CME 305: Discrete Mathematics and Algorithms

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Workshop#2 – on Friday 02/21/14

There are some common techniques to approach a discrete math problem. Let us go over these techniques and discuss how to apply them to new problems (e.g., homework assignments).

1. **Proof by induction**: most discrete math problems are associated with a positive integer (e.g., the number of edges or vertices in graphs). It is natural to consider induction.

   Examples in HWs: HW1.2.b, HW1.5.b

   Example If a graph G on n vertices contains no triangle then it contains at most $n^2/4$ edges. (at least 3 different proofs)

   Example (Turan's theorem) If a graph G on n vertices contains no copy of $K_r + 1$, the complete graph on $r + 1$ vertices, then it contains at most $(1 - 1/r)(n^2/2)$ edges. (at least 2 proofs)

2. **Proof by contradiction**: sometimes it is difficult to argue directly and simpler to assume the proposition is false and derive contradiction.

   Examples in HWs: HW1.4.

   Example

3. **Proof by enumeration**: If the number of possible cases are manageable, it is simpler to enumerate all the cases and consider them one by one.

   Example in HWs: HW1.1, HW1.6.

   Example The four color conjecture (now it becomes a theorem) that any map in a plane can be colored using four-colors in such a way that regions sharing a common boundary (other than a single point) do not share the same color. is proved by enumerating the 1,936 reducible configurations (later reduced to 1,476) which had to be checked one by one by computer and took over a thousand hours.

4. **Proof by decomposition**: Discrete objects have rich sub-structures (e.g., any undirected graph is a collection of connected components, any undirected graph without cycles is a collection of trees). By exploiting the properties of sub-structures, it is simpler to prove that the proposition holds for them as a first step, then extend to general case.

   Examples in HWs: HW1.1, HW1.3.

   Example: If a graph is bipartite if and only if each of its connected components is bipartite.

5. **Proof by construction**: The most creative way to prove a proposition is to construct an example. This is useful when the proposition has the form "there exist ...".

   Examples in HWs: HW2.1, HW2.2, HW2.6

Example The nonnegative integers $d_1, ..., d_n$ are the vertex degrees of some graph if and only if $\sum d_i$ is even.

6. **Other techniques**: 1) By assuming "something" that is as largest (smallest) as possible.

   Example in HWs: HW1.4

   Example A connected graph is Eulerian if and only if every vertex has even degree. (at least 4 proofs)

   Example: If every vertex of a graph G has degree at least 2, then G contains a cycle.

   Example: Another Solution of Triangle Free Graphs

   2) If the proposition does not hold for any n, then n_0 is the smallest number such that the proposition is not true.

Advice: for a new problem and you have no idea of how to solve it, just think about these techniques **one by one**, you will get somewhere at least. It is also possible that you need to combine several techniques.