CME 305: Discrete Mathematics and Algorithms

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1. (5 points) Prove that every tree on n nodes has a vertex cover of size at most $\lceil \frac{n-1}{2} \rceil$.

   Solution. All trees are bipartite (shown in class, but can see by rooting the tree and putting all nodes of even distance from root on one side of graph and all nodes at odd distance from root on other side).

   Each side of the bipartition, call these A and B, will be a vertex cover. So every tree has a vertex cover of size $\min\{|A|, |B|\} \leq \lfloor \frac{n}{2} \rfloor = \lceil \frac{n-1}{2} \rceil$.

2. (10 points) The long-bar-bell graph on n nodes consists of two cliques of size $n/3$ with a chain of length $n/3$ connecting the two cliques. Prove that the long-bar-bell graph has covering time $\Theta(n^3)$ (upper and lower bound needed).

   Solution. The number of edges in each clique is given by $(n/3)(n/3 - 1)/2$. The number of edges in the chain is given by $n/3$. Hence the number of edges in the graph $m = \Theta(n^2)$.

   For the upper bound recall that $C(G) \leq 2m(n-1)$. Hence $C(G) = O(n^3)$.

   For the lower bound recall that $2mR(G) \leq C(G)$. Let u be the left-most node in the chain (the joint between the left clique and the chain) and let v be the right-most node in the chain. Consider that the resistance between u and v is $n/3$. Hence $R(G) \geq n/3$. Therefore:

   $$2m(n/3) \leq 2mR(G) \leq C(G)$$

   And as $m = \Theta(n^2)$ this gives our lower bound.

3. (10 points) Prove that every tree has at most one perfect matching.

   Solution. Consider doing by induction on the number of nodes of the tree. We have for $n = 1$, no perfect matchings exist and for $n = 2$, exactly one perfect matching exists. Now assume for all trees with $\leq k$ nodes, at most one perfect matching exists.

   Now consider any tree on $k + 1$ nodes. There exists some leaf node l, which in any perfect matching must be matched with its parent node (because that is the only edge incident to l). Consider deleting l and its parent and all incident edges to those nodes from the tree. We are left with a forest, in which every tree has $\leq k-1$ nodes. We know by inductive hypothesis that each of these trees has at most one perfect matching, so the original tree has at most one perfect matching (the unique perfect matchings of each tree in the forest and the edge connecting l to its parent). By induction, we are done.

4. (15 points) The SETCOVER problem is as follows: Given a set E of elements and a collection $S_1, \ldots, S_n$ of subsets of E, is there a collection of at most k of these sets whose union equals E? Prove that SETCOVER is NP-Complete.

   Solution. First we establish that SETCOVER is in NP. Given a solution to SET-COVER, the sets S_i that cover E, we can loop over each element in E checking that it appears in some set. Hence we can check YES solutions to SETCOVER in polynomial time. Also note SETCOVER is a decision problem. Therefore SETCOVER $\in$ NP.

Second we establish that SETCOVER is at least as hard as VERTEXCOVER which is known to be NP-Complete. Given a black box for SETCOVER we show we can solve VERTEXCOVER in polynomial time.

Suppose we have an instance of VERTEXCOVER. Define E (in SETCOVER) to be the set of edges in VERTEXCOVER. For each node i we create the set S_i to be all the edges incident to node i. If there are n nodes and m edges these sets can be constructed in $O(nm)$ time.

To determine if there exists a VERTEXCOVER of size smaller than $\bar{k}$, we run SETCOVER on the sets we have just created, with k iterating from 1 to $\bar{k}$. As $\bar{k} \leq n$ this is at most n calls to the black box.

Hence SETCOVER is NP-hard. As SETCOVER $\in$ NP, it is NP-complete.

5. (10 points) Prove that a graph with minimum degree $n/2$ must have effective resistance $O(1)$.

   Solution. Consider two nodes u and v. I claim that if u and v are not adjacent then there exists a node w such that both u and v are adjacent to w.

   Suppose there does not exist such a w. So the neighborhoods of u and v are disjoint. As u and v each have degree $n/2$ or more, the union of their neighborhoods, and counting u and v, is at least $n+2$ nodes. This creates a contradiction.

   Hence either u and v are adjacent or there exists a path of length 2 between them. The effective resistance between u and v is at most the length of the shortest path between them. Hence $R_{u,v} \leq 2$.

   The above holds for all choices of u and v. Hence $R(G) \leq 2$ so $R(G) = O(1)$.