Lecture - 12 February 17, 2015

→ Topics:
1. Algebraic Graph Theory
   - Approximation algorithms
   - Applications (Machine Learning)

→ Today
- Goemans - Williamson Max-cut 0.878 approx
- Bin packing Intro

CONVEX OPTIMIZATION
→ Linear Programming (Vertex cover)
→ Semi-definite programs
→ More - convex programs

QUADRATIC PROGRAM formulation for max-cut

$\cancel{y^2 = 1}$

indicator (-1 or 1)

$$\max \sum_{(i,j) \in E} \frac{(1 - y_i y_j)}{2}$$

$$y_i^2 = 1 \quad \forall\ i = 1 \ldots n$$

cleaner version

$$\max \sum_{(i,j) \in E} w_{ij} \frac{(1 - y_i y_j)}{2}$$

$$y_i^2 = 1 \quad \forall\ i = 1 \ldots n$$

{Relaxation} $y_i \in R$

$y_i : \longrightarrow y_i \in R^n$. lift y in dimension.

(instead of just being a number, it represents n numbers)

SDP formulation

$$\max \sum_{(i,j)\in E} w_{ij} \frac{(1 - v_i^T v_j)}{2}$$

$$\text{s.t } v_i^T v_i = 1 \quad \forall i = 1, \dots, n$$

$$v_i \in R^n$$

Let $x_{ij} = v_i^T v_j$

{Need to encode that the x_{ij}'s are dot products of some n-dimensional vectors}

$$\max \sum_{(i,j)\in E} \frac{w_{ij}(1-x_{ij})}{2}$$

$\boxed{\text{can solve in polynomial time.}}$

$$\text{s.t } x_{ii} = 1$$

$$\underbrace{X \succeq 0}_{\text{positive semidefinite}} \iff \exists V \text{ s.t. } X = V^T V$$

→ Let $X = [x_{ij}]$ $\quad V = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$

$X = V^T V$

~~make it positive definite~~

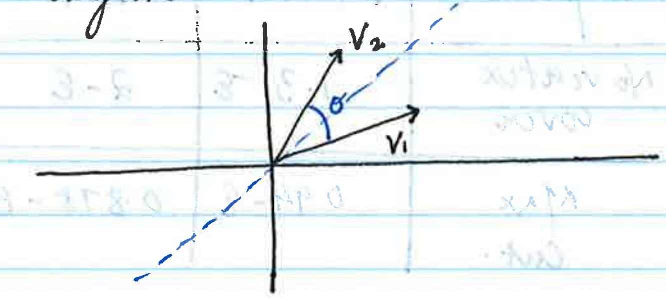
Solve it!

→ Get X.

→ Do cholesky factorization on X to get the v_i's back.

$X = V^T \cdot V$ to get v_i's.

→ Round using random hyperplane through the origin.



Probability

$$Pr[i \text{ is separated from } j] = \frac{\angle(i,j)}{\pi} = \frac{\arccos(v_i^T v_j)}{\pi}$$

$$v_i^T v_j = \underset{1}{\cancel{|v_i|}}\,\underset{1}{\cancel{|v_j|}} \cos\theta$$

$$E[\text{Cut}] = \sum_{(i,j)\in E} w_{ij} \, Pr[(i,j) \text{ is cut}] =$$

$$= \sum_{(i,j)\in E} w_{ij} \frac{\text{Arccos}(x_{ij})}{\pi}$$

(SDP OPT $\geq$ OPT)

$$\frac{E[\text{cut}]}{\text{OPT}} \geq \frac{E[\text{cut}]}{\text{SDP OPT}} = \frac{\sum_{(i,j)\in E} w_{ij} \frac{\text{Arccos}(x_{ij})}{\pi}}{\sum_{(i,j)\in E} w_{ij} \frac{(1 - x_{ij})}{2}}$$

$$\geq \min_{-1 \leq x \leq 1} \frac{\frac{\operatorname{Arccos}(x)}{\pi}}{\frac{1-x}{2}}$$

this number $\rightarrow$ 0.878

Unique Games conjecture

assumption:	P != NP	UGC
No vertex cover	$1.3 - \varepsilon$	$2 - \varepsilon$
Max Cut.	$0.94 - \varepsilon$	$0.878 - \varepsilon$