PLOTTING · Scatterplots

$$\begin{array}{c c} & 3 \\ 3 & \begin{bmatrix} | & | & | \\ x_0 & x_1 & x_2 \\ | & | & | \end{bmatrix} \end{array} \qquad 2 \begin{bmatrix} & \\ & \end{bmatrix}$$

Toeplitz.

random system $\rightarrow$ 2-D array, 1-D array x, 1-D array b.

CME 305

Lecture - 16 3/3/2015.

Agenda:

1. Sparsification Guarantee sol. Spielman / Srivastava
2. Graph Isomorphism via spectra.

Spielman - Srivastava $q = O\left(\frac{n \log(n)}{\varepsilon^2}\right)$.

For $i = 1 : q$

e = sample edge one edge from dist. $P_e = \frac{R_e}{n-1}$

add e to H with $\frac{1}{P_e q}$ additional weight

and.

This class prove:

$\forall x \in \mathbb{R}^n$: $(1-\varepsilon) \underbrace{x^T L_G x}_{\text{counts cuts.}} \leq x^T L_H x \leq (1+\varepsilon) x^T L_G x$

$\forall x \in [0,1]^n$

→ **Last Class:**

$\Pi^2 = \Pi$ $\qquad \Pi = B L^+ B^T$ $\qquad im(B) = im(\Pi)$

$\Pi_{e,e} = R_e$ $\qquad \dim(im(B)) = n - \dim(Ker(B))$

$= n - \dim(span\{1_n\})$

? $\sum_e R_e = \sum_e \Pi_{e,e} = n-1$ $\qquad = n-1$

→ Every edge has q chances to be included in H with weight $\frac{1}{P_e q}$

E [weight edge e in H] = # trials · prob. e showing with $\times$ weight of e

$= q \cdot p_e \cdot \frac{1}{q p_e} = 1.$

define

The entire analysis is for a concentration Inequality

d_e = weight of edge e as it shows up in H.

$E[S] = I_m$ $\qquad S = \begin{bmatrix} \ddots & & 0 \\ & d_e & \\ 0 & & \ddots \end{bmatrix}$

$L_H = B^T S B$

$E[L_H] = L_G$

$d_e = \sum \frac{1}{P_e q}$

Rudelson - Vershynin (Rank-1 updates).

$y \in R^m$ y is random m vector from some distribution that satisfies

$E[\|yy^T\|] \leq 1$ and $\|y\|_2 \leq M$

$$E\left[\left\|\frac{1}{q}\sum_{i=1}^{q} y_i y_i^T - E yy^T\right\|_2\right] \leq \min\left(CM\sqrt{\frac{\log(q)}{q}}, 1\right)$$

y picks out columns of π scaled by $\frac{1}{\sqrt{P_e}}$ with prob P_e.

$\|y\|_2 = \frac{1}{\sqrt{P_e}} \|\pi(\cdot, e)\|_2 = \frac{1}{\sqrt{P_e}} \sqrt{\pi(e,e)}$ →

$\pi^2 = \pi = \pi^T \pi$

$\pi^T \pi = \begin{bmatrix} | & | & | & \\ | & | & | & \\ | & | & | & \end{bmatrix} \begin{bmatrix} - & - \\ - & - \\ - & - \end{bmatrix} = \begin{bmatrix} \cdot & \\ & \end{bmatrix}$
$\qquad\qquad \pi^T \qquad\qquad \pi \qquad\qquad \pi$

$= \frac{1}{\sqrt{P_e}} \cdot \sqrt{R_e} = \frac{1}{\sqrt{\frac{R_e}{n-1}}} \sqrt{R_e} = \sqrt{n-1}$

⇓

$M \leq \sqrt{n-1}$

set $q = \frac{a c^2 n \log(n)}{\varepsilon^2}$

$$\Pi S \Pi^T = \frac{1}{q} \sum_{i=1}^{q} y_i y_i^T$$ (unravel summation in d_e)

Rudelson gives absolute error on $\Pi S \Pi$ & $\Pi \Pi$

We want: Relative error between L_G & L_H
(i.e. $B^T B$ and $B^T S B$)

By Rudelson, matrix 2norm (largest singular value)

$$E\left[\|\Pi S \Pi - \Pi\Pi\|_2\right] \le \frac{\varepsilon}{2}$$

By markov, with probability at least half

$$\|\Pi S \Pi - \Pi \Pi\|_2 \le \varepsilon$$

$\iff \forall y \in R^m$ $\quad \frac{|y^T \Pi (S-I) \Pi y|}{y^T y} \le \varepsilon$

$\Rightarrow$ $\underset{m\times 1}{y} \in im(B)$ $\quad \frac{y^T (S-I) y}{y^T y} \le \varepsilon$

$\iff$ $y = B \underset{n \times 1}{x}$ $\quad \to \frac{(Bx)^T (S-I) B x}{x^T \underbrace{B^T B}_{L_G} x} \le \varepsilon$

$x \perp \mathbb{1}_n$

$\iff$ $\forall x \perp \mathbb{1}_n$ $\quad \frac{|x^T L_H x - x^T L_G x|}{x^T L_G x} \le \varepsilon$

for $x \in span\{\mathbb{1}_n\}$ $\quad x^T L_G x = x^T L_H x = 0$

$\Rightarrow$ have guarantee for all $\forall x \in R^m$

What is this useful for?

$\to$ L_H can be computed in $O(m \log(n))$ time.

$\to$ Hiding $\log(\frac{1}{\varepsilon})$ and $\log\log(\frac{1}{\varepsilon})$ factors.

$\to$ achieved by approximating R_e

(summary: random projections onto space of dimension $\log(n)$)
(Johnson-Lindenstrauss Lemma) $\to$ Section in paper: Estimating R_e.

great preconditioner

$\to$ $L_G x = b$ $\leftarrow$ symmetric diagonally dominant (SDD) system

Every SDD linear system can be written as some L_G

$L_H^{-1} L_G$ has constant condition number.

$\Rightarrow$ Solve L_G systems quickly.

$\Rightarrow$ leads fastest known SDD systems.