Generalizing from SU(2) to SU(4) QHFM

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1 Standard SU(2) QHFM

Assume that we are working with a 2D free electron gas (FEG) which experiences an out of plane, static, spatially uniform magnetic field. We allow the electrons to have internal degrees of freedom. If this internal DOF is true, physical spin, then we know the quantum state of any single electron is described by the combination of spatial ϕ and spin χ wavefunctions:

$$|\psi\rangle = |\phi\rangle \otimes |\chi\rangle \tag{1.1}$$

I make the identification for basis vectors:

$$|\uparrow\rangle \equiv \begin{bmatrix}1\\0\end{bmatrix} \quad |\downarrow\rangle \equiv \begin{bmatrix}0\\1\end{bmatrix}, \tag{1.2}$$

so that the spinor wavefunction lives in a two-dimensional complex vector space $\mathbb{C}^2$, or more accurately, after you account for normalization, the one-dimensional complex projective space $\mathbb{CP}^1 \equiv \mathrm{Gr}(1, \mathbb{C}^2)$, which has 2 real dimensions.

$$|\chi\rangle = a \begin{bmatrix}1\\0\end{bmatrix} + b \begin{bmatrix}0\\1\end{bmatrix}. \tag{1.3}$$

Imagine that the 2D FEG of area A is threaded by N magnetic flux quantum, i.e. $BA = \Phi = N\Phi_0$, where $\Phi_0 = h/e$. Then you know that the single-particle eigenstates $|\psi\rangle$ have energies which are split up into Landau levels. The LLs are indexed by $n = 0, 1, 2, ...$ in increasing energy. Within each Landau level, there are N degenerate *spatial wavefunctions* ϕ, which you can label by a "Landau orbital index," lets call that k. Since electrons have internal DOF $|\chi\rangle$, each single Landau orbital can accommodate multiple electrons. For spin-1/2, each single Landau orbital can accommodate 2 electrons, and no more. So the single-particle eigenstates might look like

$$|\psi(k,\uparrow)\rangle = |\phi_k\rangle \otimes |\uparrow\rangle, \quad |\psi(k,\downarrow)\rangle = |\phi_k\rangle \otimes |\downarrow\rangle. \tag{1.4}$$

Let's focus on the lowest Landau level (LLL), i.e. the $n = 0$ LL, with energy $E = h\omega_c/2$. Because of the 2x spin degeneracy, the LLL can at most contain $N_e = 2N$ electrons. The $2N + 1$-th electron must go fill the $n = 1$ LL. We know from experiments that SU(2) QHFM arises when $N_e = N$, i.e. when Landau level filling factor $\nu = N_e/N = 1$, i.e. when the LLL is half filled. SU(2) QHFM here means that the many-body wavefunction of N_e electrons, instead of populating N out of the $2N$ possible single-particle states "equally" (left), or some other manner (middle), prefers to spread out in "Landau orbital index" and concentrate in "spin index" (right) – see Fig. 1.

What is the mechanism for this behavior? Lets imagine a toy system with very small magnetic field $N = 2$, and $N_e = 2$ electrons. There are two Landau orbitals, k_1 and $k_2 = -k_1$, and two spin labels, $\uparrow$ and $\downarrow$. There are a total of 4 single particle states, and we will occupy 2 out of 4 states. We know this sort of problem has eigenstates with total spin $S = 1$ (symmetric/triplet) of the form [Note I'm being sloppy and not forming full Slater determinants]

$$|\psi_{a,b}\rangle = |\psi(k,\uparrow)\rangle_a \otimes |\psi(k',\uparrow)\rangle_b \tag{1.5}$$

$$|\psi_{a,b}\rangle = |\psi(k,\downarrow)\rangle_a \otimes |\psi(k',\downarrow)\rangle_b \tag{1.6}$$

$$|\psi_{a,b}\rangle = |\psi(k,\downarrow)\rangle_a \otimes |\psi(k',\uparrow)\rangle_b + |\psi(k,\uparrow)\rangle_a \otimes |\psi(k',\downarrow)\rangle_b \tag{1.7}$$

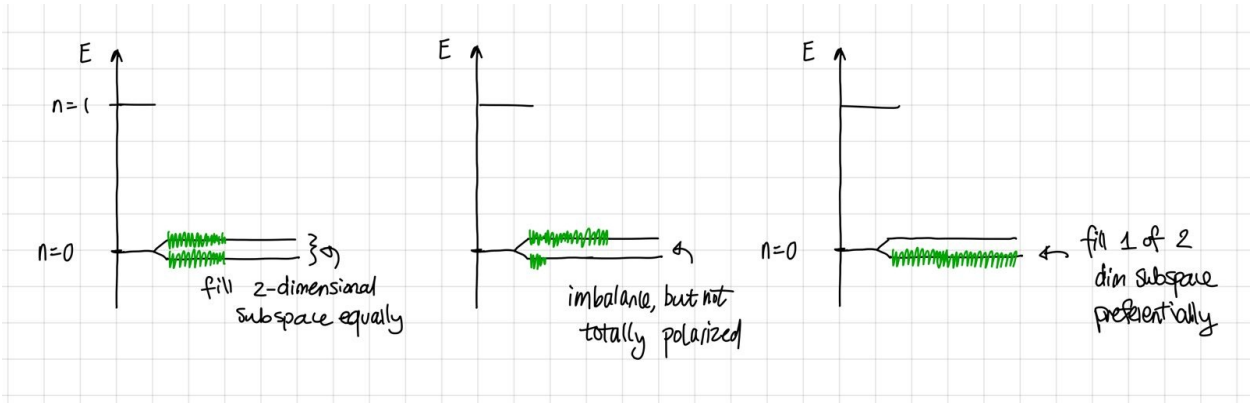

Figure 1: Different possible configurations for half-filled, SU(2) LLL

and total spin $S = 0$ (antisymmetric/singlet) of the form

$$|\psi_{a,b}\rangle = |\psi(k,\downarrow)\rangle_a \otimes |\psi(k',\uparrow)\rangle_b - |\psi(k,\uparrow)\rangle_a \otimes |\psi(k',\downarrow)\rangle_b, \tag{1.8}$$

where I label particles by a, b and Landau orbitals by $k, k' \in \{k_1, k_2\}$, $k' \neq k$. You may have seen this partition of the two-spin state space by the shorthand $2 \otimes 2 = 3 \oplus 1$.
From exchange interaction / Hund's coupling we know that Eqs. (1.5) to (1.7), is lower in energy than Eq. (1.8), because the spatial wavefunctions are antisymmetric. So the spin-polarized states are favored over the spin-unpolarized states.
The case of general N flux and N electrons follow completely analogously.
Perturbation chooses a ground state: When we only consider the SU(2)-symmetric Coulomb interaction, and no other symmetry-breaking perturbations, we are allowed to choose a symmetric configuration with any S_z, as they are all degenerate in energy. A Zeeman interaction $H' = \mu_B S_z B$ will break this symmetry and choose one S_z eigenstate as the ground state.

2 SU(4) QHFM in MLG

Let's now consider monolayer graphene (MLG) (with some 2D tight binding band structure) which experiences an out of plane, static, spatially uniform magnetic field. Now, the appropriate internal DOF is a combination of valley and physical spin indices, which we may combine to view as a SU(4) "pseudospin". The quantum state of any single electron is described by the combination of spatial ϕ and pseudospin χ wavefunctions:

$$|\psi\rangle = |\phi\rangle \otimes |\chi\rangle \tag{2.1}$$

I make the identification for pseudospin basis vectors:

$$(K,\uparrow) \equiv \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \equiv |s_1\rangle \quad (K,\downarrow) \equiv \begin{bmatrix}0\\1\\0\\0\end{bmatrix} \equiv |s_2\rangle \quad (K',\uparrow) \equiv \begin{bmatrix}0\\0\\1\\0\end{bmatrix} \equiv |s_3\rangle \quad (K',\downarrow) \equiv \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \equiv |s_4\rangle, \tag{2.2}$$

so that the pseudospinor wavefunction lives in a four-dimensional complex vector space $\mathbb{C}^4$, or more accurately, after you account for normalization, the three-dimensional complex projective space $\mathbb{CP}^3 \equiv \mathrm{Gr}(1, \mathbb{C}^4)$, which has 6 real dimensions:

$$|\chi\rangle = a \begin{bmatrix}1\\0\\0\\0\end{bmatrix} + b \begin{bmatrix}0\\1\\0\\0\end{bmatrix} + c \begin{bmatrix}0\\0\\1\\0\end{bmatrix} + d \begin{bmatrix}0\\0\\0\\1\end{bmatrix}. \tag{2.3}$$

As before, the single-particle eigenstates $|\psi\rangle$ have energies which are split up into Landau levels. The LLs are indexed by $n = 0, \pm 1, \pm 2, ...$ in increasing difference from charge neutrality point, $E = 0$. Within each

Landau level, there are N degenerate *spatial wavefunctions* ϕ, which you can label by a "Landau orbital index," lets call that k. Since electrons have internal DOF $|\chi\rangle$, each single Landau orbital can accommodate multiple electrons. For MLG, each single Landau orbital can accommodate 4 electrons, and no more. So the single-particle eigenstates might look like

$$|\psi(k,s_1)\rangle = |\phi_k\rangle \otimes |s_1\rangle, \quad |\psi(k,s_3)\rangle = |\phi_k\rangle \otimes |s_3\rangle. \tag{2.4}$$

Lets focus on the zeroth Landau level (ZLL), i.e. the $n = 0$ LL, with energy $E = 0$. Because of the 4x pseudospin degeneracy, the ZLL can at most contain $N_e = 4N$ electrons. We know from experiments that SU(4) QHFM arises when $N_e = N$, $N_e = 2N$, and $N_e = 3N$, i.e. when Landau level filling factor $\nu = N_e/N = 1, 2, 3$, i.e. when the ZLL is quarter/half/three-quarters-filled. Note: what I call $\nu = 1, 2, 3$, in graphene literature is usually called $\nu = -1, 0, 1$, so that's what I'll adopt from now on.
The different filling factors gives rise to SU(4) QHFM of different properties. First we describe $\nu = \pm 1$, then we desribe $\nu = 0$.

2.1 $\nu = \pm 1$

SU(4) QHFM here means that the many-body wavefunction of N electrons/holes, instead of populating N out of the $4N$ possible single-particle states "equally" (left), or some other manner (middle), prefers to spread out in "Landau orbital index" and concentrate in "pseudospin index" (right) – see Fig. 2.

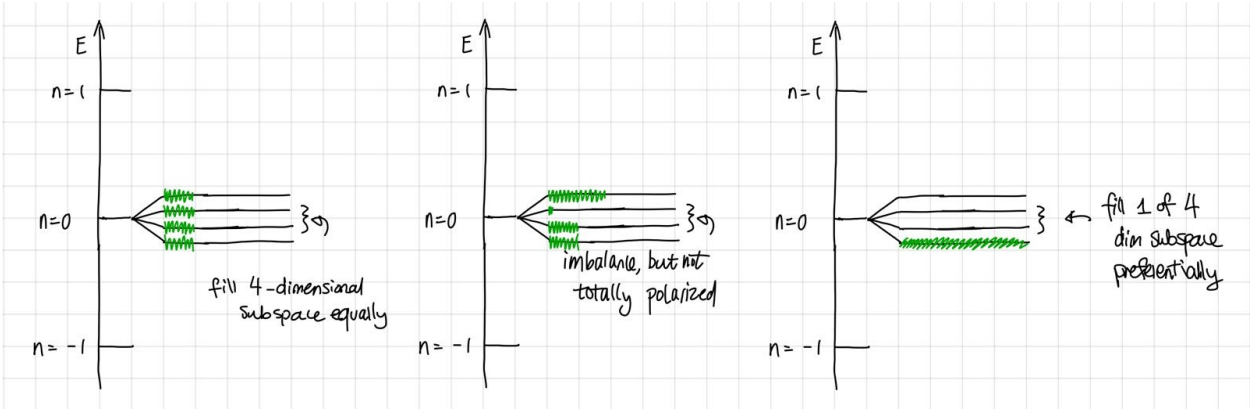

Figure 2: Different possible electron/hole configurations for quarter/three-quarters filled, SU(4) ZLL

What is the mechanism for this behavior? Lets imagine a toy system with very small magnetic field $N = 2$, and $N_e = 2$ electrons. There are two Landau orbitals, k_1 and $k_2 = -k_1$, and four pseudospin labels, $s_1, ..., s_4$. There are a total of 8 single particle states, and we will occupy 2 out of 8 states. We know this sort of problem has eigenstates with symmetric pseudospin wavefunctions of the form [Note I'm being sloppy and not forming full Slater determinants]

$$|\psi_{a,b}\rangle = |\psi(k,s)\rangle_a \otimes |\psi(k',s)\rangle_b \tag{2.5}$$

$$|\psi_{a,b}\rangle = |\psi(k,s)\rangle_a \otimes |\psi(k',s')\rangle_b + |\psi(k,s')\rangle_a \otimes |\psi(k',s)\rangle_b \tag{2.6}$$

and eigenstates with antisymmetric pseudospin wavefunctions of the form

$$|\psi_{a,b}\rangle = |\psi(k,s)\rangle_a \otimes |\psi(k',s')\rangle_b - |\psi(k,s')\rangle_a \otimes |\psi(k',s)\rangle_b, \tag{2.7}$$

where I label particles by a, b, Landau orbitals by $k \neq k' \in \{k_1, k_2\}$, and pseudospin by $s \neq s' \in \{s_1, s_2, s_3, s_4\}$. The appropriate short hand for the partition of the two-spin state space is $4 \otimes 4 = 10 \oplus 6$, standing for decomposing a tensor product into symmetric $\mathcal{S}$ and antisymmetric $\mathcal{A}$ parts.
From exchange interaction / Hund's coupling, we know that we should organize a symmetric configuration for the pseudospin wavefunction, and an antisymmetric configuration for the spatial wavefunction. So the pseudospin-polarized states are favored over the psudospin-unpolarized states.

The case of general N flux and N electrons/holes follow completely analogously.
Perturbation chooses a ground state: When we only consider the SU(4)-symmetric Coulomb interaction, and no other symmetry-breaking perturbations, we are allowed to choose any s and s' to contribute to the symmetric configuration.

2.2 $\nu = 0$

SU(4) QHFM here means that the many-body wavefunction of $2N$ electrons/holes, instead of populating $2N$ out of the $4N$ possible single-particle states "equally" (left), or some other manner (middle), prefers to spread out in "Landau orbital index" and concentrate in "pseudospin index" (right) – see Fig. 3.

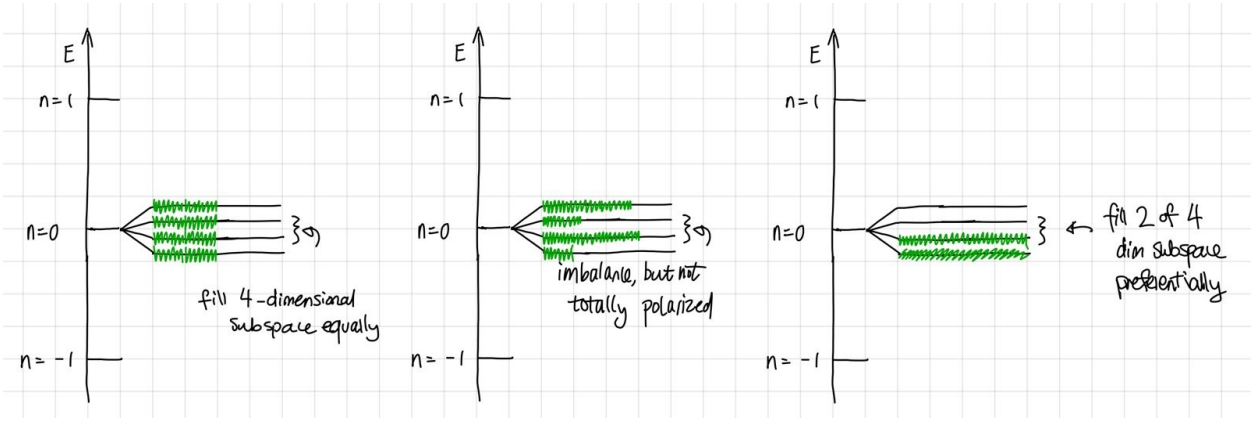

Figure 3: Different possible electron/hole configurations for half-filled, SU(4) ZLL

What is the mechanism for this behavior?
First, lets figure out the internal DOFs per Landau orbital. Each Landau orbital accomodates 2 electrons on average, so its convenient to consider two electrons together as one object, let's call it a "two-particle spinor." I make the identification for two-particle spinor basis vectors:

$$|1100\rangle \equiv |K\rangle|K\rangle \otimes (|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle) \equiv |a_1\rangle \tag{2.8}$$
$$|1010\rangle \equiv (|K\rangle|K'\rangle - |K'\rangle|K\rangle) \otimes |\uparrow\rangle|\uparrow\rangle \equiv |a_2\rangle \tag{2.9}$$
$$|1001\rangle \equiv |K\rangle|K'\rangle \otimes |\uparrow\rangle|\downarrow\rangle - |K'\rangle|K\rangle \otimes |\downarrow\rangle|\uparrow\rangle \equiv |a_3\rangle \tag{2.10}$$
$$|0110\rangle \equiv |K\rangle|K'\rangle \otimes |\downarrow\rangle|\uparrow\rangle - |K'\rangle|K\rangle \otimes |\uparrow\rangle|\downarrow\rangle \equiv |a_4\rangle \tag{2.11}$$
$$|0101\rangle \equiv (|K\rangle|K'\rangle - |K'\rangle|K\rangle) \otimes |\downarrow\rangle|\downarrow\rangle \equiv |a_5\rangle \tag{2.12}$$
$$|0011\rangle \equiv |K'\rangle|K'\rangle \otimes (|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle) \equiv |a_6\rangle \tag{2.13}$$

so that the two-particle spinor wavefunction lives in six-dimensional complex vector space $\mathcal{A}(\mathbb{C}^4 \otimes \mathbb{C}^4)$, or more accurately, after you account for normalization and phase, the Grassmanian manifold $\mathrm{Gr}(2, \mathbb{C}^4)$, which has 4 complex dimensions, or 8 real dimensions.
Then, lets imagine a toy system with very small magnetic field $N = 2$, and $N_e = 4$ electrons. There are two Landau orbitals, k_1 and $k_2 = -k_1$, and six two-particle spinor labels. There are a total of 12 two-particle states, and we will occupy 2 out of these 12 states (for a total of 4 particles).
From exchange interaction / Hund's coupling, we should organize a symmetric configuration for the two-particle spinor wavefunction, and an antisymmetric configuration for the spatial wavefunction. So the two-particle spinor-polarized states are favored over the two-particle spinor-unpolarized states.
The case of general N flux and $2N$ electrons/holes follow completely analogously.
The above argument assumes that we can consider electron pairs, rather than individual electrons. We didn't consider, for example, the scenario one Landau orbital holds 1 electron, and the other orbital holds 3 electrons. ♣ **We need an argument to show that such an arrangement can't occur, or would be higher in energy than the "pair occupation" scenario.** Intuitively, I suppose there are no reason that

electrons would favor one LL orbital over another, in the absence of translation symmetry breaking. So the pair occupation scenario is sensible. But this seems unsatisfying...

Perturbation chooses a ground state: When we only consider the SU(4)-symmetric Coulomb interaction, and no other symmetry-breaking perturbations, we are allowed to choose any two-particle spinor wavefunctions to contribute to the symmetric configuration. For more detailed analysis of possible symmetry broken states due to SU(4) anisotropic interactions, see [Kha12, JP19, ALG21]. These existing works on $\nu = 0$ QHFM categorize two-particle spinor-polarized states using what the flavor wavefunction looks like:

- Eqs. (2.8) and (2.13) are examples of **valley polarized, spin unpolarized** states, which is a family of states that include charge-density-wave (CDW) and Kekulé-distortion (KD) order. KD is also sometimes called Kekulé-bond (KB) or intervalley-coherent (IVC)[1].
- Eqs. (2.9) and (2.12) are examples of **valley unpolarized, spin polarized** states, which include ferromagnetic (F) order.
- Eqs. (2.10) and (2.11) are examples of spin-valley intertwined states which are (confusingly) called antiferromagnetic (AF) order. This state is an antiferromagnet both in spin space and in valley space.

References

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[1] Not sure this usage is common, I only see it in [LFC+22]