Stats243 Introduction to Mathematical Finance

Haipeng Xing
Department of Statistics
Stanford University
Summer 2006
Agenda

- Administrative, course description & reference, syllabus, course agenda
- Financial Products, markets and derivatives
- Expectation and arbitrage
Administrative

**Meeting time**
Monday, Wednesday, Friday 11:00am – 12:15pm
June 27, 2007 --- July 31, 2007

**Classroom**
McCullough 115

**Instructor**
Haipeng Xing ([xing@stanford.edu](mailto:xing@stanford.edu))
Office: Sequoia Hall, Room 137
Office hours: Wednesday 9:30am -- 10:45am
or by appointment

**T.A.s**
George Chang ([gtchang@stanford.edu](mailto:gtchang@stanford.edu))
Office: Sequoia Hall, Room ???
Office hours: ???
Grade Policy

1. HW will be due in class. For each assignment, 5% of the course grade will be deducted for each late day.

2. You should finish each assignment yourself, group discussion is NOT allowed.

3. Take-home final will be handed out on the class of July 31, 2007.
Topics

- Binomial tree model
- Financial derivatives, hedging and risk management
- Introduction to Ito calculus and SDE.
- Stochastic models of financial markets
- Black-Scholes pricing formula of European options
- Optimal stopping and American options
- Interest rate and discounted value
Roadmap

Financial derivatives

Pricing & hedging

Forwards, futures, options, interest rate products…

Discrete processes

Binomial models

Continuous processes

Stochastic models

Binomial representation theorem

Ito calculus & SDE

Black-Scholes models

American options

Martingale representation theorem

Interest rate models
Reference


Others:


Prerequisite: Math53, Stats116 or their equivalents
Financial Products, Markets and Derivatives

- Examples
- Financial products
  - Underlying
  - Derivatives
  - Fixed-income securities
Financial Products --- Underlying

• Equities

  – Stock or other security, which represent ownership of any asset (e.g., a company).

  – Generally, the prices of stocks are random (unpredictable). However, we can model stock prices in a probabilistic sense.

  – The holder of the stock receives dividend periodically (a portion of a company’s earnings).
Examples

- Bloomberg: IBM stock on January 12, 2006

<table>
<thead>
<tr>
<th>Stock Data</th>
<th>USD</th>
<th>Earnings</th>
<th>USD</th>
</tr>
</thead>
<tbody>
<tr>
<td>Price</td>
<td>83.84</td>
<td>Dividends</td>
<td>Quarterly</td>
</tr>
<tr>
<td>52wk High</td>
<td>96.20</td>
<td>Indicated Gross Yld</td>
<td>.95%</td>
</tr>
<tr>
<td>52wk Low</td>
<td>71.85</td>
<td>Dividend Growth</td>
<td>5YR</td>
</tr>
<tr>
<td>YTD change</td>
<td>1.64</td>
<td>Ex-Date</td>
<td>11/8/05</td>
</tr>
<tr>
<td>YTD % Change</td>
<td>2.00%</td>
<td>Type</td>
<td>Reg. Cash</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>2FA</th>
<th>Shares out</th>
<th>1579.517M</th>
<th>Earnings</th>
<th>USD</th>
<th>132426.7M</th>
</tr>
</thead>
<tbody>
<tr>
<td>Market Cap</td>
<td>USD</td>
<td>1/17/06</td>
<td>Trailing 12mo EPS</td>
<td>4.910</td>
<td></td>
</tr>
<tr>
<td>Float</td>
<td>1578.78M</td>
<td>Short Int</td>
<td>9.765M</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3TR A</td>
<td>Yr Total Return</td>
<td>11.11%</td>
<td>Est EPS</td>
<td>12/2005</td>
<td>5.158</td>
</tr>
<tr>
<td>BETA vs. SPX</td>
<td>1.05</td>
<td>P/E</td>
<td>17.08</td>
<td>LT Growth</td>
<td>10.75</td>
</tr>
<tr>
<td>4DOMON</td>
<td>Options, LEAPs, Stk Marginable</td>
<td>Est P/E</td>
<td>16.25</td>
<td>Est PEG</td>
<td>1.51</td>
</tr>
</tbody>
</table>
Financial Products --- Underlying

• **Commodities**
  – Raw products such as oil and metal that are often done on the futures market.
  – The prices of these products are unpredictable but often show seasonal effects.

• **Currencies**
  – One currency is exchanged for another (Foreign exchange, FX).
  – Some currencies are pegged to one another, and others are allowed to float freely.

• **Indices**
  – A typical index is made up from the weighted sum of a selection or basket of representative stocks.
  – Examples: Standard & Poor’s 500 (S&P500), Financial Times Stock Exchange index (FTSE100).
Financial Products --- Derivatives

• Basic derivatives (options)
  – Options give the holder the right (not the obligation) to trade in the future at a specified price (strike price).
  – A call (put) option is the right to buy (sell) an asset for an agreed amount at a specified time in the future.
  – The value of the option at expiry is a function of the underlying asset (payoff function). Let S be the stock price and E the strike, the payoff function is:
    • \( \text{Max}(S-E, 0) \) for a call option
    • \( \text{Max}(E-S, 0) \) for a put option
Financial Products --- Derivatives

- Payoff diagram for an option
Examples

- Prices of call options on IBM stocks ($84.17) at January 11, 2006

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Last</th>
<th>Change</th>
<th>Bid</th>
<th>Ask</th>
<th>Volume</th>
<th>Open Int</th>
<th>Strike Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>IBMBK.X</td>
<td>28.50</td>
<td>0.00</td>
<td>29.20</td>
<td>29.40</td>
<td>56</td>
<td>56</td>
<td>55.00</td>
</tr>
<tr>
<td>IBMBL.X</td>
<td>23.70</td>
<td>0.00</td>
<td>24.20</td>
<td>24.40</td>
<td>20</td>
<td>20</td>
<td>60.00</td>
</tr>
<tr>
<td>IBMBM.X</td>
<td>19.90</td>
<td>↑1.00</td>
<td>19.30</td>
<td>19.40</td>
<td>43</td>
<td>183</td>
<td>65.00</td>
</tr>
<tr>
<td>IBMBN.X</td>
<td>14.40</td>
<td>0.00</td>
<td>14.30</td>
<td>14.50</td>
<td>5</td>
<td>203</td>
<td>70.00</td>
</tr>
<tr>
<td>IBMO.Q.X</td>
<td>9.20</td>
<td>↑0.20</td>
<td>9.40</td>
<td>9.60</td>
<td>19</td>
<td>385</td>
<td>75.00</td>
</tr>
<tr>
<td>IBMBP.X</td>
<td>4.90</td>
<td>↓0.16</td>
<td>5.00</td>
<td>5.20</td>
<td>366</td>
<td>3,427</td>
<td>80.00</td>
</tr>
<tr>
<td>IBMBQ.X</td>
<td>1.90</td>
<td>0.00</td>
<td>1.90</td>
<td>1.95</td>
<td>4,831</td>
<td>14,447</td>
<td>85.00</td>
</tr>
<tr>
<td>IBMBR.X</td>
<td>0.50</td>
<td>↓0.05</td>
<td>0.45</td>
<td>0.50</td>
<td>1,346</td>
<td>8,690</td>
<td>90.00</td>
</tr>
</tbody>
</table>

http://finance.yahoo.com
Examples

- Bloomberg: options on IBM stock on January 12, 2006

<table>
<thead>
<tr>
<th>Option</th>
<th>Symbol</th>
<th>Last Chng</th>
<th>Vol</th>
<th>Option</th>
<th>Symbol</th>
<th>Last Chng</th>
<th>Vol</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Jan 06 80 Puts</td>
<td>MP</td>
<td>.40 unch</td>
<td>11 751</td>
<td>17 Feb 06 75 Puts</td>
<td>NO</td>
<td>.18 +.03</td>
<td>152</td>
</tr>
<tr>
<td>2 Jan 06 65 Calls</td>
<td>AQ</td>
<td>1.05 -.20</td>
<td>4 190</td>
<td>10 Feb 06 75 Calls</td>
<td>BO</td>
<td>9.00 -.20</td>
<td>150</td>
</tr>
<tr>
<td>3 Jan 06 80 Calls</td>
<td>AP</td>
<td>4.30 -.10</td>
<td>1 605</td>
<td>19 Jan 06 90 Puts</td>
<td>MR</td>
<td>6.30 -.02</td>
<td>149</td>
</tr>
<tr>
<td>4 Feb 06 65 Calls</td>
<td>BQ</td>
<td>1.70 -.20</td>
<td>1 204</td>
<td>20 Jan 06 65 Calls</td>
<td>AM</td>
<td>10.00 -.20</td>
<td>112</td>
</tr>
<tr>
<td>5 Jan 06 85 Puts</td>
<td>MQ</td>
<td>2.35 +.25</td>
<td>1 123</td>
<td>21 Feb 06 80 Calls</td>
<td>BP</td>
<td>4.70 -.20</td>
<td>104</td>
</tr>
<tr>
<td>6 Jan 06 90 Calls</td>
<td>AR</td>
<td>.15 unch</td>
<td>7 54</td>
<td>22 Apr 06 95 Calls</td>
<td>DS</td>
<td>.50 unch</td>
<td>98</td>
</tr>
<tr>
<td>7 Feb 06 90 Calls</td>
<td>BR</td>
<td>4.40 -.10</td>
<td>3 73</td>
<td>23 Jul 06 90 Calls</td>
<td>GR</td>
<td>2.65 +.05</td>
<td>97</td>
</tr>
<tr>
<td>8 Jan 06 75 Calls</td>
<td>AQ</td>
<td>8.80 -.40</td>
<td>3 50</td>
<td>24 Jan 06 75 Puts</td>
<td>MD</td>
<td>.05 unch</td>
<td>95</td>
</tr>
<tr>
<td>9 Feb 06 85 Puts</td>
<td>NQ</td>
<td>2.85 +.20</td>
<td>2 52</td>
<td>25 Feb 06 80 Puts</td>
<td>NP</td>
<td>.80 -.05</td>
<td>82</td>
</tr>
<tr>
<td>10 Jan 07 95 Calls</td>
<td>AS</td>
<td>3.40 unch</td>
<td>2 50</td>
<td>26 Jan 07 70 Calls</td>
<td>AN</td>
<td>13.50 -1.00</td>
<td>70</td>
</tr>
<tr>
<td>11 Apr 06 85 Puts</td>
<td>PQ</td>
<td>3.70 +.10</td>
<td>2 41</td>
<td>27 Apr 06 90 Calls</td>
<td>DR</td>
<td>1.45 -.10</td>
<td>69</td>
</tr>
<tr>
<td>12 Apr 06 80 Puts</td>
<td>PP</td>
<td>1.70 +.05</td>
<td>2 39</td>
<td>28 Apr 06 60 Calls</td>
<td>DL</td>
<td>24.20 -.30</td>
<td>62</td>
</tr>
<tr>
<td>13 Apr 06 85 Calls</td>
<td>DQ</td>
<td>3.20 -.17</td>
<td>2 35</td>
<td>29 Jul 06 90 Puts</td>
<td>SR</td>
<td>7.60 -.20</td>
<td>60</td>
</tr>
<tr>
<td>14 Feb 06 90 Puts</td>
<td>NR</td>
<td>6.60 +.10</td>
<td>2 27</td>
<td>30 Jul 06 95 Calls</td>
<td>GS</td>
<td>1.35 +.05</td>
<td>55</td>
</tr>
<tr>
<td>15 Jul 06 85 Calls</td>
<td>GQ</td>
<td>4.60 -.20</td>
<td>2 27</td>
<td>31 Feb 06 60 Calls</td>
<td>BL</td>
<td>24.10 +.40</td>
<td>50</td>
</tr>
<tr>
<td>16 Jul 06 75 Puts</td>
<td>SD</td>
<td>1.30 +.15</td>
<td>2 00</td>
<td>32 Feb 06 115 Puts</td>
<td>NC</td>
<td>31.10 -2.10</td>
<td>50</td>
</tr>
</tbody>
</table>

Australia 61 2 977 9600 Brazil 5511 3049 4500 Europe 44 20 7330 7500 Germany 49 69 920410 Hong Kong 852 2977 6000 Japan 01 3 3201 9900 Singapore 65 6212 1000 U.S. 1 212 319 2000 Copyright 2006 Bloomberg L.P.
Examples

• Bloomberg: details of a call option on IBM stock on January 12, 2006

---

IBM:Q 1 C85.00 $ I 1.05 -.2 I
DELAY 14:01 Vol 4,198 Op 1.25 x Hi 1.25 x Lo 1 x Prev 1.25 OpInt 63,339

---

Equity Options Description

- Ticker: IBM US 1 C85
- Strike: 85
- Type: American
- Expiration: 1/21/06

---

General Notes

- IBZ: 50 & Below
- IBM: 55 - 150.00
- IBM: 155.00 & Above
Examples

- Bloomberg: standard OV of an option on IBM stock on January 12, 2006

<table>
<thead>
<tr>
<th>Standard Option Valuation</th>
<th>Page 1/2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Price of IBM US Equity</td>
<td>93.60</td>
</tr>
<tr>
<td>Strike:</td>
<td>85.00</td>
</tr>
<tr>
<td>Exercise Type:</td>
<td>American</td>
</tr>
<tr>
<td>Put or Call:</td>
<td>Call</td>
</tr>
<tr>
<td>Time to Expiration:</td>
<td>9 01:45</td>
</tr>
<tr>
<td>Trade:</td>
<td>1/12/06 14:17</td>
</tr>
<tr>
<td>Expiration:</td>
<td>1/21/06 16:02</td>
</tr>
<tr>
<td>Settle Date:</td>
<td>1/12/06</td>
</tr>
<tr>
<td>Exercise Delay:</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Option Valuation and Risk Parameters</th>
<th>Dividends</th>
<th>Continuous</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>Percent</td>
<td>Time Value</td>
</tr>
<tr>
<td>Price</td>
<td>1.05</td>
<td>1.255%</td>
</tr>
<tr>
<td>Volatility</td>
<td>29.98%</td>
<td>Premium:</td>
</tr>
<tr>
<td>Delta</td>
<td>0.30784</td>
<td>Parity:</td>
</tr>
<tr>
<td>Gamma</td>
<td>0.09682</td>
<td>Gearing:</td>
</tr>
<tr>
<td>Vega</td>
<td>0.05062</td>
<td>Rho:</td>
</tr>
</tbody>
</table>

Stats243, Xing, Summer 2007
Financial Products

• Forwards

  – A forward contract is an agreement where one party promises to buy an asset from another party at some specified time in the future and at some specified price.
  – No money changes hands until the delivery date (maturity) of the contract.
  – The amount that is paid for the asset at the maturity is called the delivery price.
  – The Terms of the contract make it an obligation to buy the asset at the maturity.
  – As the maturity is approached, the value of the forward contract will change from initially zero to the difference between the underlying asset and the delivery price at maturity.
Financial Products

• Futures

  – A futures contract is similar to a forward contract.

  – Both are an agreement where one party promises to buy an asset from another party at a specified time in the future and at a specified price.

  – Forward contract is traded in the over-the-counter (OTC) market and there is no standard size or delivery arrangements.

  – Futures contract is traded on an exchange. The contract size and delivery dates are standard.

  – Forward contract is settled at the end of its life, while Futures contract is settled daily (the profit or loss is calculated and paid every day.)

  – Forwards and futures have two main uses in speculation and in hedging.
Financial Products

- The time value of money – interest rate
  - Simple interest: the interest you received is only based on your principal.
    \[ 1 \times (1 + r) \]
  - Compound Interest: the interest you received is based on your principal and the interest you get.
    - Discretely compounded rate
      \[ 1 \times \left(1 + \frac{r}{m}\right)^m \]
    - Continuously compounded rate
      \[ 1 \times e^{rt} \]
Financial Products

• Fixed-income securities
  – Bonds (zero coupon, coupon-bearing, floating rate, …)
  – Forward rate agreement (FRA) is an agreement between two parties that a prescribed interest rate will apply to a prescribed principal over a specified time in the future.
  – A repo is a repurchase agreement to sell some security to another party and buy it back at a fixed date and for a fixed amount. The difference between the price at which the security is bought back and the selling price is the interest rate called the repo rate. The most common repo is the overnight repo.
  – Both FRA and repo are used to lock in future interest rates.
  – Interest rate derivatives: interest rate swap, swaption, caps and floors, …
Examples

• Chart of 10-year treasury notes
Expectation and Arbitrage

- Expectation pricing
- Arbitrage pricing
Expectation Pricing

- What you are going to pay for a game that someone tosses a coin and you are paid $1 for heads and nothing for tail?
  - The expected payoff in the game is \( 0.5 \times 1 + 0.5 \times 0 = 0.5 \).
  - **Kolmogorov’s strong law of large numbers**
    A sequence of independent random numbers \( X_1, X_2, X_3, \ldots \) are sampled from the same distribution with mean \( \mu \). Then the arithmetical average of the sequence
    \[
    S_n = \sum_{i=1}^{n} \frac{X_i}{n} \longrightarrow \mu, \quad \text{as} \quad n \rightarrow \infty, \quad \text{w.p.} \quad 1.
    \]
  - The fair price of the game is $0.5.
Expectation Pricing

• **Stock model**
  – It is widely accepted that stock prices are log-normally distributed.
    \[ X = \frac{\log(S_T)}{\log(S_0)}, \quad S_T = S_0 e^X, \quad X \sim N(\mu, \sigma^2). \]

• **What is the forward price K of a forward contract?** -- Let’s try the expectation pricing

  – The value of the contract at the expiry T is \( S_T - K \).
  – The current value is \( \exp(-rT)(S_T - K) \).
  – The expected current value is \( E(\exp(-rT)(S_T - K)) \).
  – As the initial value is 0 for a forward contract, \( E(\exp(-rT)(S_T - K)) \) should be 0.
  – \( K = E(S_T) = E(S_0 \exp(X)) \)
Expectation Pricing

• What is $E(S_0e^{X})$?

$$E(S_0e^{X}) = S_0 \int_{-\infty}^{\infty} e^x f(x) \, dx = S_0 e^{\mu + \frac{1}{2} \sigma^2}$$

where

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

• This gives us the wrong answer. Why?
  – Another mechanism determines the price of a forward contract.
  – The existence of an arbitrage price overrides the strong law.
Arbitrage Pricing

• Arbitrageurs
  – Hedgers: reduce their risks with trade. For example, long a stock and a put option on it.
  – Speculators: bet the price will go up or go down.
  – Arbitrageurs: Lock in a riskless profit by simultaneously entering into two or more transactions.

• Generally, the model often assume that there are NO arbitrage opportunities.
Arbitrage Pricing

- Arbitrage pricing for the forward contract
  
  - Consider that if we are the seller of the forward contract. We could borrow $S_0$ now, buy the stock. At time $T$, we will pay back the loan $S_0 \exp(rT)$, and deliver the stock. Therefore, the forward price $K$ is at least $S_0 \exp(rT)$ for the seller.

  - Consider that if we are the buyer of the contract. We could also use the same scheme to have the stock at time $T$. Therefore, the buyer won’t pay more than $S_0 \exp(rT)$.

  - The forward price $K$ has to be $S_0 \exp(rT)$. 
Arbitrage Pricing

- Arbitrage pricing for the forward contract

  - In other words, the arbitrageur can take advantages if the forward price $K$ is not $S_0 \exp(rT)$.

  - If $K > S_0 \exp(rT)$, the arbitrageurs can buy the asset and short forward contracts on the asset.

  - If $K < S_0 \exp(rT)$, the arbitrageurs can short the asset and long forward contracts on it.