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Measurement of Small Color Differences

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A theoretical problem in color vision is to represent lights so that the discriminability of similar colors may be easily calculated. There have been many attempts to derive representations for discriminability from basic principles of visual function. For the last 100 years these attempts have mainly focused on geometric representations where lights are treated as vectors and two pairs of lights are equally discriminable when their vector differences are equal. A general characterization of vector models emphasizes the empirical rules discriminability judgments must obey in order to be described adequately by vector differences. Experimental evidence violates these rules, indicating that such geometric representations cannot succeed. Statistical representations based on maximum-likelihood decision rules provide a class of representations that emphasize the variability of both the stimulus encoding and neural decision-making processes. In these representations variability is inherent to the representation and not derived from an external source. Properties of statistical and geometric models are compared with each other and with various psychological models of choice.

The new student of color vision begins with a false sense of security. The student learns that there are simple computational rules for assigning three-dimensional coordinates to lights. This scheme assigns equal coordinates to lights only when the lights appear identical (even though the lights may be physically different). Furthermore, the coordinates assigned to a light (*a* '+' *b*) formed by mixing together lights *a* and *b* is simply the sum of the coordinates assigned to light *a* plus the coordinates assigned to *b*.

Such a scheme for assigning vectors to lights characterizes those instances when different lights have precisely the same color appearance. This leads the initiate to imagine that these same coordinate values—or some close relative—can be used to calculate small differences in color appearance. This possibility is particularly attractive because an assignment of numbers to colors that permits differences in appearance to be computed would be the basis of a true science of psychological objects—a theory of measurement being the basis of science and color appearance being a product of mind not matter.

There have been many attempts to predict small color differences based on the conventional methods of assigning coordinates to lights. Here I summarize current efforts to assign numbers to lights in such a way that identity of appearance between pairs of lights (*a* appears identical to *a*′), and identity of discriminability among quadruples of lights (the discriminability of *a* and *a*′ is equal to the discriminability of *b* and *b*′), may be computed. The difficulties in making such an assignment are well known to color theorists (e.g., Wyszecki & Stiles, 1967), though less well known to psychologists. In the past few years some of the difficulties in making such an assignment have become better understood, and new ideas as to how to proceed have been suggested.

The organization of this paper is the following: The next section presents the foundations of color measurement—the color-matching experiment. Following that are sections containing a brief review of selected attempts at the measurement of small color differences (line-element models) based on

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a principle I call the adaptation hypothesis; a framework for describing these color metrics, using notation from linear algebra; a description of an empirical test of the adequacy of the framework; alternative theoretical frameworks and further examples; and finally, a discussion of issues that are of relevance to psychological decision theories of variables other than color appearance.

Fundamentals: Motivation and Color Matching

Consider the following problem: A manufacturer wishes to produce an object with a specific color appearance. He can specify the desired appearance of the product, and he can measure the actual appearance of his product. Because of noise inherent to the production process, the actual appearance will generally not be identical to the desired appearance. The manufacturer would like to know whether the actual color is sufficiently different from the desired color so that the product must be discarded or whether the colors are sufficiently similar so that the product may be safely sold without fear of consumer complaint. Products may be anything from textiles to television pictures.

From this description of the problem, it becomes evident why theories of small color differences have been of particular interest. When the desired outcome is red and the product is green, the manufacturer has no trouble in deciding what to do. It is only when color differences are small that the manufacturer seeks guidance from standards set by science and the government.

Because the need for industrial tolerances on this question is great, governments have funded international commissions to set standards concerning the amount of deviation permissible before an object can no longer be called by a patented color name. Much of the data on discriminability of small color differences was collected by these commissions. Curves have been calculated analytically that characterize these measurements, and based on these curves color definitions have been created that provide manufacturers with tolerable, though not excellent, guidelines. Following Wyszecki and Stiles (1967), I will call attempts at strictly computational solutions to the estimation of differences in color appearance *empirical color measurement.*

In addition to analytic solutions to color scaling, there have been many attempts to predict color differences from the principles of visual organization. The main concern in this case is not primarily with discovering better formulas for predicting the visibility of small color differences but rather with testing hypotheses concerning the organization of psychological color mechanisms. In these theories the derivation of a color metric for small differences represents a natural outcome of some theory of color vision. Again, following Wyszecki and Stiles (1967) I will call attempts at creating color measurements from the principles of visual organization *inductive color measurement.*

My comments here will be generally restricted to theories of small color differences that are derived from hypotheses about the structure of the visual mechanisms, that is, to the structure of color perception. References to work on analytic measurement will be included only insofar as inductive theories may be tested against data collected for the purpose of defining analytic solutions to the scaling of small color differences. For more complete reviews of all aspects of the measurement of small color differences see Wyszecki and Stiles (1967), Judd and Wyszecki (1963), LeGrand (1970), Bouman and Walraven (1972), Krantz (1972), Vos, Friele, and Walraven (1972), and Boynton (1980).

The Color-Matching Experiment

The color-matching experiment is the foundation of color measurement. Theories of small color-difference measurements must include the case of zero difference—that is, color matches—as a special case.

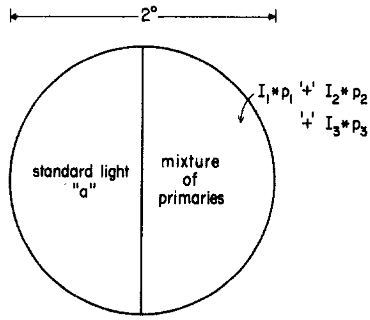
The stimuli in a typical color-matching experiment are shown in Figure 1. An observer examines a bipartite field, one half of which contains a fixed light and the other half of which contains a mixture of three lights whose intensities are under the observer's control. The entire field is confined to the central region of the observer's fovea. The observer's task is to render the appearance of the field completely uniform, that is, to match the two halves of the field in appearance. To do this the observer is free to adjust the intensity controls of the three lights, called the *color-matching primaries*, in any way whatsoever.

The importance of the color-matching experiment is that for any fixed light a normal observer can always obtain a match using the mixture of three primary lights. No fewer than three will suffice; three are sufficient. The only caveat is that for some fixed lights one or two of the primary lights may have to be added to the fixed light rather than to the variable half of the field.

Two further, and remarkable, properties of this experiment are the following: First, when a match has been obtained, multiplying the intensities of all lights on both sides of the bipartite field by equal amounts does not disturb the match. Second, adding the same amount of any light to both sides of the bipartite field does not disturb the match. These empirical observations are true to a high degree of precision over much of the visual range (see Trezona, 1953, 1954; Wyszecki & Stiles, 1967) and are generally referred to as Grassman's Laws of Color Matching (Grassman, 1853).

One direct consequence of these empirical properties is that colored lights may be represented as vectors. This fact has long been understood (Schrodinger, 1970, gives an especially clear account). An excellent recent statement of the vector representation theorems for lights from the color-matching experiment is given by Krantz (1975). The representation of colored lights may be stated quite simply in terms of the color-matching experiment itself. Any light, *a*, can be represented in a three-dimensional space where the coordinates of the space are the intensities of the three primary lights whose mixture matches the light, *a*. By convention we indicate a shift in the side of the bipartite field of the primary light by a change in sign of the intensity of the coordinate value assigned to that primary light. Thus, in effect, the settings on the three knobs of the matching primary lights are the coordinates that we assign to any arbitrary light, *a*.

A second consequence of the color-matching experiment and Grassman's laws is that



Color matching experimental arrangement

Figure 1. Spatial arrangement of a typical color matching experiment. (The subject adjusts the intensities of the primary lights on the right-hand side of the bipartite field in order to match the appearance of a standard light, *a*, on the opposite side of the field.)

when different sets of primary lights are used to perform matches the resulting vector representations will be related by a linear transformation. This, too, has been empirically demonstrated many times. It will be convenient to summarize these results in a single theorem.

COLOR MATCHING THEOREM. *Let* a *and* b *be two lights of arbitrary spectral distribution. We say that two lights,* a *and* a′, *are* isomeric *when they are physically identical and* metameric *when they are physically distinct but appear identical to the observer. We write* a is metameric to a′ *by the symbols* a ~ a′ *and we denote the physical superposition of two lights,* a *and* b, *as* a '+' b.

The premises of the color-matching theorem are (essentially) that any light, *a*, may be matched by the mixture of three lights and that Grassman's laws are correct. The conclusions of the color-matching theorem are as follows: We may represent any light, *a*, as a three-dimensional vector, $\mathbf{V}(a)$, such that

(1) $a \sim a'$ if and only if $\mathbf{V}(a) = \mathbf{V}(a')$.

(2) $\mathbf{V}(a \text{ '+' } b) = \mathbf{V}(a) + \mathbf{V}(b)$.

(3) Representations having properties (1) and (2) are unique up to a nonsingular (i.e., has an inverse matrix) linear transformation of the coordinate space.

Proofs of these conclusions are provided by Krantz (1975) and Schrodinger (1970).

The representation of lights embodied in this theorem has three central properties:

(a) Lights that are physically distinct, but perceptually identical, are mapped into the same coordinate value. (b) The physical act of mixing two lights is captured by the simple computational operation of adding vectors (see Cornsweet, 1970, for examples and a clear description of how these calculations may be carried out). And (c) all such representations are related by a linear transformation of the coordinate system.

The Extension to Color Differences

Based strictly on a consideration of the experimental procedure, the color-matching experiment can have little to say about differences in color appearance. In the color-matching experiment observers are asked only to make judgments concerning the identity of lights—they provide no information as to how different two lights a and b may appear when a and b do not match. From the color-matching experiment, therefore, it is difficult to make deductions about differences in color appearance (but see MacAdam, 1942, for an ingenious approach).

It is tempting, however, to try to make an identification between distances measured in the representation space of colors and differences in color appearance. The simplest idea is that the (Euclidean) length of the vector difference between two lights is a measure of the different appearance of the two lights. This idea is incorrect, but it will be helpful later if we specify why, precisely, this scheme cannot work.

Consider two lights, a and a '+' δ, that are just noticeably different from one another. Let $\mathbf{V}(a)$ be the vector associated with a and let $\mathbf{V}(a \text{ '+' } \delta) = \mathbf{V}(a) + \mathbf{V}(\delta)$ be the vector associated with a '+' δ. The vector difference between these two lights, relative to the representation $\mathbf{V}$, is $\mathbf{V}(\delta)$. Imagine we now increase the intensity of the light a at each wavelength by a factor I, and denote this increase as $I{*}a$. The vector difference between $I{*}a$ and $I{*}a$ '+' δ remains $\mathbf{V}(\delta)$. If the discriminability of two lights depended merely on the vector difference between those lights, $I{*}a$ '+' δ and $I{*}a$ should be just as discriminable as a '+' δ and a.

This implication can be tested and found to be false. As the intensity of light a is increased, the sensitivity of the visual system generally decreases (the Weber-Fechner relationship; but see Polden & Mollon, 1980). In the context of an increasingly intense light, $I{*}a$, the visual significance of a difference, δ, becomes vanishingly small. This rules out the possibility that vector differences from color-matching measurements may be used as a measure of color differences. The problem, then, is to devise a theory of color discrimination in the face of these objections.

Some Theories of Color Discrimination

The principle objection just raised against using vector differences of color-matching coordinates as a measure of perceived color differences is that as we vary the comparison lights the adapted state of the eye is altered. This suggests that if we could correct the metric for changes in the adapted state of the eye, we would be able to construct a color metric based on vector differences of the color-matching coordinates. I will call this possibility the *adaptation hypothesis.*

In the normal development of color representations, the adaptation hypothesis is rarely made explicit. It is more common to group color metrics by their computational formulas rather than by the principles of visual organization they represent. It will be convenient for the organization of this review, however, to group together all color metrics that assume the adaptation hypothesis. There are several inductive color metrics that are based on the adaptation hypothesis. I now review three of these.

Helmholtz's Color Metric

The dependence of an observer's sensitivity on the ambient illumination is roughly characterized by Weber's law. This law asserts that observer sensitivity is proportional to the energy in the background illumination. Helmholtz joined Weber's law of adaptation with the color-matching experiment in an attempt to devise a measure of color differences. His effort is a particularly clear example of an attempt to build an inductive color metric from principles of the visual system.

The central idea of the Helmholtz color metric is this: Helmholtz (1896) assumed that the observer has direct access to the responses of the three receptor types, ρ (red), γ (green), and β (blue). He further assumed that lights are discriminated by a comparison of the receptor signals of the three cone types generated by the two comparison lights, a and a '+' δ. Let the effect of light a on the three types of receptors be $\rho(a)$, $\gamma(a)$, and $\beta(a)$. For small δ we assume that the difference in response of each of these receptor classes between the lights a and a '+' δ is $\rho(\delta)$, $\gamma(\delta)$ and $\beta(\delta)$.

Helmholtz' color metric computed the distance, ds, between a and a '+'δ via the formula:

$$ds^2 = \left(\frac{\rho(\delta)}{\rho(a)}\right)^2 + \left(\frac{\gamma(\delta)}{\gamma(a)}\right)^2 + \left(\frac{\beta(\delta)}{\beta(a)}\right)^2.$$

The numerator of each of the terms corresponds to the difference in receptor response between lights a and a '+' δ. The denominator represents a correction for the loss of sensitivity of the receptor type due to the light a. This correction takes the form of Weber-law adaptation, where the response of the receptor type is reduced in magnitude by a factor corresponding to the quanta absorbed by that receptor type. The weighted response differences of the receptors are pooled via the Euclidean distance formula.

The principles of visual operation incorporated in such a model are quite simple and elegant. When the state of adaptation—caused by light a—is held fixed, distances between neighboring points in the color space are measured by a Euclidean distance function in the coordinate system defined by the receptor quantum catches. The effect of varying the observer's adapted state is merely to stretch the axes by amounts proportional to the quantum catch of the three distinct receptor types. This is the essence of the adaptation hypothesis: Fix the state of adaptation and the color metric is simple and Euclidean. Allow the state of adaptation to vary and we correct our distance estimates for the new state of adaptation.

Noise variance is not directly included in this representation. Certain kinds of noise—for example, independent additive noise that causes ds^2 to fluctuate—can be incorporated without changing the character of the theory. However, only noise whose distribution is independent of the different vector, $V(\delta)$, can be accommodated without significantly altering the theory.

Given knowledge of the absorption probabilities of the three receptor types, ρ, γ, and β, one can compute the discriminability of arbitrary lights. Conversely, to predict the discriminability of various lights, one must estimate these three absorption curves. Helmholtz estimated the spectral sensitivities of ρ, γ, and β required to predict the outcome of a wavelength-discrimination experiment performed by Konig and Dieterici (in Helmholtz, 1896). The derived spectral sensitivities are an implication of the model and may, therefore, serve to test it. The shapes of ρ, γ, and β that Helmholtz determined are now known to be inconsistent with the spectral absorptions of the photopigments of the receptors.

Because Helmholtz's intention was to construct a line element using the photopigment quantum absorptions as the fundamental entities, his theory of color differences cannot be entirely correct. Close relatives have been proposed, however, and I now consider some of these alternative color metrics.

Modifying the Adaptation Assumptions: The Stiles Color Metric

The Stiles (1946) color metric is a revision of the metric proposed by Helmholtz, and it too is consistent with the adaptation hypothesis. Stiles assumed that the photoreceptors are the limiting stage of chromatic discrimination, and he retained the spectral sensitivities of the three receptor types as the primitive axes of the color-metric space. He modified Helmholtz's color metric in two important ways.

First, Stiles did not accept Weber's law as a satisfactory approximation to the effect of changes in the ambient illumination (except at high intensities). Instead, he derived an empirical function, called $\zeta[\ ]$, to characterize the loss of sensitivities in the separate receptor classes as a function of their quantum absorptions. The function ζ was measured as part of a large empirical project to study the effects of ambient illumination on visual sensitivity (Stiles, 1939, 1978).

Second, Stiles assumed that the separate receptor classes are not weighted equally in their contribution toward chromatic discrimination. Again, based on his measurements of the effects of ambient illumination, Stiles assumed that the β receptors contribute little to chromatic discrimination. He made this assumption by introducing weighting coefficients for each of the terms in Helmholtz's color metric. The Stiles color metric takes on the form

$$ds^2 = \left(\frac{R\rho(\delta)}{\zeta[\rho(a)]}\right)^2 + \left(\frac{G\gamma(\delta)}{\zeta[\gamma(a)]}\right)^2 + \left(\frac{B\beta(\delta)}{\zeta[\beta(a)]}\right)^2,$$

where the R, G, and B terms are weighting constants with $B \ll R, G$. Stiles determined the size of the weighting constants by measuring the limiting value of the Weber fraction for detection mediated by each of the separate receptor classes.

A complete description of the calculations for predicting chromatic differences based on this model is provided by Wyszecki and Stiles (1967, p. 516 et seq.). They also explain why the model must be rejected. The metric has also been discussed from the point of view of statistical decision theory by Trabka (1968) and Buchsbaum and Goldstein (1979). I will return to evaluate this theory and its relatives after introducing one further type of color metric that is consistent with the adaptation hypothesis.

Modifying Assumptions About the Fundamentals

A different theoretical approach to improving the color metric is to deny the hypothesis that the observer can have direct knowledge of the output of the photoreceptors (Helmholtz's assumption) or that the photoreceptors are the limiting stage of chromatic discrimination (Stiles's assumption). Following the work of Hurvich and Jameson (1955, 1957), many authors have suggested that the limiting stage of chromatic discrimination is the neural level determined by an opponent transformation of the receptor responses. This kind of an inductive theory replaces the ρ, γ, and, β functions with the responses of opponent pathways. Examples of such theories include Friele (1961), Guth, Massof, & Benzschawel (1980), Hurvich and Jameson (1957), Ingling (1977), Koenderink, Grind, & Bouman (1972), and Vos and Walraven (1972a, 1972b). I will use the work of Guth and his colleagues as a clear example of this kind of color metric. The theoretical foundations for much of Guth's empirical analysis has been recently developed by Massof and Bird (1978a, 1978b) and Massof and Starr (1980).

Figure 2 (a) is a picture of the neural connections between the receptors and opponent-process neurons in the theory proposed by Guth et al. (1980). The receptor responses contribute to three neural mechanisms that the authors refer to as T, D, and A. The channels are similar to the conventional opponent channels of red–green, blue–yellow, and achromatic. The response of the neural mechanisms is characterized—for a fixed state of adaptation and small signals that do not significantly perturb the adapted state—by a linear transformation of the receptor signal outputs. For example, Guth et al. (1980) define the spectral response of their fundamental mechanisms under zero adaptation (i.e., no ambient illumination) via this set of equations:

$$A = (.5967\rho + .3654\gamma) \quad \text{(achromatic)} \quad (1)$$

$$T = (.9553\rho - 1.2836\gamma) \quad \text{(red–green)} \quad (2)$$

$$D = (-.0248\rho + .0483\beta) \quad \text{(blue–yellow).} \quad (3)$$

The spectral sensitivity of these three mechanisms under zero-adaptation conditions is drawn in Figure 2 (b). The spectral sensitivity of the neural mechanisms varies as the ambient illumination is changed. The sensitivity varies because of two kinds of changes within the eye. First, the sensitivity of the receptors varies with changes in the ambient illumination. This causes the coefficients within the parentheses of Equations 1–3 to vary. Second, sensitivity at neural sites beyond the point where the receptor signals are combined varies with changes in the ambient illumination. These changes are reflected in coefficients outside the parentheses of Equations 1–3. When the model's predictions are compared with data collected with an adapting light present, the values of the coefficients must be readjusted to reflect the effects of adaptation. Precise rules for these readjustments have not been determined.

When Guth and his colleagues (see Guth et al., 1980) compare the model's predictions for data collected on nonzero backgrounds, they adjust all nine coefficients in Equations 1–3 in order to bring the predictions into accord with the data.

The specific calculation of an observer's sensitivity to the difference between two lights, for a fixed adaptation level, is computed by (a) choosing a set of coefficients for Equations 1–3 and (b) computing the Euclidean distance between the coordinates of the lights via

$$ds^2 = A(\delta)^2 + T(\delta)^2 + D(\delta)^2. \quad (4)$$

While the principles of visual operation are developed formally in papers by Massof and Bird (1978a, 1978b) and Massof and Starr (1978), we may describe these principles informally by the following: First, for a fixed adapting state, small color differences may be measured by distance in a Euclidean vector space whose axes are a linear transformation of the receptor-based coordinates. This assumption is tantamount to supposing that the decision process of the visual system in discriminating weak stimuli is approximately linear. Second, the form of the linear transformation is given by the choice of coordinates in Equations 1–3. The values of these coordinates are determined by the observer's state of adaptation. The mechanisms that cause the coordinates to vary are the losses in sensitivity at the receptors and neural sites of the visual pathway. Third, just as in the case of Helmholtz's line element, noise in the decision process is independent of the lights that are being discriminated, that is, independent of the vector $V(\delta)$.

This kind of representation is still being actively considered. In the following section I generalize this representation and consider how it may be tested.

A Formal Treatment of Color Metrics

The purpose of this section is to develop a unified framework for describing metric theories listed above, a framework that emphasizes the commonality of these theories. The treatment here follows Krantz's (1975) work that characterizes the color-matching experiment by using an algebraic and measurement theoretical conception. Krantz gives three reasons for a new exposition of color measurement.

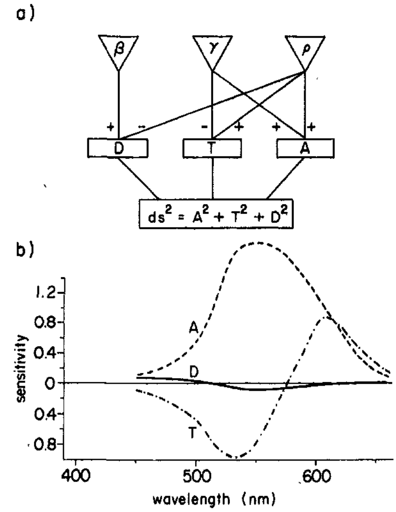


Figure 2. (a) Representation of the receptor signals (from the upside down triangles) into mechanisms that perform a linear transformation of the receptor signals. (The discriminability of different lights is determined by the value of their difference signal, ds^2, determined at the final decision site. Adapted from Guth, Massof, & Benzschawel [1980] where their model is described in detail.) (b) Spectral sensitivities of the linear transformations of the receptor outputs caused by the transformations A, T, and D. (The dashed line is the sensitivity of A, the dash-dot line is the sensitivity of T, and the smooth line is the sensitivity D. The curves are computed from the equations in the text, and the receptor sensitivities are from Smith and Pokorny [1972], whose receptor values are normalized to a peak sensitivity of 1.0.)

> First, both the language of mathematics and the standards of mathematical rigor have progressed greatly in the past century. The subject of color measurement can be communicated more clearly to new generations of students if it is cast in modern mathematical concepts. Second, many points that are obscure or difficult in traditional treatments are much clearer if a slightly more abstract standpoint is assumed. . . . Many blind alleys can be avoided if strings of equations are replaced by simple abstract arguments. Thirdly, one obtains a new perspective on the interrelation between color measurement and color theories. This makes it much easier to survey the plethora of starting points for various extant color systems or theories, and to formulate more clearly some of the main unresolved empirical problems in color perception. (p. 283)

Just as the exposition provided by Krantz has already proved useful in understanding color matching, I believe that the new exposition for color-difference measurement has already had certain benefits. The algebraic description emphasizes the crucial assumptions required to test whether an important class of metrics may be applied to discrimination.

I develop the formal notation with the following argument in mind: The color-matching theorem includes two kinds of results. First, we may map lights to the points of a vector space in a way that captures both the psychological relationship of metamerism and the additivity properties called Grassman's laws. This result is called the *representation result.* Second, all linear transformations of the vector space are equivalent in the sense that any linear transformation of the space will preserve both the metamerism relationship and the additivity relationship. This result is called the *uniqueness result.*

Together these results prompt the following question: Is there a distance formula for color discriminability with the property that small color discriminability may be predicted by the vector difference between two lights? This question includes as a special case the examples listed above where the distance formula is Euclidean. Before considering the general question, I develop the theory for this important, special case.

Invariances Across Linear Transformations

Suppose that when the observer is adapted to a light, a, there exists a set of basis coordinates in which small color differences may be represented as Euclidean vector distances. Let the space satisfying this property be denoted as $\mathbf{V}_a$ where the subscript denotes the point in space from which the discriminability of nearby points may be measured by Euclidean distance. In this definition the color matching coordinates, which we refer to as C, are a linear transformation of the space $\mathbf{V}_a$, but the metric properties of discriminability only hold for a small region of the space, near the adapting point, a. We denote the linear transformation that maps the coordinate system $\mathbf{V}_a$ into the coordinates of C as

$$A{:}\mathbf{V}_a \rightarrow C,$$

and we denote the action of A on a vector, $\mathbf{V}(a)$, as $\mathbf{V}(a)A$. We may treat A as a three by three, nonsingular (i.e., possessing inverse) matrix.

As a computational example, in the space $\mathbf{V}_a$ we measure the Euclidean distance between two not very different row vectors, a and a '+' δ, by first computing the vector difference, $\mathbf{V}(\delta)$, and then taking the inner product of the row vector $\mathbf{V}(\delta)$ and its transpose (column vector) $\mathbf{V}(\delta)^t$:

$$\begin{aligned} ds^2 &= [\mathbf{V}(a\,\text{'+'}\,\delta) - \mathbf{V}(a)]\cdot \\ &\quad [\mathbf{V}(a\,\text{'+'}\,\delta) - \mathbf{V}(a)] \\ &= \mathbf{V}(\delta)\cdot\mathbf{V}(\delta)^t \\ &= \sum_i di^2. \end{aligned} \tag{5}$$

The symbol "·" refers to inner product, whose definition is Equation 5, and the superscript, t, refers to the transpose operation. In the case of row vectors, such as $\mathbf{V}(\delta)$, $\mathbf{V}(\delta)^t$ is a column vector. In the case of a square matrix, A, each entry a_{ij} is replaced by the entry a_{ji}.

Notice that this formula depends only on the vector differences

$$\mathbf{V}(a\,\text{'+'}\,\delta) - \mathbf{V}(a).$$

The vector $\mathbf{V}(\delta)$ maps into the vector $\mathbf{V}(\delta)A$ of color-matching space. Therefore, in color-matching space the distance between two points, a and a '+' δ will be

$$\begin{aligned} ds^2 &= \mathbf{V}(\delta)A\cdot\mathbf{V}(\delta)A^t \\ &= \mathbf{V}(\delta)AA^t\mathbf{V}(\delta)^t. \end{aligned} \tag{6}$$

In the coordinate system following the transformation by A, the metric is not Euclidean but has the slightly more complex formula in Equation 6. The matrix AA^t will be symmetric, nonsingular, and positive definite. What is important about such matrices is that any such matrix, $G = AA^t$, defines a mapping from a vector space into the real numbers via Equation 6. When this mapping is explicitly written out in terms of the coefficients of the matrix $G = AA^t$ whose elements are g_{ij}, we have the formula

$$ds^2 = \sum_{ij} g_{ij}(di)(dj), \tag{7}$$

where d_i is the ith entry of the vector $\mathbf{V}(\delta)$. This formula is referred to as the line-element formula because it characterizes the (linear) distance between points in a small (elemental) region of space. Theories of color distances are, therefore, often referred to as line-element models. Conventionally (see Wyszecki & Stiles, 1967), this formula denotes the starting point for the search for a color metric, and the goal of the line-element theory is to define the coordinates g_{ij} subject only to the restriction that the choice of coordinates lead to a formula that satisfies the metric axioms. This condition is sometimes described as the condition that the matrix of elements g_{ij} be positive definite (see MacLane & Birkhoff, 1967). In the event that $g_{ij} = AA^t$, as developed here, this condition will necessarily be met.

METRIC THEOREM. *The mapping* $d{:}\mathbf{V} \times \mathbf{V} \rightarrow Re$ *defined by* $d(x, y)^2 = (x - y)AA^t(x - y)^t$ *for A, a nonsingular matrix, is a distance measure.*

PROOF: We have $d(x, x) = 0$ and $d(x, y) = d(y, x)$ by inspection of the formula. We need only demonstrate, therefore, the triangle inequality:

$$d(x, z) \leq d(x, y) + d(y, z).$$

Define the operator $\langle x, y \rangle$ as

$$\langle x, y \rangle = xAA^t y.$$

In this notation the triangle inequality is the assertion that

$$(\langle x - z, x - z \rangle)^{1/2} \leq (\langle x - y, x - y \rangle)^{1/2} + (\langle y - z, y - z \rangle)^{1/2}.$$

If we let $u = x - y$ and $v = y - z$, then this equation is equivalent to

$$\langle u + v, u + v \rangle^{1/2} \leq \langle u, u \rangle^{1/2} + \langle v, v \rangle^{1/2}.$$

Because the operator $\langle\ \ \rangle$ is bilinear, that is, from its definition $\langle x + y, v \rangle = \langle x, v \rangle + \langle y, v \rangle$, and similarly for the right-hand variable, we may write

$$\langle u + v, u + v \rangle = \langle u, u \rangle + 2\langle u, v \rangle + \langle v, v \rangle.$$

Note that $\langle u, v \rangle$ is the square of the inner product of two vectors in the space obtained by transforming $\mathbf{V}$ via the linear transformation A. We may apply the Schwarz inequality, which asserts that

$$\langle u, v \rangle \leq (\langle u, u \rangle^{1/2}) \cdot (\langle v, v \rangle^{1/2}),$$

so that by substitution

$$\begin{aligned}\langle u + v, u + v \rangle &\leq \langle u, u \rangle + 2\langle u, u \rangle^{1/2}\langle v, v \rangle^{1/2} + \langle v, v \rangle \text{ (Schwarz inequality)}\\ &= (\langle u, u \rangle^{1/2} + \langle v, v \rangle^{1/2})^2,\end{aligned}$$

which completes the proof.

From this theorem we learn that if color-matching space is a linear transformation of a special space, $\mathbf{V}$, where color discriminability may be described as the Euclidean metric, then color differences in color-matching space may be described by the formula in Equation 7. A standard theorem in linear algebra is the converse of this statement: If color differences can be measured by a formula such as in Equation 7, then there exists a linear transformation of the space such that distances may be measured using the Euclidean formula (see MacLane & Birkhoff, 1967, p. 377 et seq.). The result of this theorem is to assert that the distance formula (Eq. 7) is an invariant across linear transformations of color space when there is at least one transformation in which the distance formula is Euclidean.

A second invariant follows from a consideration of the properties of this distance metric. As I mentioned earlier, the line-element distance measure depends only on the difference between vectors. Notice that A transforms the vectors $\mathbf{V}(a)$ and $\mathbf{V}(b)$ with the property that $\mathbf{V}(a) - \mathbf{V}(b) = \mathbf{V}(\delta)$ into new vectors, $\mathbf{V}(a)A$ and $\mathbf{V}(b)A$, whose difference is $\mathbf{V}(\delta)A$. Thus, lights with equal vector differences when represented by $\mathbf{V}$ will have equal vector differences after the transformation by A into the representation $\mathbf{V}A$.

We can make use of this fact in the following way: Suppose there exists a color space, $\mathbf{V}$, in which the distance formula for discriminability depends only on the vector difference of the two lights to be discriminated. In such a space equally discriminable lights are those with equal difference vectors. If the color space in color-matching coordinates is a linear transformation, via the mapping A of the space V, then pairs of points whose difference vector is $\mathrm{V}(\delta)$ in the space V will have a difference vector $\mathrm{V}(\delta)A$ in the space defined by the color-matching coordinates. Therefore, pairs of points that are equidiscriminable, that is, separated by a constant amount $\mathrm{V}(\delta)$ in V, will be represented by pairs of points that are separated by a constant amount $\mathrm{V}(\delta)A$ in color-matching coordinates.

We have just proved the following theorem.

PARALLELISM INVARIANCE THEOREM. *First, we define two subsets, L and L′, of a vector space as* parallel *if and only if there exists a translation,*

$$T{:}\mathrm{V}(a) \rightarrow \mathrm{V}(a) + \mathrm{V}(\text{constant}),$$

that maps the elements of L onto the elements of L′.

*Now consider a set of lights L = {a} to each of whose members we associate a second light, a '+' i(a)*δ where i(a) is the intensity of the light δ that permits the subjects to just noticeably discriminate light a from light a '+' i(a)*δ. We define the set L′ = {a '+' i(a)*δ}.*

We may conclude that if discriminability depends only on the vector difference of the lights, then the set L′ is parallel to the set L when these sets are represented as vectors in any *linear transformation of color space. As a special case, note that if L is a line in color-matching coordinates, then L′ is a line parallel to L.*

This theorem is stated in terms of the entire color space; however, because of adaptation we do not expect that equidiscriminable lights will fall in parallel sets for arbitrarily large regions of color space. Rather, parallelism should hold for a restricted range of lights—those that do not perturb the adapted state.

Summary of Results

I want to emphasize three conclusions. First, the treatment here does not single out any subset of color space as privileged with respect to any other subset of color space. This is different from the usual treatment (see Boynton, 1980) where theories of chromatic discrimination are developed for the special case of lights falling within a so-called equiluminance plane. This plane is defined by an empirical procedure—flicker photometry—whose relationship to the color-matching experiment is not yet well understood.

The restriction of working within the equiluminance plane is replaced by the restriction of using test lights sufficiently weak so that the adapted state of the observer does not vary. In adopting this assumption we have followed the approach of Stiles (1946, 1978) in his two-color threshold measurements and associated line-element model. A question that will arise is what is meant by *small.* I defer this issue to the next section.

Second, notice that there is no essential difference between line-element formulas for distance and the Euclidean formula for distance except for a renaming of the coordinate system. The Euclidean distance formula becomes Equation 7, the line-element distance formula, under a linear transformation of the coordinate system. And, in turn, the Euclidean distance formula is but a special case of Equation 7, which we therefore treat as an invariant across linear transformations.

Finally, I emphasize that a general invariant across linear transformations is parallelism. *Parallelism* refers to sets of vectors that differ by a constant vector. If there exists a color-metric space for discriminability, V, where the discriminability of lights depends only on their vector difference, lights represented by parallel sets will be equally discriminable from one another. Because parallelism is invariant under linear transformations, these sets of equidiscriminable points will remain parallel if the color-matching coordinate system is a linear transformation of the space V. Furthermore, if the coordinate system of color-matching space is (locally) a linear transformation of a vector space V, where color discriminability can be measured by vector differences, then equidiscriminable lights will (locally) form parallel sets in the coordinate system of color-matching space. I describe a test of this hypothesis in the next section.

A Test of Vector Representations and the Adaptation Hypothesis

Suppose an observer gazes at a large, uniform adapting field. The purpose of this field is to fix the observer's state of adaptation. We introduce a weak perturbation of the observer's visual system by presenting a pedestal flash, call it light a (see Figure 3). In color space we may think of the background light as determining the region of color space in which small discriminations are performed. The pedestal is treated as a small vector whose direction is defined by the chromaticity of the pedestal and whose length is defined by the intensity of the pedestal. We assume that pedestals near threshold do not disturb the adapted state of the observer. For this reason we consider near-threshold pedestals to be small. As we increase the intensity of the pedestal, but not so far as to disturb the observer's state of adaptation, the size of the perturbation due to the pedestal vector will increase. As the intensity, k, of the pedestal flash, a, grows, the observer's visual system will be moved out along a line in color space, defined by $k*a$. A line parallel to $k*a$ may be constructed by adding a very small second component to the pedestal flash. Call this test flash δ. The points $k*a$ '+' δ will form a second line, parallel to the line $k*a$ (see Figure 4).

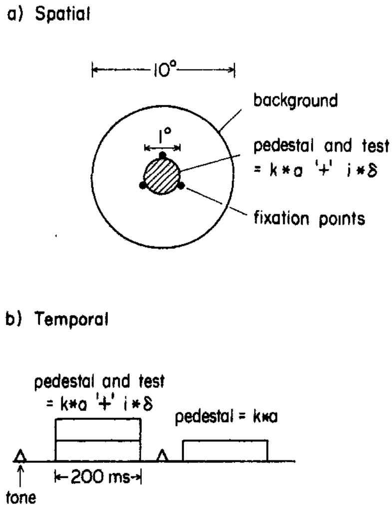


Figure 3. (a) Spatial arrangement of the stimuli for the pedestal experiment. (The test and pedestal were presented on the center of a large background. Both the test and pedestal were 1° in diameter.) (b) Temporal arrangement of the stimuli for the pedestal experiment. (The pedestal plus test occurred in either the first or the second interval, with equal probability. The pedestal alone occurred in the other interval. The stimuli were presented simultaneously for a duration of 200 ms [milliseconds]. Each interval was preceded by a short warning tone.)

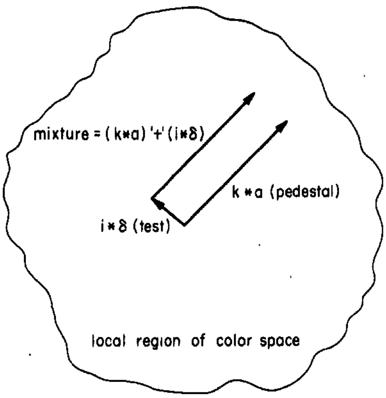


Figure 4. Vectors in a local region of color-matching coordinates for the pedestal ($k*a$) and the test ($i*\delta$) and for the mixture of the test and pedestal. (As the intensity, k, of the pedestal is increased, but the intensity of the test, i, is held constant, the resultant vector $k*a$ '+' $i*\delta$ forms a line that is parallel to the line formed by the pedestal light itself, $k*a$.)

If the adaptation hypothesis is correct, then there exists a linear transformation of this region of the color-matching coordinates to a space where vector differences may be used to compute discriminability. By parallelism invariance the lines $k*a$ and $k*a$ '+' δ will be parallel in any linearly transformed space. Because the vector differences of the points $k*a$ and $k*a + \delta$ are equal for all values of k, the lights $k*a$ and $k*a$ '+' δ will be equally discriminable for small values of k and for all choices of the pedestal, a.

A second way of stating this prediction, which is more convenient for empirical testing, is the following: Fix an intensity, k, of the pedestal flash. Measure the intensity of δ—call it $i(k*a)$—that is required in order to render $k*a$ '+' $i(k*a)*\delta$ just barely discriminable from $k*a$. The adaptation hypothesis asserts that as k varies $k*a$ and $k*a + i(k*a)*\delta$ will be parallel lines in any linear transformation of the color-matching coordinates. This is equivalent to saying that $i(k*a)$ is constant for small k.

Loftus and I set out to test this hypothesis

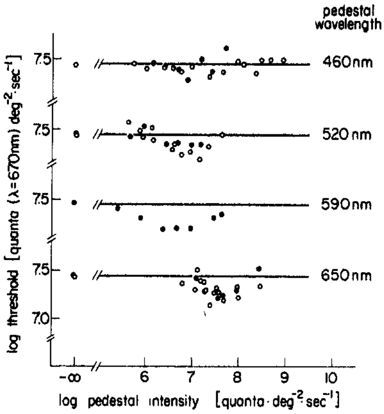


Figure 5. Results of the pedestal experiment. (The horizontal axis plots the intensity of the pedestal flash in units of physical intensity: log quanta per square degree per second. Each curve represents the data from a pedestal of a different wavelength. The vertical axis plots threshold intensity to the 670-nm test flash. In many instances the presence of the pedestal reduces the amount of energy required to detect the presence of the test flash. Pedestal thresholds were [SE_M and pedestal wavelength in brackets; units are log quanta $\deg^{-2} \sec^{-1}$] 7.2 [.04; 650 nm], 6.2 [.03; 590 nm], 6.0 [.02; 520 nm], and 6.0 [.11; 460 nm], respectively. Open and filled circles indicate different observers.)

(Wandell & Loftus, Note 1). Examples of data we collected are shown in Figure 5. For these particular measurement conditions the background field was 10°, yellow (580 nm), and moderately intense (8.931 log quanta/$\deg^{-2} \sec^{-1}$). The pedestal, light a, was a 200 msec, 1.1° spot viewed by the central fovea. The pedestal wavelength is the parameter of the separate curves, which are arbitrarily displaced relative to the vertical axis. The intensity of the pedestal is indicated on the horizontal axis, and the threshold values of each of the pedestals is indicated on the curves themselves.

The test flash, δ, was a 670 nm, 200 msec, 1.1° flash superimposed on the pedestal. The observer's task was to decide in which of two temporal intervals the test flash was presented on the pedestal, that is, to discriminate pedestal from test flash plus pedestal. The intensity of the test flash at threshold, for each of the various pedestal wavelengths and intensities, corresponds to the value $\hat{\imath}(k*a)$. The prediction of the adaptation hypothesis is that this value should be constant for pedestals near threshold visibility. As is evident from the curves, there are systematic departures from this prediction.

These results are inconsistent with the adaptation hypothesis. We must, therefore, return to re-examine the assumptions of the adaptation hypothesis. Two kinds of issues may be raised.

First we may ask whether the empirical test of the adaptation hypothesis is an adequate operationalization of the theory. Several kinds of objections are possible. For example, the color-matching experiment takes place under essentially steady state conditions. No lights are flashed. However, in the present test of the adaptation hypothesis, derived from measurements based on the color-matching experiment, we use flashed pedestals and tests. Although 200 msec is comparable to the duration of a fixation, perhaps we cannot treat these 200 msec flashes as comparable to the steady fields used in the color-matching experiment. A second objection is that in the Wandell and Loftus (Note 1) experiments threshold judgments were made over time not space. In the usual color-matching experiments, judgments are made across the border of the bipartite field, rather than across time as in the two-interval, forced-choice design. This is another reason to object to the experiment.

A further objection may be raised: Are the intensities of the pedestal flash small in the sense required by the theory? There is no objective means of answering this question. Notice, however, that the threshold of the pedestal flashes alone (indicated in the figure caption) are below threshold when deviations from the prediction begin. This means that if the theory is correct, it can only apply to discriminability of lights that are invisible.

If the pedestal experiments are viewed as adequate tests of the theory, the possibility of achieving any metric representation based on vector differences is placed in doubt. In view of the results of the pedestal experiment (as well as the closely related results of Nachmias & Kocher, 1970; Nachmias & Sansbury, 1973; Whittle & Swanston, 1974), one must find new justifications for the adaptation hypothesis. It is not premature, therefore, to consider alternatives and extensions of the geometric theory. I consider one alternative form of representation in the following section.

Alternative Conceptions

The adaptation hypothesis is but one principle that has served to guide the development of a color metric. A second conception, complementary to the adaptation hypothesis though seldom distinguished from it, is a hypothesis I will call the *fluctuation hypothesis* of chromatic discrimination. Various forms of this hypothesis have been stated and discussed (see Buchsbaum & Goldstein, 1979; Kranda & King-Smith, 1979; Trabka, 1968; Vos & Walraven, 1972a, 1972b).

The fluctuation hypothesis assumes that color discriminability is limited by noise (either quantal or physiological) that contaminates the decision process. Fluctuation theories incorporate the noise as an integral part of the representation of colored lights. Thus, although a theory consistent with the adaptation hypothesis may suppose the existence of noise, this noise is viewed as external to the representation and not a part of color science per se; rather, it is an annoyance that must be confronted.

Formulas derived from the fluctuation hypothesis often have the same functional form (Eq. 7) as the formulas derived from the adaptation hypothesis (see Buchsbaum & Goldstein, 1979; Vos & Walraven, 1972b), but the principles that give rise to the formulas are quite different. For example, Vos and Walraven (1972b) derive a distance formula consistent with Equation 7 from principles of quantal fluctuation. On this derivation, however, the weighting coefficients g_{ij} are determined by statistical signal-to-noise considerations. Specifically, the weighting coefficient assigned to a term is larger when that term reflects the action of a mechanism with a larger signal-to-noise ratio. Therefore, the meaning of the notation is entirely different when the derivation is based on the fluctuation hypothesis rather than the adaptation hypothesis.

If the fluctuation hypothesis is correct, the physical process of discrimination acts very differently from the physical process that one would suppose based on the adaptation hypothesis. Because the signal-to-noise ratio of different mechanisms depends on the test lights being compared as well as the state of adaptation, the coefficients g_{ij} will depend on the test lights that are being discriminated. Even in a fixed state of adaptation, the weight assignments, g_{ij}, will vary depending on the signal-to-noise properties for the particular test stimuli. On the adaptation hypothesis the weight assignments, g_{ij}, depend only on the observer's adapted state.

Because the interpretation of the symbols in Equation 7 is different when derived from the fluctuation hypothesis as compared to the adaptation hypothesis, tests of the fluctuation hypothesis require different sorts of experiments. I now describe the principles of fluctuation hypotheses more completely so that I may later describe testable properties of models based on the idea of fluctuation.

The Fluctuation Hypothesis

Trabka (1968), Vos and Walraven (1972a, 1972b), and Buchsbaum and Goldstein (1979) have discussed models in which chromatic discriminations are characterized as dependent on the information in three physiological channels, corresponding to the three receptor types or simple transformations of the receptors. Each channel has (Poisson) noise characteristics that limit discriminability of chromatic stimuli. The source of this noise may be the quantum fluctuations of the light itself or noise inherent to the physiological mechanisms. I will describe the work of Buchsbaum and Goldstein (1979) because their presentation is representative of the fluctuation theories.

Buchsbaum and Goldstein (1979) treat the receptors as the three noisy channels that limit discriminability. The rate parameter of the Poisson process on each channel depends on the light presented to the observer and is computed via a linear functional (derived from the receptor spectral sensitivity curve) on the space of wavelength distributions. A particular set of linear functionals—

the estimated spectral sensitivities of the three receptors from Thomson and Wright (1953)—is used for the purpose of calculations, though this is not crucial to their approach.

To evaluate this framework Buchsbaum and Goldstein (1979) keep the assumptions listed above fixed. Attention is focused on the selection of decision statistics, computed from the responses of the three channels, and on which the maximum-likelihood calculation is performed. Examples of two different decision rules are (a) counting the number of events in a fixed interval on each channel (see also Luce & Green, 1972; McGill, 1967; Wandell, 1977) and (b) estimating the total interevent interval for a fixed number of counts (Luce & Green, 1972; Wandell, 1977).

Using these and closely related decision statistics, Buchsbaum and Goldstein (1979) show that traditional line-element formulas (Eq. 7), derived from geometric models based on the color-matching experiment, are equivalent in mathematical form to the discriminability predictions derived by choosing different decision statistics in the maximum-likelihood model. This had been shown in the special case of the Stiles line-element by Trabka (1968). Buchsbaum and Goldstein (1979) extended the result to include other line-element models. It is important to note that equivalence of the mathematical form for predictions of color discrimination does not imply that the models are equivalent with respect to all possible experiments.

In the maximum-likelihood approach, color discriminability decisions are modeled as follows: On each trial the subject is presented a stimulus and asked to decide whether the stimulus is one of two possible lights. One assumes that the observer has knowledge of the probable effects of the two lights. In particular, each light has some probability of causing the observer to receive a sense impression, S, from the space of all possible sense impressions, X (i.e., S is an element of X). From the color-matching experiment we may assert that any sense impression may be represented by a three-dimensional vector whose coordinates correspond to the values of the decision statistics computed on the three channels. For each light, a, there is a probability that it will give rise to the particular impression, S. Call the likelihood that the observer has sensation S given light a, $L(S|a)$.

From the sensation, S, observed on the experimental trial, the observer computes the relative likelihood that the observed sense impression is caused by light a versus the likelihood that it is caused by light b. This estimate is made as

$$\frac{L(S|a)}{L(S|b)} = \text{likelihood estimate.}$$

If this value is sufficiently large, the observer responds that the light was a, and otherwise he or she responds that the light was b.

For convenience I will refer to models derived from the fluctuation hypothesis as statistical models. Models derived from the adaptation hypothesis will be called geometric models.

Comparison of Statistical and Geometric Models

An important difference between the geometric and statistical models is the following: In the geometric models lights are identified with points (sensations) in a three-dimensional coordinate system. The outcome of an experiment must be explained by the relationships among the points assigned to the lights. In the statistical models lights are identified with functions defined across the space of all points (sensations). Thus, in the Buchsbaum and Goldstein (1979) formulation, lights are not defined by mapping into specific sensory effects, S. Rather, lights are assigned to likelihood functions $L(S|a)$, defined across the space of all sensory impressions. This is a much more general description of a light than is allowed in the geometric model. To emphasize how profound this difference is, notice that in the geometric model the number of parameters defining the effect of a light is three, namely, the coordinates of the light. In the statistical models lights are described by an infinite list of numbers, namely, the values taken on by their likelihood distributions across the entire space.

There is a cost to this more general definition. When we combine two lights by forming the mixture of light a with light b as a '+' b, we cannot write down a simple relationship between their associated representations. Thus, if we know the distribution $L(S|a)$ and the distribution $L(S|b)$, we cannot always find a simple formula of the form

$$L(S|a\,'+'\,b) = F\{L(S|a), L(S|b)\}.$$

Rather, we must return to first principles in order to define the likelihood of the mixture. In some instances the rule will be simple, in others not. This, too, is different from the vector-representation theories because the color-matching theorem assures that the vector representation of a '+' b is given by

$$\mathbf{V}(a\,'+'\,b) = \mathbf{V}(a) + \mathbf{V}(b).$$

A further difference demonstrates an advantage of the statistical form of representation. Let us consider the likelihood decision rule where the subject is unbiased such that

$$\text{If}\quad L(S|a)\begin{cases} >L(S|b), \text{ respond } a \\ \leq L(S|b), \text{ respond } b \end{cases}.$$

Let us further simplify the arguments by supposing that prior beliefs are already incorporated into the likelihood functions and that we may treat the likelihood functions as if they were probability densities over the space of outcomes, X. With the unbiased decision rule, the probability of correctly discriminating a and b, which I denote as $p(a, b)$, may be computed in the following way:

$$p(a, b) = \int_X [P(a)L(S|a)P(\text{respond } a|S) + P(b)L(S|b)P(\text{respond } b|S)],$$

where $P(a)$ is the probability of presenting a; $P(b)$ is the probability of presenting b. If we let $P(a) = P(b) = .5$ for convenience and we define the subset, A, where the subject responses "a" via

$$A = \{S\text{: respond } a = S\text{: } L(S|a) > L(S|b)\},$$

we can simplify the expression to the form

$$p(a, b) = \frac{1}{2}\left[\int_A L(S|a) + \int_{X-A} L(S|b)\right].$$

This formula has the property that when a and b are equal, the probability of choosing a rather than b is $1/2$. When the distribution $L(S|b)$ is zero for all values of S where $L(S|a)$ is greater than zero, $p(a, b)$ is equal to one (i.e., the lights are perfectly discriminable).

We have assumed that $L(S|b)$ is a density, so the integral over X of $L(S|b)$ is unity. We may further simplify to

$$p(a, b) = \frac{1}{2}\left[\int_A (L(S|a) - L(S|b)\right] + \frac{1}{2}.$$

We conclude that all lights a, a', b, b' such that

$$\int_A [L(S|a) - L(S|b)] = \int_A [L(S|a') - L(S|b')] \qquad (8)$$

are equally discriminable. The functional computation on each side of Equation 8 plays a role equivalent to the computation of the vector difference in the geometric model. We define, therefore, the distance between two points, $\mathrm{d}(a, b)$, via

$$\mathrm{d}(a, b) = \int_A [L(S|a) - L(S|b)].$$

We use the term *distance* to describe the computation of $\mathrm{d}(a, b)$ because this function satisfies the metric axioms. To see this notice that the set A is precisely the set where $L(S|a) - L(S|b) > 0$. Therefore, $\mathrm{d}(a, b)$ will always be positive. By definition, $p(a, b)$ is symmetric and, therefore, $\mathrm{d}(a, b)$ will be. To check the triangle inequality notice that

$$\begin{aligned}\mathrm{d}(a, c) &= \int_A [L(S|a) - L(S|c)] \\ &= \int_A [L(S|a) - L(S|b)] \\ &\quad + \int_A [L(S|b) - L(S|c)] \\ &= \mathrm{d}(a, b) + \int_A [L(S|b) - L(S|c)].\end{aligned}$$

Recall that B is precisely the set of points where $L(S|b) - L(S|c) > 0$ so that integrating over any other set must yield a value smaller than integrating over the set B. That is,

$$\int_A [L(S|b) - L(S|c)] \leq \int_B [L(S|b) - L(S|c)] = \mathrm{d}(b, c).$$

We may substitute the above in the previous equation, changing the equality sign to an inequality sign and write

$$\mathrm{d}(a, c) \leq \mathrm{d}(a, b) + \mathrm{d}(b, c) \text{ (triangle inequality).}$$

Let us summarize our characterization of the statistical theories representation of discrimination. We can do this by contrasting the statistical representation with geometric models.

Properties of geometric models:

1. Lights are assigned values as three-dimensional vectors.
2. The assignment of vector quantities is additive with respect to the mixture of lights in the sense that the point that represents the light a '+' b is $\mathrm{V}(a) + \mathrm{V}(b)$.
3. Pairs of equally discriminable lights are those quadruples (a, b) and (a', b') such that $\mathrm{V}(a) - \mathrm{V}(b) = \mathrm{V}(a') - \mathrm{V}(b')$.

Properties of statistical models:

1. Lights are assigned values as functions over a three-dimensional space.
2. There is no additivity of the representation with respect to light mixtures because there is no simple relationship between representations of a, b, and a '+' b.
3. Pairs of equally discriminable lights are those quadruples (a, b) and (a', b') where

$$\int_A [L(S|a) - L(S|b)] = \int_{A'} [L(S|a') - L(S|b')].$$

The question now arises whether statistical models may explain the results of color-discrimination experiments. In the next section I will consider experimental tests of the adequacy of statistical models. I will discuss the ways in which their structure differs from that of geometric models and the ways in which their structure is related to other phychological measures of similarity and choice.

Relationship to Other Psychological Theories

Two kinds of issues arise in evaluating the geometric and statistical formulations. First are the theoretical issues concerning the implications of the representational forms themselves. In this section I will ask what is the relationship to other schemes for measuring the psychological structure imposed on physical stimuli. I will treat the problem of defining cardinal psychological dimensions when geometric representations are used.

Second, I will begin an inquiry into further properties of statistical representations that will permit us to test their adequacy. Because vector theories of color discrimination have been studied for many years, the relationships between these models and empirical procedures has already received some attention. Statistical models have not received as much study. We must, therefore, develop empirical procedures to serve as the foundations for such a theory.

Geometric Representations

Psychologists have attempted to find geometric representations of various stimuli—words, emotions, attitudes—for more than 20 years (see Shepard, 1980, for a review of much of this work). One important goal of some of this work is to define cardinal directions, or principal psychological axes, that characterize the effect of stimuli on people. Surprisingly, the general geometric model, based on the line-element metric, includes a well-defined notion of axis or dimension only in certain special cases.

To understand how a model may fail to allow a definition of cardinal axes, it is useful to consider a special model where cardinal directions are well defined—Helmholtz's color metric. Helmholtz assumed that the cardinal directions of color space are defined by the receptor primaries. This is reflected by the fact that the only transformations of color space incorporated into Helmholtz's metric were transformations that stretched the space to compensate for Weber's law, applied separately to the receptor classes. This is represented in the metric by the fact that the matrix of coefficients, G, in Helmholtz's line element is a diagonal matrix whose entries are

$$g_{ij} = \begin{cases} 0 & \text{if } i \text{ not equal to } j \\ x_i & \text{otherwise} \end{cases},$$

where x_i is the rate of quantum absorption in the ith receptor class. The transformations of space, therefore, are merely transformations that stretch the cardinal axes to compensate for changes in sensitivity.

In terms of psychological assumptions, this means that for all stimulus comparisons the directions defined by the receptor spectral sensitivities have a special significance. The *receptor spectral sensitivities* are the directions along which functions of differences between stimuli are summed. In Helmholtz's color metric the directions along which small differences between stimuli are added do not vary when different stimuli are compared or when the state of adaptation is changed. It is therefore sensible to single out the receptor spectral sensitivities as having special psychological—and in this case physiological—significance.

I suggest that what is meant by *cardinal directions* generally in psychological scaling is precisely those directions along which small signals are summed when distances between stimuli are measured. Notice that when we consider metrics whose representation includes nonzero off-diagonal entries, g_{ij}, the basis vectors on which addition of small differences occurs will vary as the g_{ij} vary. Guth et al.'s (1980) model is an example of a line-element model with no cardinal directions because there is no set of axes across which small measures are always summed in order to estimate distance. The directions across which small differences are summed varies depending on the state of adaptation during the measurement. In general, privileged axes exist only when the line-element transformation, G, contains only diagonal entries. If arbitrary linear transformations, G, are allowed, the line-element model does not permit the definition of a special set of axes.

We conclude, therefore, that there exist line-element representations of stimuli that do not permit the definition of privileged axes. Line-element metrics are examples of representations that contain local metric properties without a concomitant definition of cardinal axes that characterizes a privileged coordinate system throughout the space. What we have learned, then, is that the ability to measure does not imply the ability to identify psychological dimensions.

Statistical Representations

I now discuss two aspects of statistical representations. First, I will discuss the relationship between the statistical representation and two models of choice (Luce, 1959; Tversky, 1972a, 1972b). Other choice theories, notably those based on Thurstonian scaling (e.g., Marley, 1971), are related to the statistical representation, but these models usually depend importantly on the assumption that the stimuli may be characterized as random fluctuations of a one-dimensional variable (Marley, 1971). Because color discrimination depends on a multidimensional encoding of the stimulus, the Thurstonian theories are not of great help in elucidating the statistical representations.

Second, I will speculate on the properties that a complete, statistical representation will have. In this section I again take up the problem of defining psychological dimensions.

Properties of statistical representations. For ease of discourse imagine that the effect of a threshold light on the observer may be characterized by a sensation, S, that is an ordered pair rather than an ordered triple. This will not result in any loss of generality in our attempt to describe properties of the statistical representation. The ordered pair characterizes the stimulus effect on two receptor classes. Further, imagine that in a discrimination experiment the observer has a likelihood, $L(S|a)$, that an ordered pair, S, will arise given a stimulus, a. For each stimulus we can draw a picture of its likelihood function where the horizontal plane contains the possible sensations, S, and the height of the function above the plane is the likelihood of observing the outcome in the

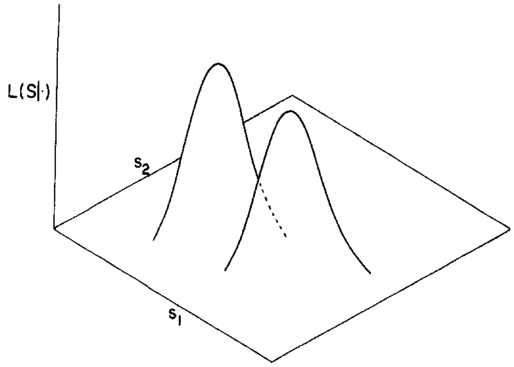


Figure 6. Schematic diagram of the functions describing the likelihood of observing a sensation, $S = (s_1, s_2)$, due to one of two stimuli, represented by the separate peaked distributions. (The horizontal plane refers to the stimulus sensations and the vertical axis refers to the degree of likelihood.)

horizontal plane. The maximum-likelihood rule assigns responses to sensations in different regions of the horizontal plane by the rule that the stimulus whose likelihood function is highest over that sensation is the one the subject selects as the response (see Figure 6).

This decision rule effectively partitions the horizontal plane into regions associated with the different responses. We may summarize the likelihood rule by a simpler, planar picture: The effect of a stimulus on the observer will be to generate a pattern of sensations that are distributed in the horizontal sensation plane. It is convenient to stretch the sensation plane so that the probability of a stimulus causing a sensation in any small region of the stretched plane is equal. The nature of the stretching may be complicated (i.e., nonlinear), but it is possible. We can then partition the new plane into two regions, corresponding to the region in which the likelihood function $L(S|a)$ is higher and the region in which $L(S|b)$ is higher. For a single experiment the probability of responding light "a" is equal to the probability of randomly selecting a point on this plane and discovering that the point is in area A. Notice that the probabilities we are computing here are choice probabilities not the probability of responding correctly, which we computed in the previous section (see Figure 7).

Consider the outcomes of two experiments where the subject is shown lights a and b and presented with a series of stimuli and asked to decide whether the presented stimulus was light a or light b. He is then shown lights b and c as alternatives and presented the same series of stimuli (up to a permutation of order) and asked to decide whether the presented stimulus was light b or light c. There will be two partitions of the horizontal plane: one that divides it into response areas A and B and a second that divides it into B and C. We ask whether from knowledge of the paired choice probabilities between a and b and the paired choice probabilities between b and c can we restrict the possible choice probabilities between a and c?

In the paired comparison experiment let $P(S \text{ in } \{A > B\})$ denote the probability that the sensation S occurs in the region of the graph denoted by A, when the alternative choice is stimulus b. The probability that the subject responds "a" in a choice between a and c is at least as great as the probability that the stimulus outcome falls in a region where the likelihood of a dominates $b[L(S|a) > L(S|b)]$ and the likelihood of b dominates $c[L(S|b) > L(S|c)]$. This is because the region of the sensation plane where a dominates b and b dominates c is but a subset of the region where a dominates c. Thus, $P(S \text{ in } \{A > C\}) \geq P(S \text{ in } \{A > B \text{ and } B > C\})$. By simple rules of probability we can rewrite this as

$$
\begin{aligned}
P(S \text{ in } \{A > C\}) &\geq 1 - P(S \text{ in } \{B > A \text{ or } C > B\}) \\
&= 1 - P(S \text{ in } \{B > A\}) - P(S \text{ in } \{C > B\}) + P(S \text{ in } \{B > A \text{ and } C > B\}) \\
&\geq 1 - P(S \text{ in } \{B > A\}) - P(S \text{ in } \{C > B\}). \quad (9)
\end{aligned}
$$

Equation 9 represents a constraint that all paired choice probabilities must satisfy when the stimulus presentation probabilities are fixed. Notice that we can rewrite Equation 9 as

$$
\begin{aligned}
P(S \text{ in } \{A > C\}) &\geq 1 - \{[1 - P(S \text{ in } \{A > B\}] - [1 - P(S \text{ in } \{B > C\})]\} \\
&= P(S \text{ in } \{A > B\}) + P(S \text{ in } \{B > C\}) - 1, \quad (10)
\end{aligned}
$$

which is an extremely weak transitivity condition. The weakest transitivity condition usually considered by choice theorists is called *weak stochastic transitivity*. This condition asserts that if $P(S \text{ in } \{A > B\})$ and $P(S \text{ in } \{B > C\})$ are greater than one half, then $P(S \text{ in } \{A > C\})$ is greater than one half. Equation 10 does not imply weak stochastic transitivity because when both terms on the right-hand side are equal to one half we are guaranteed nothing about the left-hand term. The condition does, however, constrain binary choice probabilities. For example, if $P(S \text{ in } \{A > B\})$ is .75 and $P(S \text{ in } \{B > C\})$ is .75, then $P(S \text{ in } \{A > C\})$ must be at least one half.

A second constraint that must be satisfied comes from considering increasing the stimulus set from a paired choice probability to a selection from among three alternatives. When we introduce a third likelihood function into the graph, it will not change the ordering of likelihoods between stimuli *a* and *b* at any points. The effect of introducing a third alternative is to assign some of the possible stimulus events, *S*, to the new choice *c*, or leaving them assigned as they were. In terms of the partition this is like laying down a third set, on top of the fixed sets *A* and *B*, that removes some of the area from *A* and some of the area from *B*, putting them in set *C*, but does not change the remaining areas. This may change the relative probabilities of choosing *a* and *b*, but the absolute probability of responding "*a*" or responding "*b*" cannot increase when *c* is introduced. This condition is called regularity, and it further constrains what may happen in choice experiments if the statistical representation is valid.

What remain unknown are the necessary and sufficient conditions to ensure the existence of a statistical representation via likelihood functions. Especially, can we discover a constructive procedure that will permit us to construct the likelihood functions of the representations?

It is instructive to compare the properties of the statistical decision rules with two choice models that have a natural set theoretic interpretation: the elimination-by-aspects model of Tversky (1972a, 1972b) and Luce's (1959) choice model. The elimination-by-aspects model associates sets of aspects with the offered choices much in the way that the statistical decision model associates sets of sensations with the offered choices. The elimination-by-aspects model differs from the statistical decision model in these two ways: First, in the elimination-by-aspects model, a new choice added to the offered set may introduce new aspects (sensations) that are not present in the other choices in the offered set. Second, different choices may be associated with sets that share common aspects so that the sets of aspects associated with the choices have nonempty intersections.

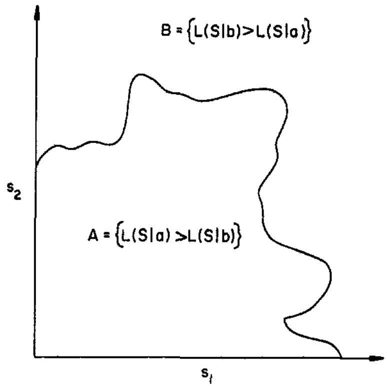


Figure 7. Schematic of the horizontal, sensation plane where the s_1 and s_2 axes may have been stretched so that the probability of generating a sensation at any point in the plane is equally likely. (The region denoted by *A* refers to the parts of the plane where the likelihood function associated with light *a* exceeds the likelihood function associated with light *b*. The region denoted by *B* refers to those parts of the plane where light *b*'s likelihood function is larger than that of light a.)

The statistical representations differ in both respects from the elimination-by-aspects model. First, the possible sensations are not altered as different choices are offered. Only changing the colored lights used as stimuli can introduce new sensations (aspects). The set of sensations (aspects) is neither augmented nor decreased as new choices are offered to the subject. Second, the statistical decision model imposes a partition of the sensations (aspects) into disjoint sets representing the offered choices.

Luce's (1959) choice model is a special case of the elimination-by-aspects model (Tversky, 1972a). In Luce's model different choices are represented by sets that partition the aspects into disjoint sets—similar to the statistical models here. In Luce's model, however, the structure of the sets associated with different choices is constrained by the rule that the ratio of the probabilities of choosing light *a* and light *b* must remain constant as new choices are offered. This is accomplished by supposing that the introduction of a new choice brings with it entirely new aspects, disjoint from those already present in the original alternatives. As I noted above, the statistical representation does not allow new aspects to be introduced by new elements of the offered set. In the statistical decision model, introducing new choices will generally cause the relative probabilities of selecting light *a* or light *b* to vary, violating Luce's model.

Goals in defining a statistical representation. Because statistical representations are relatively new in the scaling of small color differences, it seems useful to speculate briefly on what the goals of defining a statistical representation might be.

First, statistical representations do not impose strong structure at the level of the elements of the space. We treat the possible sensations as a set of primitives, having little structure. The relationships among the sensations themselves are not of great significance. Stimuli are defined by distributions across the whole set of primitives—the stimulus distributions $L(S|a)$—and experimental outcomes are defined by rules involving computations on these distributions across all sensations.

For example, in the case of metameric matching the identity of two stimuli is represented by the rule that two lights are judged metameric only if there is an identity of the distributions $L(S|a) = L(S|b)$. In the case of discriminability, the rule has the form that the discriminability of *a* and *b* depends monotonically on

$$\int_A [L(S|a) - L(S|b)].$$

Thus, empirical observations are captured not by relationships among points, S, but rather by the rules defined on the likelihood distributions associated with different stimuli. Thus, the dimensionality of the space of points, S, is not a significant part of this kind of representation. And, as for the general case of the line-element formulas, it is difficult to sensibly define cardinal psychological dimensions.

What is the goal of devising such a representation? First, for each experiment we would like to be able to define a rule that characterizes the results of the experiment. Further, across experiments, the distributions that characterize the effects of the stimuli, $L(S|a)$, ought to be fixed.

Second, we would like to restrict the set of possible operators in some principled way. Rules concerning the class of permissible operators ought to be defined based on our understanding of the mechanisms of the process under study.

Third, a new question of interest arises for this kind of representation as compared to the geometrical representations. What are the relationships among the rules themselves? Consider the experimental problem of measuring the identification probabilities when a stimulus light, *a*, is identified from a set of *n* alternative lights. This is a generalization of the operator described for the discrimination experiment when the alternative set of lights contains only $n - 1$ lights. The operators that characterize the observables from the various *n* alternative experiments should be simply related to one another. And the nature of this relationship should provide us with an insight as to the processing strategies of the human observer.

Final Remarks

Constructing an inductive color metric has not proven to be simple. The extension of the color-matching experiment based on judgments of the identity of stimuli to discrimination experiments based on judgments of stimulus differences has forced us to reconsider the nature of visual function.

In studying the benefits and shortcomings of representations of discriminability, I have come to the following view of representation for color discriminability. There are two difficulties with vector models: First, neural stimulus processing is given too passive a role. On the geometric approach a fixed set of mechanisms is stimulated by a light, and based on this response a fixed calculation is performed. If one response mechanism is particularly strongly stimulated, then that mechanism contributes a stronger response only because of the stimulus effect. The nervous system does not weight the channel with the strongest signal more heavily. Statistical decision rules are more flexible. The maximum-likelihood rule includes the case where the nervous system weights more heavily those elements that respond more strongly. This will, generally, allow superior performance. The statistical decision rules foster the idea of an active nervous system, where processing may be more intelligently allocated to meet the stimulus conditions.

A second feature of vector models that is attractive because of its simplicity, but is probably unrealistic, is the implicit assumption that noise in discrimination judgments occurs because of factors external to the representation. The vector representation assumes that when two lights have equal vector differences they will be equally discriminable. This is equivalent to supposing the existence of a noise source—a random variable—that is added to the difference vector, causing the nondeterministic behavior. If the added noise is external to the representation and therefore the same for all discriminations, then equal vector differences must lead to equal discriminability. If we complicate the vector model by permitting the noise to vary with the vector difference, then the noise becomes part of the representation and the vector model becomes a statistical model. The incorporation of randomness into the representation is a second reason why statistical models may prove of more value in providing an accurate measure of small color differences.

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