CME 305: Discrete Mathematics and Algorithms

Instructor: Reza Zadeh (rezab@stanford.edu)

HW#1 – Due at the beginning of class Thursday 01/21/16

1. Prove that at least one of G and $\overline{G}$ is connected. Here, $\overline{G}$ is a graph on the vertices of G such that two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G.

2. A vertex in G is *central* if its greatest distance from any other vertex is as small as possible. This distance is the *radius* of G.

   (a) Prove that for every graph G

   $$\text{rad } G \leq \text{diam } G \leq 2 \text{ rad } G$$

   (b) Prove that a graph G of radius at most k and maximum degree at most $d \geq 3$ has fewer than $\frac{d}{d-2}(d-1)^k$ vertices.

3. A random permutation π of the set $\{1, 2, \ldots, n\}$ can be represented by a directed graph on n vertices with a directed arc (i, π_i), where π_i is the i'th entry in the permutation. Observe that the resulting graph is just a collection of distinct cycles.

   (a) What is the expected length of the cycle containing vertex 1?

   (b) What is the expected number of cycles?

4. Let $v_1, v_2, \ldots, v_n$ be unit vectors in $\mathbb{R}^n$. Prove that there exist $\alpha_1, \alpha_2, \ldots, \alpha_n \in \{-1, 1\}$ such that

   $$||\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n||_2 \leq \sqrt{n}$$

5. Consider a graph G on $2n$ vertices where every vertex has degree at least n. Prove that G contains a perfect matching.

6. Let $G = (V, E)$ be a graph and $w : E \to R^+$ be an assignment of nonnegative weights to its edges. For $u, v \in V$ let $f(u, v)$ denote the weight of a minimum $u - v$ cut in G.

   (a) Let $u, v, w \in V$, and suppose $f(u, v) \leq f(u, w) \leq f(v, w)$. Show that $f(u, v) = f(u, w)$, i.e., the two smaller numbers are equal.

   (b) Show that among the $\binom{n}{2}$ values $f(u, v)$, for all pairs $u, v \in V$, there are at most $n - 1$ distinct values.

7. Let T be a spanning tree of a graph G with an edge cost function c. We say that T has the *cycle property* if for any edge $e' \notin T$, $c(e') \geq c(e)$ for all e in the cycle generated by adding e' to T. Also, T has the *cut property* if for any edge $e \in T$, $c(e) \leq c(e')$ for all e' in the cut defined by e. Show that the following three statements are equivalent:

   (a) T has the cycle property.

   (b) T has the cut property.

   (c) T is a minimum cost spanning tree.

Remark 1: Note that removing $e \in T$ creates two trees with vertex sets V_1 and V_2. A *cut* defined by $e \in T$ is the set of edges of G with one endpoint in V_1 and the other in V_2 (with the exception of e itself).

8. Prove that there is an absolute constant $c > 0$ with the following property. Let A be an $n \times n$ matrix with pairwise distinct entries. Then there is a permutation of the rows of A so that no column in the permuted matrix contains an increasing subsequence of length $c\sqrt{n}$.

9. At lunchtime it is crucial for people to get to the food trucks as quickly as possible. The building is represented by a graph $G = (V, E)$, where each room, landing, or other location is represented by a vertex and each corridor or stairway is represented by an edge. Each corridor has an associated capacity c, meaning that at most c people can pass through the corridor at once. Traversing a corridor from one end to the other takes one timestep and people can decide to stay in a room for the entire timestep. Suppose all people are initially in a single room s, and that the building has a single exit t. Give a polynomial time algorithm to find the fastest way to get everyone out of the building.