A Brief Introduction to Special Relativity

Gary Oas
Head of Physics - EPGY
oas@epgy.stanford.edu

June 9, 2009

1 The Need for a New Theory

The following concepts of classical physics are necessary to understand the reason for the necessity of special relativity.

[Kin] Rules of kinematics.

[N3] Newton's three laws of motion.

[PGR] Principle of Galilean Relativity.

[Max] Maxwell's Theory of Electromagnetism.

The last two exhibit a conflict that can only be resolved by introducing relativity.

[Kin] Kinematics

These are the rules that quantitatively describe how objects move. The principle measurements that one can make are positions $\vec{x}$ and times t labeling these positions ('the object is at position $\vec{x}$ at time t'). Out of these basic measurements on can describe objects' motions by velocity $\vec{v} = \frac{\text{change in position}}{\text{change in time}} = \frac{\Delta \vec{x}}{\Delta t}$ and acceleration $\vec{a} = \frac{\text{change in velocity}}{\text{change in time}} = \frac{\Delta \vec{v}}{\Delta t}$.

[N3] Newton's 3 laws of motion, Dynamics

Newton's three laws of motion (1687) ties together kinematics with dynamics. Dynamics is concerned with quantities that describe why objects move the way they do. Mass is a measure of an object's inertia in classical physics and force is an entity that produce a change in the motion of an object. Newton specified precisely the relation between these entities and the kinematical entities, relations that had been debated for centuries prior.

1^{st} An object will continue in its state of motion or rest unless acted upon by an external net force. An object's momentum ($p = mv$) will remain the same unless an outside force acts upon it.

2^{nd} The rate of change of an object's momentum is directly proportional to the net force acting upon the object: $\vec{F} = \frac{d\vec{p}}{dt}$.

3^{rd} For every force that acts on an object there is an equal and oppositely directed force acting on the entity supplying the original force.

[PGR] Principle of Galilean Relativity

The laws of mechanics ([N3]) are the same in every inertial reference frame.

or

It is not possible to distinguish an IRF with a frame that is at rest.

As first noted by Galileo (and hence the name) the physics occurring in a frame of reference fixed to the surface of the Earth is identical to that which occurs in a frame that moves with respect to the first frame with a constant velocity (this assumes that neither frame is accelerating). Put another way, there is no way to determine whether a frame of reference is at rest or moving with constant velocity by performing mechanical physics experiments. The way to convince yourself of this principle is to perform some simple mechanics experiment the next time you are on an airplane. If the plane has reached cruising altitude and the air is smooth, toss an object into the air, swing an object on the end of a string, or pour some scalding hot coffee into a cup. Though you are traveling hundreds of miles an hour faster than when on the ground, the behavior of these experiments will be exactly as when done on the ground.

[Max] Maxwell's Theory of Electromagnetism

Maxwell's Theory of electromagnetism is one of the most important achievement in physics that unified various phenomena into one theory. In short, Maxwell's four equations completely describe the behavior of electric (E) and magnetic (B) fields. The significant result of this is that, from the second two equations:

A time-varying E field can produce a B field and a time-varying B field can produce an E field.

Maxwell realized, if done appropriately, this process can produce an *electromagnetic wave*. Maxwell's equations leads to a relation describing this wave. Important here is that

the speed of an electromagnetic wave is determined by the theory and is equal to $c = 3 \times 10^8 \frac{m}{s}$***, the same as light***.

Conflict between [PGR] and [Max]

It soon becomes clear that there is a fundamental conflict between Maxwell's theory of electromagnetic waves and the Galilean Principle of Relativity ([Max] and [PGR]). With Maxwell's discovery that the fundamental relations of electric and magnetic fields set the speed of electromagnetic waves to one speed, the principle that all inertial frames that move with constant speed are equivalent is rendered untenable. For, if in one IRF the speed of light is measured to be the usual $c = 3 \times 10^8 \frac{m}{s}$, in another IRF that moves at a different speed will yield a different speed for light. This is in violation of [PGR] if extended to electromagnetic phenomena. It becomes more dramatic if one envisions two frames, one with light traveling at c and another moving along with the light beam (where one would conclude it is at rest).

There are several thought experiments that can be developed that highlight this fundamental conflict (see Appendix). ***Something needs to be done to resolve this.***

---

2 The Theory of Special Relativity

In Einstein's 1905 paper[**?**] that introduced special relativity, he basically extends the [PGR] to include all physical laws, not just mechanics[1]. In so doing, the previous conflict between [PGR] and [Max] is eliminated immediately (basically he said that [Max] is correct and [PGR] and [Kin] need to be modified).

[PSR] The ***Principle of Special Relativity*** is nothing more than the extension of [PGR] to ***all*** physics (not just mechanics). This is a natural suggestion in hindsight. If [PGR] works for mechanics, why should it not work for all laws of physics? Indeed, it would seem unnatural

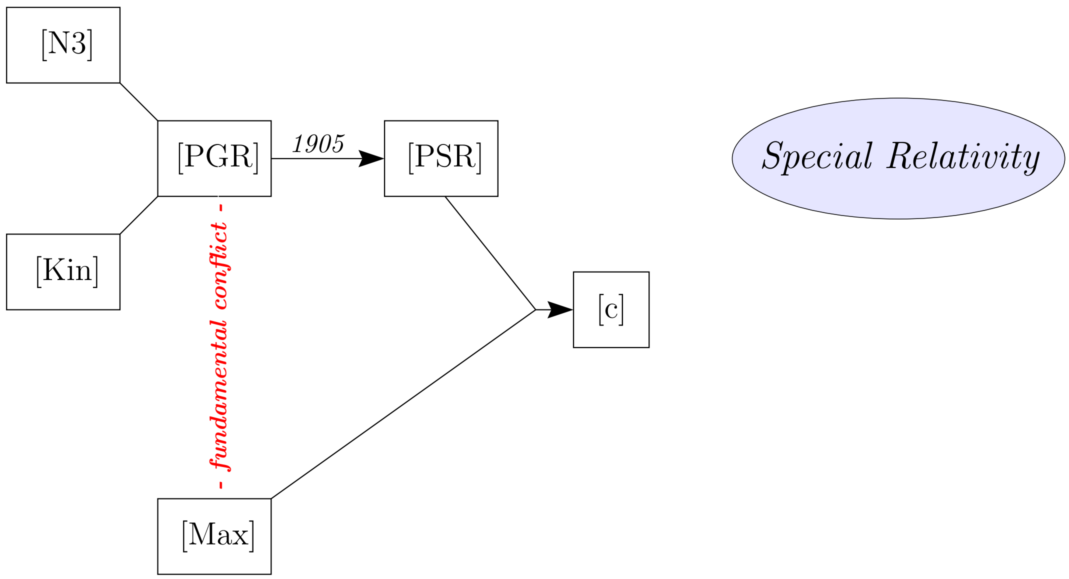
[1] In Einstein's paper he proposes two principles, [PSR] and [c]. However, [c] is a result of [PSR] and [Max]. There is some debate about whether these are two independent principles or not. Here, we take the view that [c] is a natural result of considering [Max] under [PSR].

to have certain laws of physics be invariant in different IRFs but not others (indeed one could probably coax a logical inconsistency).

This principle immediately clarifies the conflict with [Max], for now [Max] is valid in all IRFs. This leads immediately to an entirely new (and radical) prediction:

[c] ***Invariance of the speed of light in all IRFs***. Since [Max] states that electromagnetic waves travel at $c = 3 \times 10^8 \frac{m}{s}$, *from the theory*, and this theory must be the same in all IRFs, then it must be that the speed of light, c, is the same in all IRFs.

This latter result is sufficient to effect the change of all kinematics and dynamics.



---

3 Special Relativistic Kinematics

Up to now we have been concerned with why classical must be changed. Now we want to see what the consequences of [PSR] leads to. In what follows we do not need to be concerned with [PSR] but rather the resultant principle [c]. From this one principle and one additional basic principle, the nature of space and time will be seen to be entirely different than what was before. The additional principle is one that is taken for granted in classical physics but must now be narrowed and stipulated precisely. Its justification will be elaborated in an appendix as a necessity to maintain logical consistency.

[PIE] Principle of the Invariant Event. A minor principle, assumed in classical physics: ***what occurs in one frame (a fact) is the same in ANY other frame***, restricted here to events.

What is new here is the restriction to observations that have no extent in time or space. Only *events* occurring at one instant of time and one position must be seen as the same. Thus, observations like electric charge, number of molecules, two objects in contact, a bomb having been detonated, the structure of DNA, these must be observed to be the same. Observations that require time or require a ruler to measure, like length or size of an object, speed, color, the principle can not say anything.

[RV] Principle of reciprocal velocity. This is a simple principle that holds in [PGR] and and still holds in relativistic scenarios. Consider two IRFs that move with respect to each other. If frame A observes frame B to pass by with speed v, then frame B must observe frame A to pass at the same speed but in the opposite direction.

With [c] and [PIE] we can immediately proceed to see how kinematics must change (and become relativistic kinematics). The effects we will discuss to start off with are the following.

[TD] Time Dilation

[LC] Length Contraction

[Sim] Lack of Universal Simultaneity

[AV] Addition of Velocity

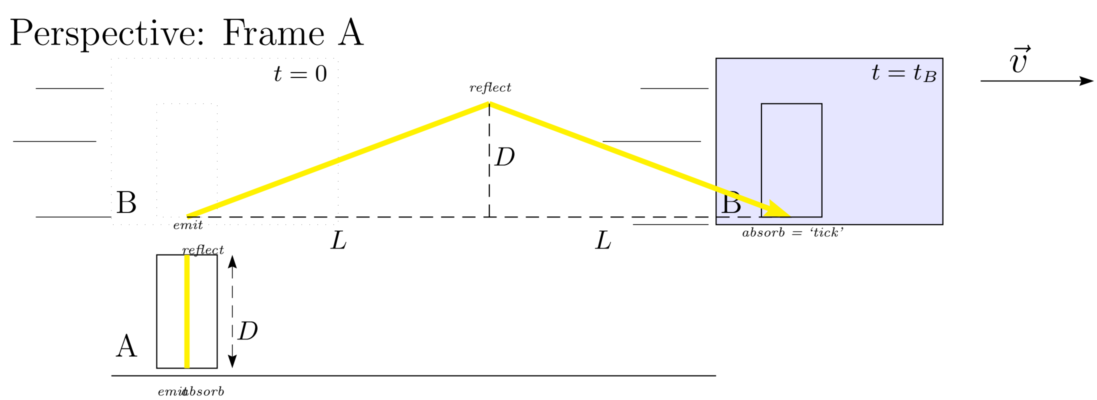
[Noc] Speed of Light is the Ultimate Speed Limit.

[TD] Time Dilation

Consider two inertial frames of reference, A and B and for the moment concentrate on the perspective of an observer in frame A, Alice (she is at rest in this frame). Frame B moves from left to right at a constant but high speed. Observer B, Bob, stands within this frame (you may consider it to be a rocketship). They both have in their possession extremely precise and identical lightclocks. A lightclock is a device to measure time by having a pulse of light travel up to a mirror and return to a detector, this being one 'tick' of the clock. As shown in the image, each have the light clock orientated perpendicular to their relative velocity. If at rest next to each other they tick at the same rate. Let's see what happens when they are in relative motion.

Lightclock A as seen by Alice: Since the roundtrip is $2D$ and light travels at speed c, the time between each tick of clock A is (from $time = \frac{distance}{speed}$), $\Delta t_A = \frac{2D}{c}$.

But now note that Alice observes the pulse within lightclock B to travel a greater distance, since it is moving to the right (see figure).



Lightclock B as seen by Alice: The distance that Alice sees the light pulse to travel can be read of the diagram to be $x = 2\sqrt{D^2 + L^2}$. And with [c] the speed of this light pulse *is exactly the same as for her own lightclock*, thus the time between ticks is $\Delta t_B = \frac{2\sqrt{D^2+L^2}}{c}$ ***which is clearly greater than for her own clock!*** Since the time between ticks is greater for light clock B then, as Alice observes, it runs at a slower rate than her own clock. This is ***time dilation***; time runs slower within frames that are in motion.

Note well that *who is in motion* is a relative term. From Alice's perspective she is in motion and Bob is at rest, while from Bob's perspective Alice is in motion and he is at rest. Both views are correct from [PSR].

Lightclocks as seen by Bob: Thus analyzing this situation from Bob's perspective you get the same result; lightclock B ticks are $\Delta t'_B = \frac{2D}{c}$ seconds long while lightclock A (which moves with equal speed in the opposite direction, from [RV]) has ticks that are $\Delta t'_A = \frac{2\sqrt{D^2+L^2}}{c}$ long.[2] The scenario is symmetric.

[2] We use unprimed variables to represent frame A (t_A, t_B) and primed variables to represent frame B (t'_A, t'_B).

To get a more quantitative grasp on this effect we can do some algebra to give a more general relation to the two time rates. Note that (from Alice's perspective) lightclock B moves to the right at speed v and takes a time Δt_B to tick. From the figure we see that the length $2L$ is equal to the speed of B times the time it takes: $2L = v\Delta t_B$. So we have (noting $2D = c\Delta t_A$).

$$\begin{aligned}
\Delta t_B &= \frac{\sqrt{4D^2 + v^2\Delta t_B^2}}{c} = \frac{\sqrt{c^2\Delta t_A^2 + v^2\Delta t_B^2}}{c} = \sqrt{\Delta t_A^2 + \frac{v^2}{c^2}\Delta t_B^2} \\
\Delta t_B^2 &= \Delta t_A^2 + \frac{v^2}{c^2}\Delta t_B^2 \\
\Delta t_B^2(1 - \frac{v^2}{c^2}) &= \Delta t_A^2 \qquad (1)
\end{aligned}$$

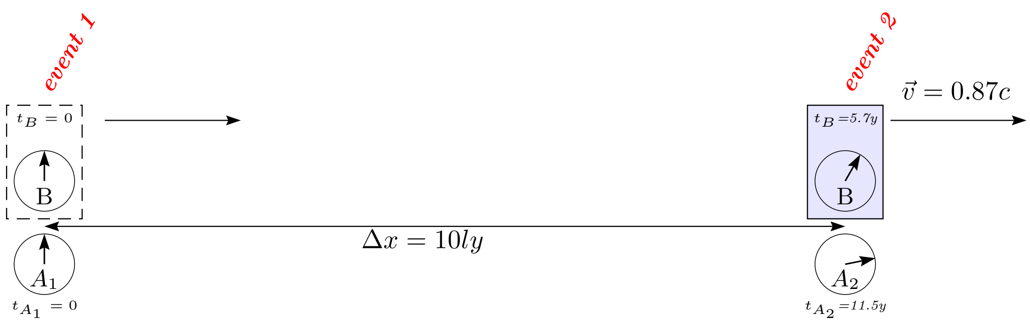
Cleaning up this last expression and solving for Δt_B we have

$$\Delta t_B = \frac{\Delta t_A}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad (2)$$

This is the quantitative formula that tells by how much clocks run slow. The term is so important, and recurs many times in special relativity, that it is given a special symbol: $\gamma \equiv \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$. (So that the [TD] formula reads $\Delta t_B = \gamma\Delta t_A$). Note the extreme limits of this expression: when $v = 0$ we get $\Delta t_B = \Delta t_A$ as we expect (this is our classical result); and when $v = c$ we get that $\Delta t_B = \frac{1}{0} = \infty$, this means that clock B is *never* seen to tick in frame A. In the following discussion we will use a speed where time dilation makes the rate $\frac{1}{2}$ of the rest frame clock, this occurs at a speed of 87% of c, or $\frac{v}{c} = 0.87$ (and $\gamma = 2$).

[LC] Length Contraction

Returning to our two frames, suppose that frame B is centered on a rocket ship that is observed to travel at $v = 87\%$ of c (or $0.87\frac{lightyears}{year}$) in frame A between two points, A_1 and A_2, that are 10 light years apart.

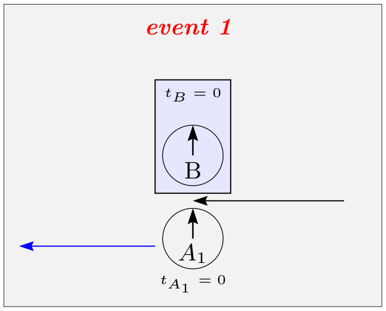


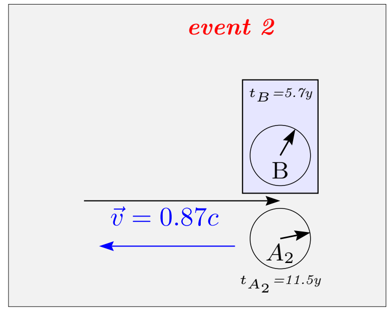
Let's tabulate what the observers in frame A determine for the ship.

- The clock on board the ship is progressing at $\frac{1}{2}$ the rate of their own clock by [TD]. ($\gamma = 2$).
- The ship travels 10 light years. (We'll call this distance Δx.)
- The ship takes a time $\Delta t = \frac{\Delta x}{v} = \frac{10ly}{0.87\frac{ly}{y}} = 11.5y$ to travel 10 ly.
- The clocks onboard the ship only register $\frac{\Delta t}{\gamma} = 5.7y$ to have passed during this trip.

The two important events here are <u>event 1</u>, the departure, where the hand of clock on the ship points to $0y$, and <u>event 2</u>, the arrival, where the hand on the clock on board the ship is reading $5.7y$. These are facts that all must agree on by [PIE].

Now consider the view from frame B, which observes frame A to be traveling at 87% of c in the opposite direction by [RV]. On board the spaceship the observer passes the first event (departure) at time $0y$ and arrives time $5.7y$.





- The clocks in frame A are progressing at $\frac{1}{2}$ the rate of their own clock by [TD]. ($\gamma = 2$).
- The ship takes a time of 5.7 y to reach the second event.
- The distance between the points as observed in B is $\Delta x' = v\Delta t' = 0.87\frac{ly}{y} \times 5.7y = 5ly$.
- (The clocks in A only register $\frac{5.7y}{2} = 2.85y$ to pass. *We'll leave this result for the next section*).

The third item is the one of interest here. The distance between the two points is 10 ly in frame A but is only measured to be 5 ly in frame B - this is the effect of *Length Contraction*. Objects that are moving ***are*** shorter in the direction of travel then when they are viewed at rest. Again, since observer B experiences a shorter time to lapse between the two points, and the relative speeds are seen to be the same, then it must see a shorter distance between them.

[Sim] Lack of Universal Simultaneity

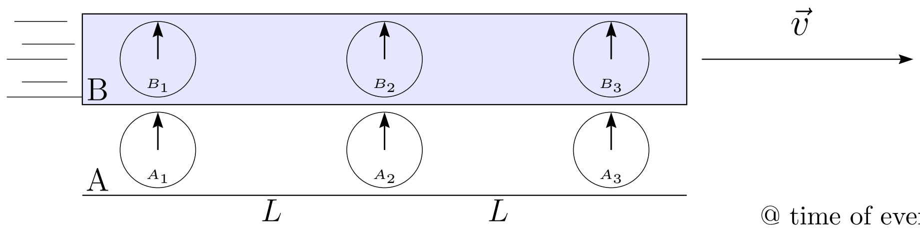
In the previous section we observed that frames that observed to move experience time dilation and length contraction but the last item seemed a bit odd.

- The clocks in A only register $\frac{5.7y}{2} = 2.85y$ to pass.

But we know from [PIE] that the clock at event 1 (call it A_1) reads $0y$ when the ship passes and the clock at event 2 (A_2) reads $10y$ when the ship arrives, that seems to disagree with this last result. There is an underlying assumption here that the two clocks are synchronized from the viewpoint of B - they are not.

To convince you of this, consider the two frames as before but each with three clocks spread out in the direction of travel. They are arranged ahead of time such that, from the perspective of frame A, at one instant of time (say time $t = 0$) all of the clocks read the same value and the two trios line up (see figure). This defines three *simultaneous* events, which we can call 1,2, and 3 (event 1 corresponding to when A_1 and B_1 are at the same location, etc.).

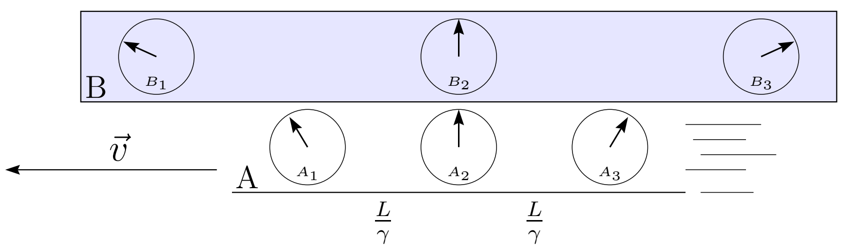
Perspective: Frame A



@ time of events 1,2, and 3

Let's now change perspective and see how these series of events appear in frame B. First, from B's perspective, the clocks in frame A will appear closer together (by a factor of γ) since they are moving, [LC]. Second, the clocks in frame B, now being at rest, must be farther apart than they are observed to be in frame A. (I.e. they are now $L' = \gamma L$ apart). The view at event 2 in frame B is shown in the next figure.

Perspective: Frame B



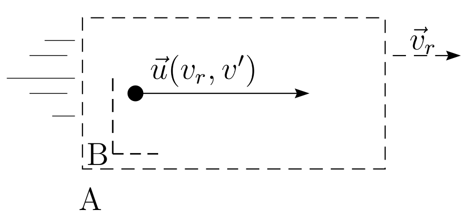
@ time of event 2

Notice that in this frame it is physically impossible to have the events 1,2, or 3 occur at the same time. What are three simultaneous events in frame A are three events that occur at different times in B. This is a key result in relativity and often goes under the overly simplistic term *simultaneity*: what is simultaneous in one frame need not be simultaneous in a different frame, moving with respect to the first. Here we will call it lack of universal simultaneity [Sim].

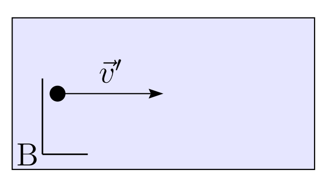
The other important point to note is that from frame A's perspective, all of the clocks were synchronized, yet in frame B no two clocks remain synchronized. Thus we see that simple processes like setting up a series of synchronized clocks is a non-trivial matter under relativistic scenarios. In the image (at event 2) event 3 has already occurred (thus its clock is past $t_{B_3} = 0$) and event 1 has not yet occurred.

[AV] Addition of Velocities

In this section we want to see how velocities known in different frames add together. Consider the perspective of frame A, in which a small ball is observed to travel from left to right at speed $\vec{u}$. This ball was thrown forward by an observer in frame B at speed $\vec{v}'$ (as measured in frame B). The frame (rocketship say) is traveling at speed $\vec{v}_r$ (as measured in frame A). How do you combine these two velocities to obtain $\vec{u}(v_r, v')$? For classical physics it is simple, you merely do the vector some of the two velocities (in this case $u(v_r, v') = v_r + v'$). What is this relation in a relativistic scenario?



Perspective: Frame A



Perspective: Frame B

We will not prove the relation but will find it by inspection (to save time). The addition of velocity formula must obey 5 constraints, listed below. The first three are the same as the classical expression and we can conclude from them that the relation is of the form, $u(v_r, v') = f(v_r, v')(v_r + v')$, where the function f needs to be determined. The last two expressions are the new relativistic rules that stem from [c] (if any of the velocities are c, all must see it at c).

1. $u(0, 0) = 0$
2. $u(0, v') = v'$ and $u(v_r, 0) = v_r$.
3. $u(v, -v) = 0$

4 $u(v_r, c) = u(c, v) = c$.

5 $u(c, c) = c$.

To determine f we just need to insert are result into these constraints. Item 2 gives $u(0, v') = f(0, v')v' = v'$ and $u(v_r, 0) = f(v_r, 0)v_r = v_r$. Thus $f(0, v') = f(v_r, 0) = 1$. Now consider the fourth expression (the last will not be necessary to find the final form but we can use it as verification),

$$u(0, c) = f(v_r, c)(v_r + c) = c \quad \& \quad u(c, v') = f(c, v')(c + v') = c$$

This tells us that,

$$f(v_r, c) = \frac{c}{v_r + c} = \frac{1}{1 + \frac{v_r}{c}}$$
$$f(c, v') = \frac{c}{v' + c} = \frac{1}{1 + \frac{v'}{c}}$$

With a little trial and error you could arrive at the expression, to save time the result is $f(v_r, v') = \frac{1}{1+\frac{v_r v'}{c^2}}$, giving the addition of formula relation,

$$u(v_r, v') = \frac{v_r + v'}{1 + \frac{v_r v'}{c^2}} \tag{3}$$

You should verify for yourself that this expression satisfies the above constraints.

Appendices

Maxwell's Equations

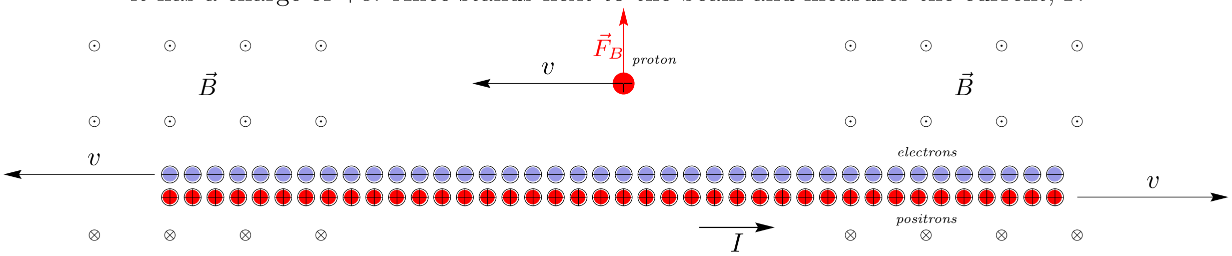
$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$ Gauss law.
$\nabla \cdot \vec{B} = 0$ No Magnetic monopoles.
$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ Faraday's law.
$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$ Ampere's law (extended).

Electromagnetic Wave Equation from Maxwell's Equations

$$\frac{\partial^2 \vec{E}^2}{\partial x^2} + \frac{1}{c^2} \frac{\partial^2 \vec{E}^2}{\partial t^2} = 0$$
$$\frac{\partial^2 \vec{B}^2}{\partial x^2} + \frac{1}{c^2} \frac{\partial^2 \vec{B}^2}{\partial t^2} = 0 \tag{4}$$

Logical Inconsistency of [PGR] and [Max]

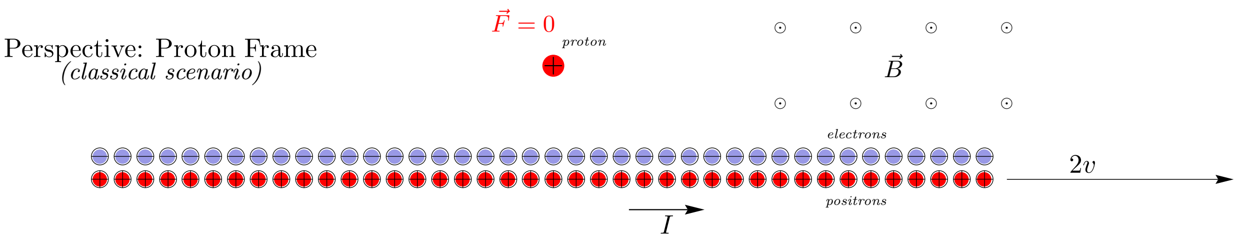
A very long beam of electrons is formed at SLAC consisting of 1.0×10^{10} electrons per meter, all in a line, travelling at $v = 2.0 \times 10^8$ m/s to the left. In the opposite direction (to the right) is another beam of positrons with the same density, travelling at the same speed, and very close to the electron beam. A positron is the anti particle of the electron. It is identical in every way except it has a charge of $+e$. Alice stands next to the beam and measures the current, I.



From Alice's perspective there is an equal density of charge due to the electrons and positrons, being of opposite charge, the net charge is zero. If there is no electric charge in this region (due to the beams) there is no electric field (E) and no electric force if a charge is placed near the beam. There is a current however, and thus a magnetic (B) field that circulates around the beam (and is out of the page above the beam).

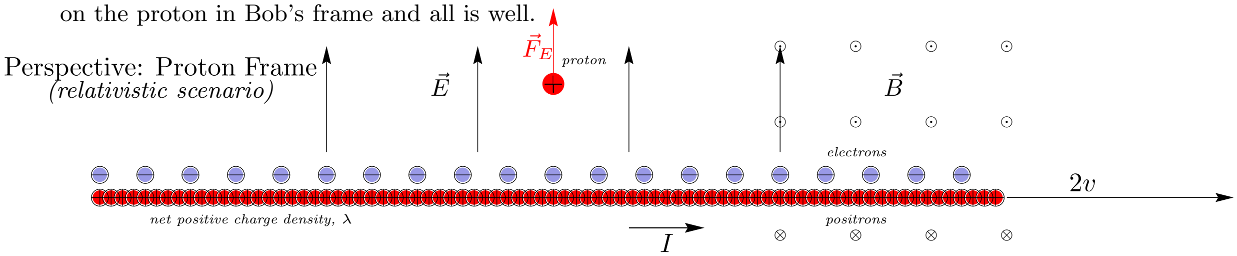
Now consider a proton passing near the beam, parallel to it (to the left) at the same velocity as the electrons. Again, since there is no E field there is no electric force acting on this proton but there is a magnetic force upon it. The nature of the magnetic force is a little complicated ($\vec{F} = e\vec{v} \times \vec{B}$ in case you're interested) and with the field out of the page at the point in the figure and the velocity to the left, the magnetic force on this proton is up, away from the beam. Thus the proton will accelerate away from the beam.

Let's see what happens in a frame that moves with the same velocity as the proton, thus in this frame it is at rest. There are still equal and opposite charge densities in the beam, thus no E field and no electric force. Even though the velocities of the particles in the beam are different now (electrons at rest, positrons moving at twice the speed) the current in the beam is the same as it was in Alice's frame. Thus there is still a magnetic field at the location of the proton, out of the page. However now there is no magnetic force acting on the proton because it is at rest. The magnetic force only acts on charges *that are moving through the B field* (see previous equation). Thus there is ***no*** force acting upon the proton and it just stays put in this frame.



What's the problem? Well a logical inconsistency results from the two different behaviors in the frames. In Alice's frame the proton accelerates away from the beam and after a long while will be very far from it, whereas in the proton's frame (frame P) it will still be next to the beam. This is a problem. To see why, place a detector just above the proton which is connected to a very large bomb. As viewed in the proton's frame, the proton will never hit the detector and there will be no explosion. In Alice's frame, the proton hits the detector destroying SLAC and, sadly, Alice. If we have an observer riding along the proton (call him Bob) in a rocketship, he would never see the bomb go off, could turn off the beam, decelerate and come to rest next to Alice. During this time there is no mechanism that allows the bomb to detonate. Thus he talks to Alice about this experiment. But, from someone (Andrea) faraway who watches this experiment and is in Alice's frame, a terrible explosion occurs. Thus when she joins Bob, something that can not happen occurs. Andrea and Bob are at rest with respect to each other and Andrea sees a charred landscape and no Alice but Bob sees a functional SLAC and Alice standing there. This is a logical inconsistency and points to the fact that Maxwell's theory of electromagnetism can not exist alongside the Principle of Galilean Relativity (and its related transformation laws).

How does relativity fix this? Well, [LC] saves the day. By changing the lengths, the charge densities of the electrons will change (in going to Bob's frame the electron's will be further spaced apart and the positrons will be closer together thus there will now be a net electric charge). Now there is a electric force on the proton in Bob's frame and all is well.



References

[1] A. Einstein *On the Electrodynamics of Moving Bodies* Ann. Phys. **17** 891- 921(1905).

[2] E. Taylor, J. Wheeler, *Spacetime Physics* Freeman Worth 1992. (This is a classic text that approaches relativity from the geometric (metric) approach.)

[3] T. Moore, *Six Ideas That Shaped Physics: Unit R*, McGraw-Hill, 2003. (The only other introductory textbook that approaches the topic in the modern geometrical approach.)