)D_BX^{-1}$ by associativity
- $= XD_AD_BX^{-1}$ since $X^{-1}X = I$
- $= XD_BD_AX^{-1}$ by Litte Fact 1
- $= XD_B(X^{-1}X)D_AX^{-1}$ – we've inserted $X^{-1}X = I$
- $= (XD_BX^{-1}(XD_AX^{-1}) = BA$.

Commutativity

Little Facts 1 and 2 above *also say the same thing*.

Problem 11 Show: if two matrices A and B can be diagonalized by the same matrix S, then A and B commute.

Reason:

- $AB = (XD_AX^{-1})(XD_BX^{-1})$,
- $= XD_A(X^{-1}X)D_BX^{-1}$ by associativity
- $= XD_AD_BX^{-1}$ since $X^{-1}X = I$
- $= XD_BD_AX^{-1}$ by Litte Fact 1
- $= XD_B(X^{-1}X)D_AX^{-1}$ – we've inserted $X^{-1}X = I$
- $= (XD_BX^{-1}(XD_AX^{-1}) = BA$.

A key fact is that the *converse* is true.

Theorem 1 *If A and B are both diagonalizable, then they are commutative if and only if they have a common eigenbasis.*

Eigenbases

Let's go back to the notion of an eigenbasis, that is,

Eigenbases

Let's go back to the notion of an eigenbasis, that is,

a set $\{v_1, \ldots, v_n\}$ of distinct eigenvectors.
An eigenbasis exists IFF a matrix is diagonalizable.

Eigenbases

Let's go back to the notion of an eigenbasis, that is,

a set $\{v_1, \ldots, v_n\}$ of distinct eigenvectors.
An eigenbasis exists IFF a matrix is diagonalizable.

$$A = [v_1 \mid v_2 \mid \ldots\ v_n] \begin{bmatrix} \lambda_1 & 0 & 0 & \ldots & 0 \\ 0 & \lambda_2 & 0 & \ldots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \ldots & 0 & \lambda_n \end{bmatrix} [v_1 \mid v_2 \mid \ldots\ v_n]^{-1}$$

Eigenbases

Let's go back to the notion of an eigenbasis, that is,

a set $\{v_1, \ldots, v_n\}$ of distinct eigenvectors.
An eigenbasis exists IFF a matrix is diagonalizable.

$$A = [v_1 \mid v_2 \mid \ldots \; v_n] \begin{bmatrix} \lambda_1 & 0 & 0 & \ldots & 0 \\ 0 & \lambda_2 & 0 & \ldots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \ldots & 0 & \lambda_n \end{bmatrix} [v_1 \mid v_2 \mid \ldots \; v_n]^{-1}$$

where $Av_i = \lambda_i v_i$.

Eigenbases

Let's go back to the notion of an eigenbasis, that is,

a set $\{v_1, \ldots, v_n\}$ of distinct eigenvectors.
An eigenbasis exists IFF a matrix is diagonalizable.

$$A = [v_1 \mid v_2 \mid \ \ldots \ v_n] \begin{bmatrix} \lambda_1 & 0 & 0 & \ldots & 0 \\ 0 & \lambda_2 & 0 & \ldots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \ldots & 0 & \lambda_n \end{bmatrix} [v_1 \mid v_2 \mid \ \ldots \ v_n]^{-1}$$

where $Av_i = \lambda_i v_i$.

Of course, the v_i and λ_i might (have to) be complex, even if A is real.

Eigenbases

Let's go back to the notion of an eigenbasis, that is,

a set $\{v_1, \ldots, v_n\}$ of distinct eigenvectors.
An eigenbasis exists IFF a matrix is diagonalizable.

$$A = [v_1 \mid v_2 \mid \ldots\ v_n] \begin{bmatrix} \lambda_1 & 0 & 0 & \ldots & 0 \\ 0 & \lambda_2 & 0 & \ldots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \ldots & 0 & \lambda_n \end{bmatrix} [v_1 \mid v_2 \mid \ldots\ v_n]^{-1}$$

where $Av_i = \lambda_i v_i$.

Of course, the v_i and λ_i might (have to) be complex, even if A is real. We'll come back to this.

Eigenbases

Now, suppose that A is diagonalizable,

Eigenbases

Now, suppose that A is diagonalizable, and has $\{v_1, \ldots, v_n\}$ as $n = dim(A)$ independent eigenvectors.

Eigenbases

Now, suppose that A is diagonalizable, and has $\{v_1, \ldots, v_n\}$ as $n = dim(A)$ independent eigenvectors.

Collect them in groups of equal eigenvalues, say, in decreasing order:

Eigenbases

Now, suppose that A is diagonalizable, and has $\{v_1, \ldots, v_n\}$ as $n = dim(A)$ independent eigenvectors.

Collect them in groups of equal eigenvalues, say, in decreasing order:

$v^1_{\lambda_1}, v^2_{\lambda_1}, \ldots, v^{n_1}_{\lambda_1}$ have eigenvalue $\lambda_1 = \lambda_{max}$

Eigenbases

Now, suppose that A is diagonalizable, and has $\{v_1, \ldots, v_n\}$ as $n = dim(A)$ independent eigenvectors.

Collect them in groups of equal eigenvalues, say, in decreasing order:

$v^1_{\lambda_1}, v^2_{\lambda_1}, \ldots, v^{n_1}_{\lambda_1}$ have eigenvalue $\lambda_1 = \lambda_{max}$

$v^1_{\lambda_2}, v^2_{\lambda_2}, \ldots, v^{n_2}_{\lambda_2}$ have eigenvalue $\lambda_2 < \lambda_1$

Eigenbases

Now, suppose that A is diagonalizable, and has $\{v_1, \ldots, v_n\}$ as $n = dim(A)$ independent eigenvectors.

Collect them in groups of equal eigenvalues, say, in decreasing order:

$v^1_{\lambda_1}, v^2_{\lambda_1}, \ldots, v^{n_1}_{\lambda_1}$ have eigenvalue $\lambda_1 = \lambda_{max}$

$v^1_{\lambda_2}, v^2_{\lambda_2}, \ldots, v^{n_2}_{\lambda_2}$ have eigenvalue $\lambda_2 < \lambda_1$

. . . .

Eigenbases

Now, suppose that A is diagonalizable, and has $\{v_1, \ldots, v_n\}$ as $n = dim(A)$ independent eigenvectors.

Collect them in groups of equal eigenvalues, say, in decreasing order:

$v^1_{\lambda_1}, v^2_{\lambda_1}, \ldots, v^{n_1}_{\lambda_1}$ have eigenvalue $\lambda_1 = \lambda_{max}$

$v^1_{\lambda_2}, v^2_{\lambda_2}, \ldots, v^{n_2}_{\lambda_2}$ have eigenvalue $\lambda_2 < \lambda_1$

. . . .
$v^m_{\lambda_m}, v^2\lambda_m, \ldots, v^{n_m}_{\lambda_m}$ have eigenvalue $\lambda_m = \lambda_{min}$.

Eigenbases

Now, suppose that A is diagonalizable, and has $\{v_1, \ldots, v_n\}$ as $n = dim(A)$ independent eigenvectors.

Collect them in groups of equal eigenvalues, say, in decreasing order:

$v^1_{\lambda_1}, v^2_{\lambda_1}, \ldots, v^{n_1}_{\lambda_1}$ have eigenvalue $\lambda_1 = \lambda_{max}$

$v^1_{\lambda_2}, v^2_{\lambda_2}, \ldots, v^{n_2}_{\lambda_2}$ have eigenvalue $\lambda_2 < \lambda_1$

. . . .

$v^m_{\lambda_m}, v^2\lambda_m, \ldots, v^{n_m}_{\lambda_m}$ have eigenvalue $\lambda_m = \lambda_{min}$.

$m =$ number of distinct eigenvalues, and
$n_1 + n_2 + \ldots + n_m = \sum_i n_i = dim(A) = n$.

Eigenbases

We can think of A (after some change of basis) as

Eigenbases

We can think of A (after some change of basis) as

$$A = \begin{bmatrix} \lambda_1 I_{n_1} & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 I_{n_2} & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \dots & 0 & \lambda_m I_{n_m} \end{bmatrix}.$$

Eigenbases

We can think of A (after some change of basis) as

$$A = \begin{bmatrix} \lambda_1 I_{n_1} & 0 & 0 & \ldots & 0 \\ 0 & \lambda_2 I_{n_2} & 0 & \ldots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \ldots & 0 & \lambda_m I_{n_m} \end{bmatrix}.$$

Each block $\lambda_i I_{n_i}$ is the *eigenspace* associated with eigenvalue λ_i.

Eigenbases

We can think of A (after some change of basis) as

$$A = \begin{bmatrix} \lambda_1 I_{n_1} & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 I_{n_2} & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \dots & 0 & \lambda_m I_{n_m} \end{bmatrix}.$$

Each block $\lambda_i I_{n_i}$ is the *eigenspace* associated with eigenvalue λ_i.

Problem 12 Show: for each i, A_{λ_i} is a vector space with dimension n_i.

Eigenbases

We can think of A (after some change of basis) as

$$A = \begin{bmatrix} \lambda_1 I_{n_1} & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 I_{n_2} & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \dots & 0 & \lambda_m I_{n_m} \end{bmatrix}.$$

Each block $\lambda_i I_{n_i}$ is the *eigenspace* associated with eigenvalue λ_i.

Problem 12 Show: for each i, A_{λ_i} is a vector space with dimension n_i.

Reason: linear combinations of eigenvectors with the same eigenvalue are also eigenvectors with that eigenvalue.

Eigenbases

We can think of A (after some change of basis) as

$$A = \begin{bmatrix} \lambda_1 I_{n_1} & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 I_{n_2} & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \dots & 0 & \lambda_m I_{n_m} \end{bmatrix}.$$

Each block $\lambda_i I_{n_i}$ is the *eigenspace* associated with eigenvalue λ_i.

Problem 12 Show: for each i, A_{λ_i} is a vector space with dimension n_i.

Reason: linear combinations of eigenvectors with the same eigenvalue are also eigenvectors with that eigenvalue.
For the same reason, any basis of A_{λ_i} is equivalent to any other, for the purposes of diagonalization.

Behavioral Eigen-Analysis

Let’s get back to the issue of complex and real eigenvalues.

Behavioral Eigen-Analysis

Let's get back to the issue of complex and real eigenvalues. Suppose all the eigenvalues of A are real. Then

Behavioral Eigen-Analysis

Let's get back to the issue of complex and real eigenvalues. Suppose all the eigenvalues of A are real. Then

$A = SDS^{-1}$ where

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \dots & 0 & \lambda_n \end{bmatrix}$$

is a diagonal matrix with all real entries along the diagonal.

Behavioral Eigen-Analysis

Let's get back to the issue of complex and real eigenvalues. Suppose all the eigenvalues of A are real. Then

$A = SDS^{-1}$ where

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \dots & 0 & \lambda_n \end{bmatrix}$$

is a diagonal matrix with all real entries along the diagonal.

Question: what does a diagonal matrix with real entries correspond to? (don't forget some could be negative)

Behavioral Eigen-Analysis

Let's get back to the issue of complex and real eigenvalues. Suppose all the eigenvalues of A are real. Then

$A = SDS^{-1}$ where

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \dots & 0 & \lambda_n \end{bmatrix}$$

is a diagonal matrix with all real entries along the diagonal.

Question: what does a diagonal matrix with real entries correspond to? (don't forget some could be negative)
Answer: stretching along various directions, with a flip as well if negative.

Behavioral Eigen-Analysis

Hm, what does a complex eigenvalue correspond to?

Behavioral Eigen-Analysis

Hm, what does a complex eigenvalue correspond to?

Problem 13 Compute the eigenvalues and eigenvectors of

$$R_\theta = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix}.$$

Behavioral Eigen-Analysis

Hm, what does a complex eigenvalue correspond to?

Problem 13 Compute the eigenvalues and eigenvectors of

$$R_\theta = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix}.$$

Answer: $v_1 = [1\ i]$, with eigenvalue $cos(\theta) - isin(\theta)$ and $v_2 = [i\ 1]$ with eigenvalue $cos(\theta) + isin(\theta)$.

Behavioral Eigen-Analysis

Hm, what does a complex eigenvalue correspond to?

Problem 13 Compute the eigenvalues and eigenvectors of

$$R_\theta = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix}.$$

Answer: $v_1 = [1\ i]$, with eigenvalue $cos(\theta) - isin(\theta)$ and $v_2 = [i\ 1]$ with eigenvalue $cos(\theta) + isin(\theta)$.

Now, notice three facts:

Behavioral Eigen-Analysis

Hm, what does a complex eigenvalue correspond to?

Problem 13 Compute the eigenvalues and eigenvectors of

$$R_\theta = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix}.$$

Answer: $v_1 = [1\ i]$, with eigenvalue $cos(\theta) - isin(\theta)$ and $v_2 = [i\ 1]$ with eigenvalue $cos(\theta) + isin(\theta)$.

Now, notice three facts:

- Rotation here corresponds to complex eigenvalues/eigenvectors.

Behavioral Eigen-Analysis

Hm, what does a complex eigenvalue correspond to?

Problem 13 Compute the eigenvalues and eigenvectors of

$$R_\theta = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix}.$$

Answer: $v_1 = [1\ i]$, with eigenvalue $cos(\theta) - isin(\theta)$ and $v_2 = [i\ 1]$ with eigenvalue $cos(\theta) + isin(\theta)$.

Now, notice three facts:

- Rotation here corresponds to complex eigenvalues/eigenvectors.
- The eigenvalues are complex conjugates of each other.

Behavioral Eigen-Analysis

Hm, what does a complex eigenvalue correspond to?

Problem 13 Compute the eigenvalues and eigenvectors of

$$R_\theta = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix}.$$

Answer: $v_1 = [1\ i]$, with eigenvalue $cos(\theta) - isin(\theta)$ and $v_2 = [i\ 1]$ with eigenvalue $cos(\theta) + isin(\theta)$.

Now, notice three facts:

- Rotation here corresponds to complex eigenvalues/eigenvectors.
- The eigenvalues are complex conjugates of each other.
- Conjugate pair corresponds to real 2x2 with equal diagonal elements, $\pm$ off-diagonals.

Behavioral Eigen-Analysis

Let's see why these facts are general.

Behavioral Eigen-Analysis

Let's see why these facts are general.

Problem 14 Show that of λ is an eigenvalue of a real matrix, so is $\bar{\lambda}$.

Behavioral Eigen-Analysis

Let's see why these facts are general.

Problem 14 Show that of λ is an eigenvalue of a real matrix, so is $\bar{\lambda}$.

Reason:

Behavioral Eigen-Analysis

Let's see why these facts are general.

Problem 14 Show that of λ is an eigenvalue of a real matrix, so is $\bar{\lambda}$.

Reason:

- If $Ax = \lambda x$, then conjugating gives:

Behavioral Eigen-Analysis

Let’s see why these facts are general.

Problem 14 Show that of λ is an eigenvalue of a real matrix, so is $\bar{\lambda}$.

Reason:

- If $Ax = \lambda x$, then conjugating gives:
- $\overline{Ax} = \overline{\lambda x} = \bar{A}\bar{x} = \bar{\lambda}\bar{x}$.

Behavioral Eigen-Analysis

Let's see why these facts are general.

Problem 14 Show that of λ is an eigenvalue of a real matrix, so is $\bar{\lambda}$.

Reason:

- If $Ax = \lambda x$, then conjugating gives:
- $\overline{Ax} = \overline{\lambda x} = \bar{A}\bar{x} = \bar{\lambda}\bar{x}$.
- since A is real, $\bar{A} = A$, so

Behavioral Eigen-Analysis

Let's see why these facts are general.

Problem 14 Show that of λ is an eigenvalue of a real matrix, so is $\bar{\lambda}$.

Reason:

- If $Ax = \lambda x$, then conjugating gives:
- $\overline{Ax} = \overline{\lambda x} = \bar{A}\bar{x} = \bar{\lambda}\bar{x}$.
- since A is real, $\bar{A} = A$, so
- $A\bar{x} = \bar{\lambda}\bar{x}$

Behavioral Eigen-Analysis

Let's see why these facts are general.

Problem 14 Show that of λ is an eigenvalue of a real matrix, so is $\bar{\lambda}$.

Reason:

- If $Ax = \lambda x$, then conjugating gives:
- $\overline{Ax} = \overline{\lambda x} = \bar{A}\bar{x} = \bar{\lambda}\bar{x}$.
- since A is real, $\bar{A} = A$, so
- $A\bar{x} = \bar{\lambda}\bar{x}$
- so $\bar{\lambda}$ is an eigenvalue by definition.

Behavioral Eigen-Analysis

Hence, the eigenvalues of A can be listed in two groups.

Behavioral Eigen-Analysis

Hence, the eigenvalues of A can be listed in two groups.

Real values: $r_1, r_2, \ldots, r_k$ and

Behavioral Eigen-Analysis

Hence, the eigenvalues of A can be listed in two groups.

Real values: $r_1, r_2, \ldots, r_k$ and non-real values: $c_1, \bar{c}_1, c_2, \bar{c}_2, \ldots, c_l, \bar{c}_l$.

Behavioral Eigen-Analysis

Hence, the eigenvalues of A can be listed in two groups.

Real values: $r_1, r_2, \ldots, r_k$ and non-real values: $c_1, \bar{c}_1, c_2, \bar{c}_2, \ldots, c_l, \bar{c}_l$.
So, when diagonalized,

Behavioral Eigen-Analysis

Hence, the eigenvalues of A can be listed in two groups.

Real values: $r_1, r_2, \ldots, r_k$ and non-real values: $c_1, \bar{c_1}, c_2, \bar{c_2}, \ldots, c_l, \bar{c_l}$.
So, when diagonalized,

$$A = \begin{bmatrix} r_1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 & \ldots & 0 & 0 & 0 \\ \vdots & 0 & 0 & \ldots & 0 & 0 & \vdots & \\ 0 & \ldots & r_k & 0 & \ldots & 0 & 0 & \\ 0 & 0 & 0 & c_1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{c_1} & 0 & \ldots & 0 \\ \vdots & & & & & & & \vdots \\ 0 & & & & & & c_l & 0 \\ 0 & & & & & & 0 & \bar{c_l} \end{bmatrix}.$$

Behavioral Eigen-Analysis

But consider those complex pair-blocks

$$\text{Block}_i = \begin{bmatrix} c_i & 0 \\ 0 & \bar{c}_i \end{bmatrix}.$$

Behavioral Eigen-Analysis

But consider those complex pair-blocks

$$\text{Block}_i = \begin{bmatrix} c_i & 0 \\ 0 & \bar{c}_i \end{bmatrix}.$$

In fact, they correspond to real matrices. Namely:

Behavioral Eigen-Analysis

But consider those complex pair-blocks

$$\text{Block}_i = \begin{bmatrix} c_i & 0 \\ 0 & \bar{c}_i \end{bmatrix}.$$

In fact, they correspond to real matrices. Namely: Block_i is the same up to change of basis as

Behavioral Eigen-Analysis

But consider those complex pair-blocks

$$\text{Block}_i = \begin{bmatrix} c_i & 0 \\ 0 & \bar{c}_i \end{bmatrix}.$$

In fact, they correspond to real matrices. Namely: Block_i is the same up to change of basis as

$$\begin{bmatrix} a_i & -b_i \\ b_i & a_i \end{bmatrix}$$

where $a_i = Re(c_i)$ and $b_i = Im(c_i)$.

Behavioral Eigen-Analysis

But consider those complex pair-blocks

$$\text{Block}_i = \begin{bmatrix} c_i & 0 \\ 0 & \bar{c}_i \end{bmatrix}.$$

In fact, they correspond to real matrices. Namely: Block_i is the same up to change of basis as

$$\begin{bmatrix} a_i & -b_i \\ b_i & a_i \end{bmatrix}$$

where $a_i = Re(c_i)$ and $b_i = Im(c_i)$.

Problem 15 This is a constant times a rotation matrix. Which one? (Hint: use the definition of cosine.)

Behavioral Eigen-Analysis

But consider those complex pair-blocks

$$\text{Block}_i = \begin{bmatrix} c_i & 0 \\ 0 & \bar{c}_i \end{bmatrix}.$$

In fact, they correspond to real matrices. Namely: Block_i is the same up to change of basis as

$$\begin{bmatrix} a_i & -b_i \\ b_i & a_i \end{bmatrix}$$

where $a_i = Re(c_i)$ and $b_i = Im(c_i)$.

Problem 15 This is a constant times a rotation matrix. Which one? (Hint: use the definition of cosine.)

Answer: rotation angle is $\theta_i = cos^{-1}(a_i/\sqrt{a_i^2 + b_i^2})$

Behavioral Eigen-Analysis

Thus, we have a TOTAL behavioral understanding of diagonalizable real matrices:

Behavioral Eigen-Analysis

Thus, we have a TOTAL behavioral understanding of diagonalizable real matrices:

Theorem 2 *All such matrices can be written as* SDS^{-1} *where* D *has diagonal elements corresponding to real-eigenvalue dilations or 2x2 blocks corresponding to complex-eigenvalue rotation-dilations.*

Behavioral Eigen-Analysis

Thus, we have a TOTAL behavioral understanding of diagonalizable real matrices:

Theorem 2 *All such matrices can be written as* SDS^{-1} *where* D *has diagonal elements corresponding to real-eigenvalue dilations or 2x2 blocks corresponding to complex-eigenvalue rotation-dilations.*

Moreover: the rotation rate and stretch multiple are controlled by the eigenvalues.

Behavioral Eigen-Analysis

Thus, we have a TOTAL behavioral understanding of diagonalizable real matrices:

Theorem 2 *All such matrices can be written as SDS^{-1} where D has diagonal elements corresponding to real-eigenvalue dilations or 2x2 blocks corresponding to complex-eigenvalue rotation-dilations.*

Moreover: the rotation rate and stretch multiple are controlled by the eigenvalues.
But what if the matrix is *not* diagonalizable? i.e

Behavioral Eigen-Analysis

Thus, we have a TOTAL behavioral understanding of diagonalizable real matrices:

Theorem 2 *All such matrices can be written as SDS^{-1} where D has diagonal elements corresponding to real-eigenvalue dilations or 2x2 blocks corresponding to complex-eigenvalue rotation-dilations.*

Moreover: the rotation rate and stretch multiple are controlled by the eigenvalues.
But what if the matrix is *not* diagonalizable? i.e

What if we can't find n linearly independent eigenvectors?

Nilpotency

We started with one property of scalars that matrices didn't share (commutativity).

Nilpotency

We started with one property of scalars that matrices didn't share (commutativity). Here's another:

Nilpotency

We started with one property of scalars that matrices didn't share (commutativity). Here's another:

For all real (or complex) numbers a, if $a^k = 0 \;\Rightarrow\; a = 0$.

Nilpotency

We started with one property of scalars that matrices didn't share (commutativity). Here's another:

For all real (or complex) numbers a, if $a^k = 0 \Rightarrow a = 0$.

Problem 16 Find a non-zero 2x2 matrix A such that $A^2 = 0$.

Nilpotency

We started with one property of scalars that matrices didn't share (commutativity). Here's another:

For all real (or complex) numbers a, if $a^k = 0 \;\Rightarrow\; a = 0$.

Problem 16 Find a non-zero 2x2 matrix A such that $A^2 = 0$.

Answer: the standard answer is

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Nilpotency

We started with one property of scalars that matrices didn't share (commutativity). Here's another:

For all real (or complex) numbers a, if $a^k = 0 \Rightarrow a = 0$.

Problem 16 Find a non-zero 2x2 matrix A such that $A^2 = 0$.

Answer: the standard answer is

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Problem 17 Compute the eigenvalues and eigenvalues of this A.

Nilpotency

We started with one property of scalars that matrices didn't share (commutativity). Here's another:

For all real (or complex) numbers a, if $a^k = 0 \Rightarrow a = 0$.

Problem 16 Find a non-zero 2x2 matrix A such that $A^2 = 0$.

Answer: the standard answer is

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Problem 17 Compute the eigenvalues and eigenvalues of this A.

Answer: Trick question. A has no non-trivial eigenvectors.

Nilpotency

A matrix is said to be *nilpotent* if $A^k = 0$ for some k.

Nilpotency

A matrix is said to be *nilpotent* if $A^k = 0$ for some k.

Problem 18 Find an n-by-n matrix such that $A^{n-1} = 0$ but $A^i \neq 0$ for $i < n - 1$ (A is the said to be "nilpotent of order $n - 1$").

Nilpotency

A matrix is said to be *nilpotent* if $A^k = 0$ for some k.

Problem 18 Find an n-by-n matrix such that $A^{n-1} = 0$ but $A^i \neq 0$ for $i < n-1$ (A is the said to be "nilpotent of order $n-1$").

Answer: Standard answer is

$$N_n = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & & & & & \vdots \\ 0 & \dots & & & 0 & 1 \\ 0 & \dots & & & & 0 \end{bmatrix}.$$

Nilpotency

A matrix is said to be *nilpotent* if $A^k = 0$ for some k.

Problem 18 Find an n-by-n matrix such that $A^{n-1} = 0$ but $A^i \neq 0$ for $i < n-1$ (A is the said to be “nilpotent of order $n-1$”).

Answer: Standard answer is

$$N_n = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & & & & & \vdots \\ 0 & \dots & & & 0 & 1 \\ 0 & \dots & & & & 0 \end{bmatrix}.$$

This matrix has *no* non-trivial eigenvectors.

Nilpotency

A matrix is said to be *nilpotent* if $A^k = 0$ for some k.

Problem 18 Find an n-by-n matrix such that $A^{n-1} = 0$ but $A^i \neq 0$ for $i < n-1$ (A is the said to be "nilpotent of order $n-1$").

Answer: Standard answer is

$$N_n = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & & & & & \vdots \\ 0 & \dots & & & 0 & 1 \\ 0 & \dots & & & & 0 \end{bmatrix}.$$

This matrix has *no* non-trivial eigenvectors. It's behavior is like a step-by-step "collapser":

Nilpotency

A matrix is said to be *nilpotent* if $A^k = 0$ for some k.

Problem 18 Find an n-by-n matrix such that $A^{n-1} = 0$ but $A^i \neq 0$ for $i < n-1$ (A is the said to be "nilpotent of order $n-1$").

Answer: Standard answer is

$$N_n = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & & & & & \vdots \\ 0 & \dots & & & 0 & 1 \\ 0 & \dots & & & & 0 \end{bmatrix}.$$

This matrix has *no* non-trivial eigenvectors. It's behavior is like a step-by-step "collapser":

$$e_n \to e_{n-1} \to \dots \to e_1 \to 0.$$

Nilpotency

Problem 19 Show that any nilpotent matrix has no non-trivial eigenvectors.

Nilpotency

Problem 19 Show that any nilpotent matrix has no non-trivial eigenvectors.

Reason:

Nilpotency

Problem 19 Show that any nilpotent matrix has no non-trivial eigenvectors.

Reason:

- Suppose $Nx = \lambda x$ and N is nilpotent.

Nilpotency

Problem 19 Show that any nilpotent matrix has no non-trivial eigenvectors.

Reason:

- Suppose $Nx = \lambda x$ and N is nilpotent.
- Then $N^l x = \lambda^l x$ for all l.

Nilpotency

Problem 19 Show that any nilpotent matrix has no non-trivial eigenvectors.

Reason:

- Suppose $Nx = \lambda x$ and N is nilpotent.
- Then $N^l x = \lambda^l x$ for all l.
- But $\lambda^l \neq 0$ for all l,

Nilpotency

Problem 19 Show that any nilpotent matrix has no non-trivial eigenvectors.

Reason:

- Suppose $Nx = \lambda x$ and N is nilpotent.
- Then $N^l x = \lambda^l x$ for all l.
- But $\lambda^l \neq 0$ for all l,
- which conflicts with $N^k = 0$ for some k.

Nilpotency

Problem 19 Show that any nilpotent matrix has no non-trivial eigenvectors.

Reason:

- Suppose $Nx = \lambda x$ and N is nilpotent.
- Then $N^l x = \lambda^l x$ for all l.
- But $\lambda^l \neq 0$ for all l,
- which conflicts with $N^k = 0$ for some k.

So we might suspect: nilpotent matrices fill in the "hole" left by the non-diagonalizable parts of arbitrary matrix.

Nilpotency

Problem 19 Show that any nilpotent matrix has no non-trivial eigenvectors.

Reason:

- Suppose $Nx = \lambda x$ and N is nilpotent.
- Then $N^l x = \lambda^l x$ for all l.
- But $\lambda^l \neq 0$ for all l,
- which conflicts with $N^k = 0$ for some k.

So we might suspect: nilpotent matrices fill in the “hole” left by the non-diagonalizable parts of arbitrary matrix.

The lack of eigenvectors of nilpotent matrices could make up for the missing dimensions in a non-diagonal matrix.

Nilpotency

Theorem 3 *All nilpotent matrices can be put into the standard form – that is all zeros, except 1s on the "super-diagonal".*

Nilpotency

Theorem 3 *All nilpotent matrices can be put into the standard form – that is all zeros, except 1s on the “super-diagonal”.*

That is, if an n-by-n matrix A is nilpotent of order $n-1$, then there is an invertible matrix S such that

Nilpotency

Theorem 3 *All nilpotent matrices can be put into the standard form – that is all zeros, except 1s on the "super-diagonal".*

That is, if an n-by-n matrix A is nilpotent of order $n-1$, then there is an invertible matrix S such that

$$A = S \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & & & & & \vdots \\ 0 & \dots & & & 0 & 1 \\ 0 & \dots & & & & 0 \end{bmatrix} S^{-1} = S N_n S^{-1}.$$

Nilpotency

Theorem 3 *All nilpotent matrices can be put into the standard form – that is all zeros, except 1s on the "super-diagonal".*

That is, if an n-by-n matrix A is nilpotent of order $n-1$, then there is an invertible matrix S such that

$$A = S \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & & & & & \vdots \\ 0 & \dots & & & 0 & 1 \\ 0 & \dots & & & & 0 \end{bmatrix} S^{-1} = SN_nS^{-1}.$$

Lower-order nilpotency $\Rightarrow$ some smaller N_i blocks.

Nilpotency

Theorem 3 *All nilpotent matrices can be put into the standard form – that is all zeros, except 1s on the "super-diagonal".*

That is, if an n-by-n matrix A is nilpotent of order $n-1$, then there is an invertible matrix S such that

$$A = S \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & & & & & \vdots \\ 0 & \dots & & & 0 & 1 \\ 0 & \dots & & & & 0 \end{bmatrix} S^{-1} = S N_n S^{-1}.$$

Lower-order nilpotency $\Rightarrow$ some smaller N_i blocks.

To fill in the "missing eigenvector" gap, let's add diagonal matrices to nilpontent matrices.

A Simple Form

Let λ be any complex number.

A Simple Form

Let λ be any complex number. Now, let $J_n^\lambda = \lambda I_n + N_n$.

A Simple Form

Let λ be any complex number. Now, let $J_n^\lambda = \lambda I_n + N_n$. This is the sum of the simplest diagonal and nilpotent matrices.

A Simple Form

Let λ be any complex number. Now, let $J_n^\lambda = \lambda I_n + N_n$. This is the sum of the simplest diagonal and nilpotent matrices.

$$J_n^\lambda = \begin{bmatrix} \lambda & 1 & 0 & 0 & \dots & 0 \\ 0 & \lambda & 1 & 0 & \dots & 0 \\ \vdots & & & & & \vdots \\ 0 & \dots & & & \lambda & 1 \\ 0 & \dots & & & & \lambda \end{bmatrix}.$$

A Simple Form

Let λ be any complex number. Now, let $J_n^\lambda = \lambda I_n + N_n$. This is the sum of the simplest diagonal and nilpotent matrices.

$$J_n^\lambda = \begin{bmatrix} \lambda & 1 & 0 & 0 & \dots & 0 \\ 0 & \lambda & 1 & 0 & \dots & 0 \\ \vdots & & & & & \vdots \\ 0 & \dots & & & \lambda & 1 \\ 0 & \dots & & & & \lambda \end{bmatrix}.$$

Problem 20 What are the eigenvalues/vectors of J_n^λ?

A Simple Form

Let λ be any complex number. Now, let $J_n^\lambda = \lambda I_n + N_n$. This is the sum of the simplest diagonal and nilpotent matrices.

$$J_n^\lambda = \begin{bmatrix} \lambda & 1 & 0 & 0 & \dots & 0 \\ 0 & \lambda & 1 & 0 & \dots & 0 \\ \vdots & & & & & \vdots \\ 0 & \dots & & & \lambda & 1 \\ 0 & \dots & & & & \lambda \end{bmatrix}.$$

Problem 20 What are the eigenvalues/vectors of J_n^λ?

Answer: $[1\ 0\ \dots\ 0]$ is the only eigenvector, with eigenvalue λ.

A Simple Form

Ok, $[1\ 0\ \ldots\ 0]$ was the only eigenvector of J_n^λ, so it's not diagonalizable, etc...

A Simple Form

Ok, $[1\ 0\ \ \ldots\ \ 0]$ was the only eigenvector of J_n^λ, so it's not diagonalizable, etc... but,

Problem 21 Compute

$$(J_n^\lambda - \lambda I_n)^{n-1}.$$

Fast.

A Simple Form

Ok, $[1\ 0\ \ldots\ 0]$ was the only eigenvector of J_n^λ, so it's not diagonalizable, etc... but,

Problem 21 Compute

$$(J_n^\lambda - \lambda I_n)^{n-1}.$$

Fast.

Answer: 0, b/c the stuff inside the parentheses is just N_n, the nilpotent matrix.

A Simple Form

Ok, $[1\ 0\ \ldots\ 0]$ was the only eigenvector of J_n^λ, so it's not diagonalizable, etc... but,

Problem 21 Compute

$$(J_n^\lambda - \lambda I_n)^{n-1}.$$

Fast.

Answer: 0, b/c the stuff inside the parentheses is just N_n, the nilpotent matrix.

Hence, all vectors are “generalized" eigenvectors;

A Simple Form

Ok, $[1\ 0\ \ldots\ 0]$ was the only eigenvector of J_n^λ, so it's not diagonalizable, etc... but,

Problem 21 Compute

$$(J_n^\lambda - \lambda I_n)^{n-1}.$$

Fast.

Answer: 0, b/c the stuff inside the parentheses is just N_n, the nilpotent matrix.

Hence, all vectors are "generalized" eigenvectors; x is a *generalized eigenvector* of A with *generalized eigenvalue* λ if there is a k such that $(A - \lambda I_n)^k(x) = 0$.

A Simple Form

Ok, $[1\ 0\ \ldots\ 0]$ was the only eigenvector of J_n^λ, so it's not diagonalizable, etc... but,

Problem 21 Compute

$$(J_n^\lambda - \lambda I_n)^{n-1}.$$

Fast.

Answer: 0, b/c the stuff inside the parentheses is just N_n, the nilpotent matrix.

Hence, all vectors are "generalized" eigenvectors; x is a *generalized eigenvector* of A with *generalized eigenvalue* λ if there is a k such that $(A - \lambda I_n)^k(x) = 0$.

(.... it a power of k where there was 1 in the original definition)

A Simple Form

Let's investigate the behavior of J_n^λ.

A Simple Form

Let's investigate the behavior of J_n^λ.

Problem 22 Compute

$$(J_3^\lambda)^l \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}^l \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

A Simple Form

Let's investigate the behavior of J_n^λ.

Problem 22 Compute

$$(J_3^\lambda)^l \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}^l \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Answer:

$$\begin{bmatrix} l\lambda^{l-1} \\ \lambda^l \\ 0 \end{bmatrix}.$$

A Simple Form

Let's investigate the behavior of J_n^λ.

Problem 22 Compute

$$(J_3^\lambda)^l \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}^l \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Answer:

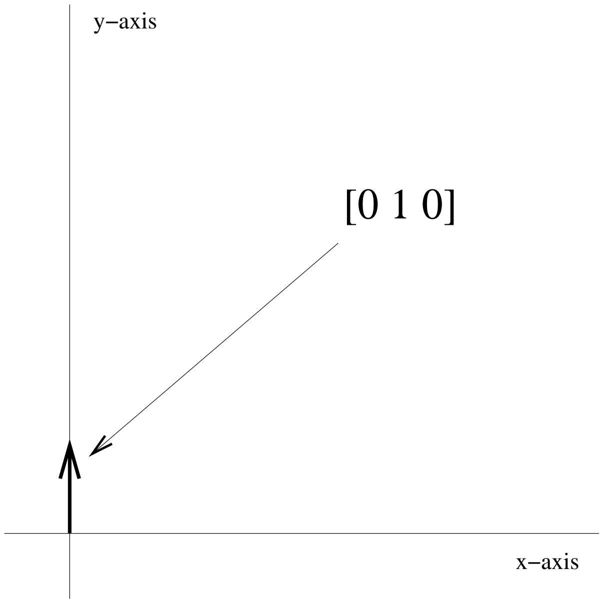
$$\begin{bmatrix} l\lambda^{l-1} \\ \lambda^l \\ 0 \end{bmatrix}.$$

(Aside: doesn't it remind you of derivatives?)

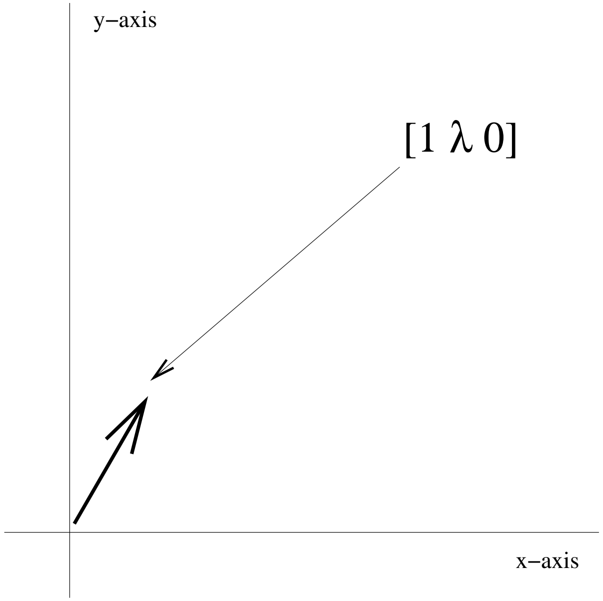
A Simple Form

Pictorially:

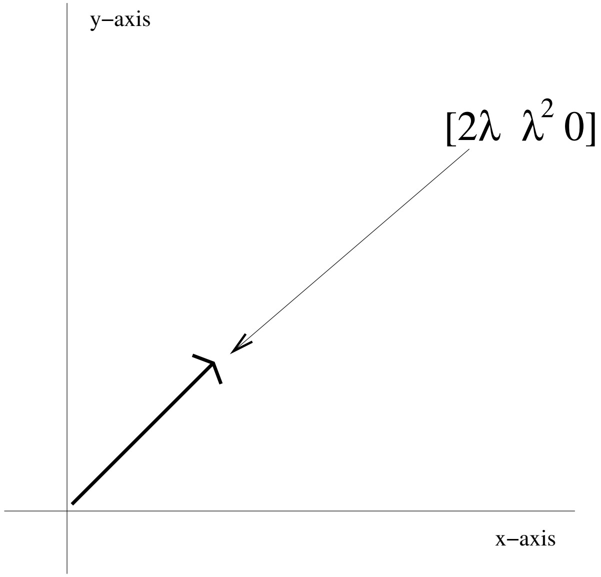
A Simple Form



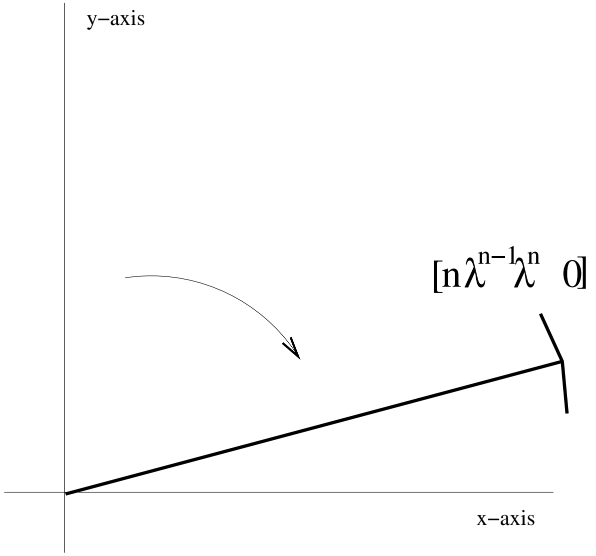
A Simple Form



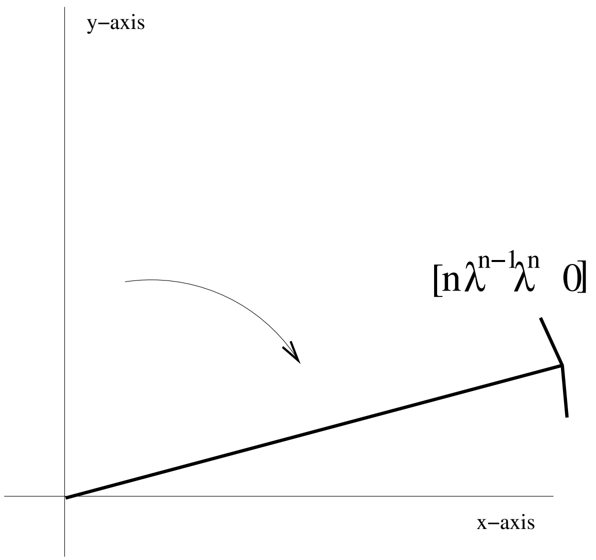
A Simple Form



A Simple Form

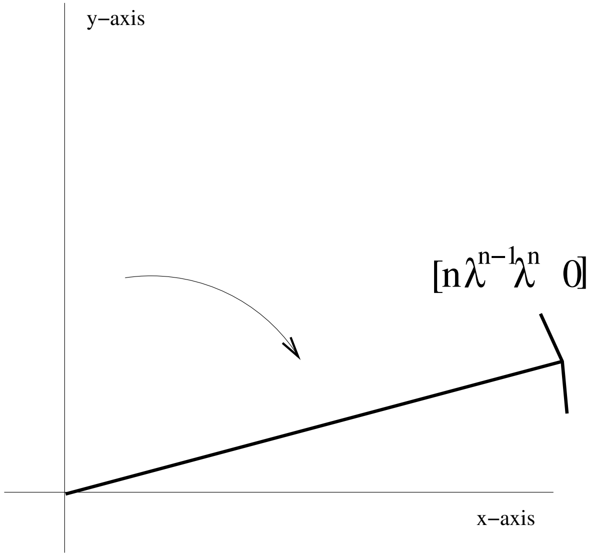


A Simple Form



Problem 23 What angle does this make with the x-axis, as $l \to \infty$?

A Simple Form



Problem 23 What angle does this make with the x-axis, as $l \to \infty$?

Answer:

$$\lim_{l \to \infty} cos^{-1}\left(\frac{n\lambda^{n-1}}{\sqrt{(n\lambda^{n-1})^2 + (\lambda^n)^2}} = \frac{n}{\sqrt{n^2+\lambda^2}}\right) = cos^{-1}(1) = 0.$$

A Simple Form

Now, if you compute

A Simple Form

Now, if you compute

$$(J_3^\lambda)^l \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}^l \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

A Simple Form

Now, if you compute

$$(J_3^\lambda)^l \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}^l \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

You get

$$\begin{bmatrix} (l(l-1)/2)\lambda^{l-2} \\ l\lambda^{l-1} \\ \lambda^l \end{bmatrix} \propto \begin{bmatrix} 1 \\ O(\frac{\lambda}{n}) \\ O(\frac{\lambda^2}{n^2}) \end{bmatrix}.$$

A Simple Form

Now, if you compute

$$(J_3^\lambda)^l \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}^l \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

You get

$$\begin{bmatrix} (l(l-1)/2)\lambda^{l-2} \\ l\lambda^{l-1} \\ \lambda^l \end{bmatrix} \propto \begin{bmatrix} 1 \\ O(\frac{\lambda}{n}) \\ O(\frac{\lambda^2}{n^2}) \end{bmatrix}.$$

Just as above, as $l \to \infty$, the angle with x-axis moves toward zero.

A Simple Form

Now, if you compute

$$(J_3^\lambda)^l \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}^l \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

You get

$$\begin{bmatrix} (l(l-1)/2)\lambda^{l-2} \\ l\lambda^{l-1} \\ \lambda^l \end{bmatrix} \propto \begin{bmatrix} 1 \\ O(\frac{\lambda}{n}) \\ O(\frac{\lambda^2}{n^2}) \end{bmatrix}.$$

Just as above, as $l \to \infty$, the angle with x-axis moves toward zero. Conclusion: J_n^λ “pushes" all the generalized eigenvectors down (asymptotically) to a true eigenvector.

Putting it All Together

The most celebrated and powerful result of linear algebra is now in reach.

Putting it All Together

The most celebrated and powerful result of linear algebra is now in reach.

Theorem 4 *(The Jordan Normal Form) Given any matrix A, there is an invertible matrix S such that $A = SDS^{-1}$, where*

$$D = \begin{bmatrix} J_{n_1}^{\lambda_1} & 0 & 0 & \dots & 0 \\ 0 & J_{n_2}^{\lambda_2} & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & \dots & & 0 & J_{n_k}^{\lambda_k} \end{bmatrix}.$$

Putting it All Together

The most celebrated and powerful result of linear algebra is now in reach.

Theorem 4 *(The Jordan Normal Form) Given any matrix A, there is an invertible matrix S such that $A = SDS^{-1}$, where*

$$D = \begin{bmatrix} J_{n_1}^{\lambda_1} & 0 & 0 & \dots & 0 \\ 0 & J_{n_2}^{\lambda_2} & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & \dots & & 0 & J_{n_k}^{\lambda_k} \end{bmatrix}.$$

I.e.: in the right basis, all matrices are of block-diagonal form, where the blocks are sums of constant and standard nilpotent matrices.

Putting it All Together

The most celebrated and powerful result of linear algebra is now in reach.

Theorem 4 *(The Jordan Normal Form) Given any matrix A, there is an invertible matrix S such that $A = SDS^{-1}$, where*

$$D = \begin{bmatrix} J_{n_1}^{\lambda_1} & 0 & 0 & \dots & 0 \\ 0 & J_{n_2}^{\lambda_2} & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & \dots & & 0 & J_{n_k}^{\lambda_k} \end{bmatrix}.$$

I.e.: in the right basis, all matrices are of block-diagonal form, where the blocks are sums of constant and standard nilpotent matrices.
D is the as-diagonalized-as-possible version of A.

Putting it All Together

$$\begin{bmatrix} J_{n_1}^{\lambda_1} & 0 & 0 & \dots & 0 \\ 0 & J_{n_2}^{\lambda_2} & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & \dots & & 0 & J_{n_k}^{\lambda_k} \end{bmatrix}$$

Putting it All Together

$$\begin{bmatrix} J_{n_1}^{\lambda_1} & 0 & 0 & \dots & 0 \\ 0 & J_{n_2}^{\lambda_2} & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & \dots & & 0 & J_{n_k}^{\lambda_k} \end{bmatrix}$$

If A is actually diagonalizable, $n_i =$ block size $= 1$, λ_i are actual eigenvalues; rows with same eigenvalues collect into eigenspaces.

Putting it All Together

$$\begin{bmatrix} J_{n_1}^{\lambda_1} & 0 & 0 & \dots & 0 \\ 0 & J_{n_2}^{\lambda_2} & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & \dots & & 0 & J_{n_k}^{\lambda_k} \end{bmatrix}$$

If A is actually diagonalizable, $n_i =$ block size $= 1$, λ_i are actual eigenvalues; rows with same eigenvalues collect into eigenspaces.

In general, λ_i are generalized eigenvalues and like-valued blocks collect into *generalized* eigenspaces.

Putting it All Together

$$\begin{bmatrix} J_{n_1}^{\lambda_1} & 0 & 0 & \dots & 0 \\ 0 & J_{n_2}^{\lambda_2} & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & \dots & & 0 & J_{n_k}^{\lambda_k} \end{bmatrix}$$

If A is actually diagonalizable, $n_i =$ block size $= 1$, λ_i are actual eigenvalues; rows with same eigenvalues collect into eigenspaces.

In general, λ_i are generalized eigenvalues and like-valued blocks collect into *generalized* eigenspaces.

Restated,

Putting it All Together

$$\begin{bmatrix} J_{n_1}^{\lambda_1} & 0 & 0 & \dots & 0 \\ 0 & J_{n_2}^{\lambda_2} & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & \dots & & 0 & J_{n_k}^{\lambda_k} \end{bmatrix}$$

If A is actually diagonalizable, $n_i =$ block size $= 1$, λ_i are actual eigenvalues; rows with same eigenvalues collect into eigenspaces.

In general, λ_i are generalized eigenvalues and like-valued blocks collect into *generalized* eigenspaces.

Restated, all vectors in a given J-block correspond to a single true eigenvector with eigenvalue λ

Putting it All Together

$$\begin{bmatrix} J_{n_1}^{\lambda_1} & 0 & 0 & \dots & 0 \\ 0 & J_{n_2}^{\lambda_2} & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & \dots & & 0 & J_{n_k}^{\lambda_k} \end{bmatrix}$$

If A is actually diagonalizable, $n_i =$ block size $= 1$, λ_i are actual eigenvalues; rows with same eigenvalues collect into eigenspaces.

In general, λ_i are generalized eigenvalues and like-valued blocks collect into *generalized* eigenspaces.

Restated, all vectors in a given J-block correspond to a single true eigenvector with eigenvalue λ and multiplicity n_i.

Putting it All Together

$$\begin{bmatrix} J_{n_1}^{\lambda_1} & 0 & 0 & \dots & 0 \\ 0 & J_{n_2}^{\lambda_2} & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & \dots & & 0 & J_{n_k}^{\lambda_k} \end{bmatrix}$$

If A is actually diagonalizable, $n_i =$ block size $= 1$, λ_i are actual eigenvalues; rows with same eigenvalues collect into eigenspaces.

In general, λ_i are generalized eigenvalues and like-valued blocks collect into *generalized* eigenspaces.

Restated, all vectors in a given J-block correspond to a single true eigenvector with eigenvalue λ and multiplicity n_i.

Problem 24 In the above terms, how many true eigenvectors does D have?

Putting it All Together

$$\begin{bmatrix} J_{n_1}^{\lambda_1} & 0 & 0 & \dots & 0 \\ 0 & J_{n_2}^{\lambda_2} & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & \dots & & 0 & J_{n_k}^{\lambda_k} \end{bmatrix}$$

If A is actually diagonalizable, $n_i =$ block size $= 1$, λ_i are actual eigenvalues; rows with same eigenvalues collect into eigenspaces.

In general, λ_i are generalized eigenvalues and like-valued blocks collect into *generalized* eigenspaces.

Restated, all vectors in a given J-block correspond to a single true eigenvector with eigenvalue λ and multiplicity n_i.

Problem 24 In the above terms, how many true eigenvectors does D have?

Answer: $k =$ number of blocks.

The Final Eigen-Analysis

$$D = \begin{bmatrix} J_{n_1}^{\lambda_1} & 0 & 0 & \dots & 0 \\ 0 & J_{n_2}^{\lambda_2} & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & \dots & & 0 & J_{n_k}^{\lambda_k} \end{bmatrix}.$$

The Final Eigen-Analysis

$$D = \begin{bmatrix} J_{n_1}^{\lambda_1} & 0 & 0 & \dots & 0 \\ 0 & J_{n_2}^{\lambda_2} & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & \dots & & 0 & J_{n_k}^{\lambda_k} \end{bmatrix}.$$

The final behavioral analysis:

The Final Eigen-Analysis

$$D = \begin{bmatrix} J_{n_1}^{\lambda_1} & 0 & 0 & \dots & 0 \\ 0 & J_{n_2}^{\lambda_2} & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & \dots & & 0 & J_{n_k}^{\lambda_k} \end{bmatrix}.$$

The final behavioral analysis:

- Generalized eigenvectors get pushed “up" their J-block, asymptotically collapsing to a corresponding true eigenvector.

The Final Eigen-Analysis

$$D = \begin{bmatrix} J_{n_1}^{\lambda_1} & 0 & 0 & \dots & 0 \\ 0 & J_{n_2}^{\lambda_2} & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & \dots & & 0 & J_{n_k}^{\lambda_k} \end{bmatrix}.$$

The final behavioral analysis:

- Generalized eigenvectors get pushed “up" their J-block, asymptotically collapsing to a corresponding true eigenvector.
- Eigenvectors of real eigenvalues get stretched.

The Final Eigen-Analysis

$$D = \begin{bmatrix} J_{n_1}^{\lambda_1} & 0 & 0 & \dots & 0 \\ 0 & J_{n_2}^{\lambda_2} & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & \dots & & 0 & J_{n_k}^{\lambda_k} \end{bmatrix}.$$

The final behavioral analysis:

- Generalized eigenvectors get pushed “up" their J-block, asymptotically collapsing to a corresponding true eigenvector.
- Eigenvectors of real eigenvalues get stretched.
- Eigenvectors of complex eigenvalues get rotated and stretched.

The Final Eigen-Analysis

$$D = \begin{bmatrix} J_{n_1}^{\lambda_1} & 0 & 0 & \dots & 0 \\ 0 & J_{n_2}^{\lambda_2} & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & \dots & & 0 & J_{n_k}^{\lambda_k} \end{bmatrix}.$$

The final behavioral analysis:

- Generalized eigenvectors get pushed “up" their J-block, asymptotically collapsing to a corresponding true eigenvector.
- Eigenvectors of real eigenvalues get stretched.
- Eigenvectors of complex eigenvalues get rotated and stretched.
- Collapse, dilation, and rotation rates controlled by eigenvalues.

The Final Eigen-Analysis

$$D = \begin{bmatrix} J_{n_1}^{\lambda_1} & 0 & 0 & \dots & 0 \\ 0 & J_{n_2}^{\lambda_2} & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & \dots & & 0 & J_{n_k}^{\lambda_k} \end{bmatrix}.$$

The final behavioral analysis:

- Generalized eigenvectors get pushed “up" their J-block, asymptotically collapsing to a corresponding true eigenvector.
- Eigenvectors of real eigenvalues get stretched.
- Eigenvectors of complex eigenvalues get rotated and stretched.
- Collapse, dilation, and rotation rates controlled by eigenvalues.

These are the *only possible behaviors of a linear system*.