CME 305: Discrete Mathematics and Algorithms

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Workshop#2 – on Friday 02/21/14

There are some common techniques to approach a discrete math problem. Let us go over these techniques and discuss how to apply them to new problems (e.g., homework assignments).

1. **Proof by induction**: most discrete math problems are associated with a positive integer (e.g., the number of edges or vertices in graphs). It is natural to consider induction.

   Examples in HWs: HW1.2.b, HW1.5.b

   Example If a graph G on n vertices contains no triangle then it contains at most $n^2/4$ edges.

   Solution: We proceed by induction on n. For $n = 1$ and $n = 2$, the result is trivial, so assume that we know it to be true for $n-1$ and let G be a graph on n vertices. Let x and y be two adjacent vertices in G. As above, we know that $d(x) + d(y) \leq n$. The complement H of x and y has $n - 2$ vertices and since it contains no triangles must, by induction, have at most $(n - 2)^2/4$ edges. Therefore, the total number of edges in G is at most

   $$e(H) + d(x) + d(y) - 1 \leq (n - 2)^2/4 + n - 1 \leq n^2/4$$

   where the -1 comes from the fact that we count the edge between x and y twice.

   Example (Turan's theorem) If a graph G on n vertices contains no copy of $K_r + 1$, the complete graph on $r + 1$ vertices, then it contains at most $(1 - 1/r)(n^2/2)$ edges.

2. **Proof by contradiction**: sometimes it is difficult to argue directly and simpler to assume the proposition is false and derive contradiction.

   Examples in HWs: HW1.4.

   Example

3. **Proof by enumeration**: If the number of possible cases are manageable, it is simpler to enumerate all the cases and consider them one by one.

   Example in HWs: HW1.1, HW1.6.

   Example The four color conjecture (now it becomes a theorem) that any map in a plane can be colored using four-colors in such a way that regions sharing a common boundary (other than a single point) do not share the same color. is proved by enumerating the 1,936 reducible configurations (later reduced to 1,476) which had to be checked one by one by computer and took over a thousand hours.

4. **Proof by decomposition**: Discrete objects have rich sub-structures (e.g., any undirected graph is a collection of connected components, any undirected graph without cycles is a collection of trees). By exploiting the properties of sub-structures, it is simpler to prove that the proposition holds for them as a first step, then extend to general case.

Examples in HWs: HW1.1, HW1.3.

Example: If a graph is bipartite if and only if each of its connected components is bipartite.

5. **Proof by construction**: The most creative way to prove a proposition is to construct an example. This is useful when the proposition has the form "there exist ...".

   Examples in HWs: HW2.1, HW2.2, HW2.6

   Example The nonnegative integers $d_1, ..., d_n$ are the vertex degrees of some graph if and only if $\sum d_i$ is even.

   Solution: Necessity: obvious

   Sufficiency: Suppose that $\sum d_i$ is even. We construct a graph with vertex set $v_1, ..., v_n$ and $d(v_i) = d_i$ for all i. Since $\sum d_i$ is even, the number of odd values is even. First form an arbitrary pairing of the vertices in $\{v_i\colon\ d_i\}$ is odd. For each resulting pair, form an edge having these two vertices as its endpoints. The remaining degree needed at each vertex is even and nonnegative; satisfy this for each i by place $\lfloor d_i/2 \rfloor$ loops at v_i.

6. **Other techniques**: 1) By assuming "something" that is as largest (smallest) as possible.

   Example in HWs: HW1.4

   Example A connected graph is Eulerian if and only if every vertex has even degree.

   Solution: The degree condition is clearly necessary: a vertex appearing k times in an Euler tour (or $k+1$ times, if it is the starting and finishing vertex and as such counted twice) must have degree 2k.

   Conversely, let G be a connected graph with all degrees even, and let

   $$W = v_0 e_0 ... e_{l-1} v_l$$

   be a longest walk in G using no edge more than once. Since W cannot be extended, it already contains all the edges at v. By assumption, the number of such edges is even. Hence , so W is a closed walk. Suppose W is not an Euler tour. Then G has an edge e outside W but incident with a vertex of W, say $e = uv_i$. Then the walk

   $$u e v_i v_0 e_0 ... e_{l-1} v_l$$

   is longer than W, a contradiction.

   Example: If every vertex of a graph G has degree at least 2, then G contains a cycle.

   Another Solution: Let A be the largest independent set in the graph G. Since the neighborhood of every vertex x is an independent set, we must have $d(x) \leq |A|$. Let B be the complement of A. Every edge in G must meet a vertex of B. Therefore, the number of edges in G satisfies

   $$e(G) \leq \sum_{x \in B} d(x) \leq |A||B| \leq (|A| + |B|)^2/4 \leq n^2/4$$

One can also find the condition when the equality holds.

2) If the proposition does not hold for any n, then n_0 is the smallest number such that the proposition is not true.

Advice: for a new problem and you have no idea of how to solve it, just think about these techniques **one by one**, you will get somewhere at least. It is also possible that you need to combine several techniques.