Asymptotics of Random Perfect Matchings on Rail Yard Graphs

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Perfect matching: definition

- $G = (V, E)$: $|V| < \infty$, $|E| < \infty$
- Dimer configuration (perfect matching): subset of edges such that each vertex is incident to exactly one edge.
- Edge weight: $w : E(G) \to \mathbb{R}^+ \cup \{0\}$;
- $\mathrm{P}(M) = \frac{1}{Z} \prod_{e \in M} w(e)$;
- Partition function $Z = \sum_M \prod_{e \in M} w(e)$.
- When $w(e) = 1$, $\forall e \in E(G)$, Z is the total number of perfect matchings.

Perfect matching, dimer, and tiling

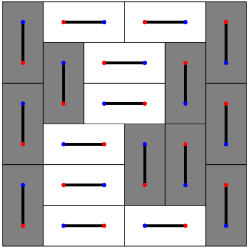


Figure: Perfect matching on square grid and domino tiling

Asymptotic behavior

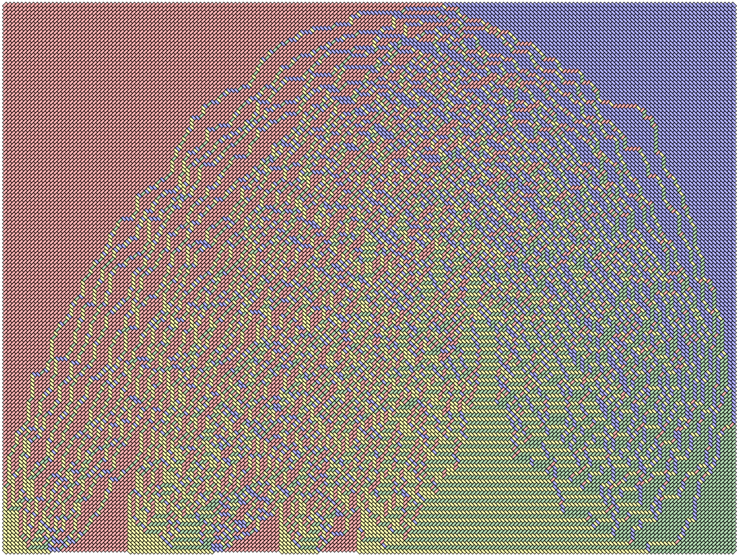


Figure: A domino tiling of a large Aztec rectangle with 1×4 periodic weights, and a boundary partition with parts taking 4 distinct values. It

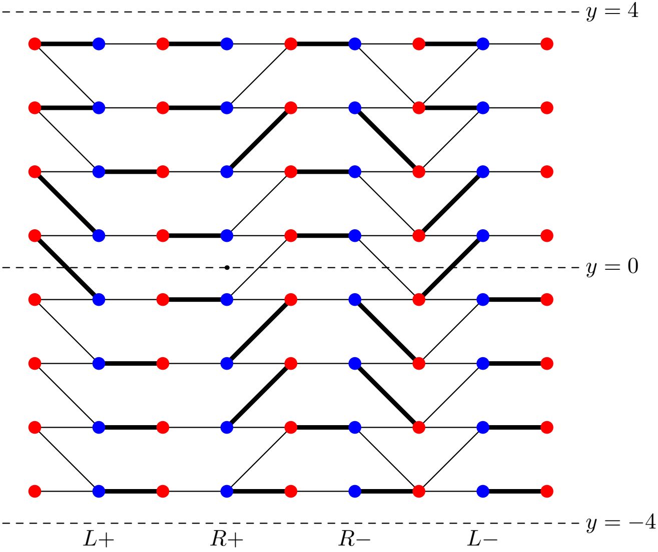
Previous work: Kastelyn-Temperley-Fisher approach

- (Kastelyn-Temperley-Fisher, 1961) Number of perfect matchings on a plane graph can be computed by the Pfaffian of a weighted adjacency matrix
- (Kenyon 1997-2001) Local statistics of perfect matchings can be computed by a minor of the inverse weighted adjacency matrix; conformal invariance, height fluctuation converges to GFF
- (Cohn, Kenyon and Propp 2001) A variational principle for domino tilings
- (Kenyon, Okounkov 2007) Explicitly solved the variational problem, obtain limit shape and frozen boundary for uniform lozenge tilings in polygon domains
- (Kenyon, Prause 2020) Limit height function can be obtained by the harmonic extension of boundary values with respect to a certain intrinsic coordinate

Previous work: Perfect matchings as Schur processes

- (Okounkov, Reshetikhin 2001) Correlation function of Schur process with application to local geometry of a random 3-dimensional Young diagram
- (Petrov, Gorin, Panova, Novak, Bufetov, Knizel, Dimitrov, Ahn 2012-2018) Perfect matchings on hexagon lattice or square grid with certain boundary conditions: limit shape and height fluctuation, GUE minor processes
- (Borodin and Ferrari 2015) Random tiling on tower graph and Markov chains
- (Boutillier, Bouttier, Chauy, Corteel, Ramassamy, 2015) Dimers on rail yard graph as a Schur process, correlation function

Rail-yard graph



Examples

- hexagon lattice: $LLLLL\dots$ or $RRRRR\dots$
- square grid: $LRLRLR\dots$ or $RLRLRL\dots$

Dimer coverings

- Each inner vertex is incident to exactly one edge in the subset;
- Each left or right boundary vertex is incident to at most one edge in the subset;
- only a finite number of diagonal edges are present in the subset.

Pure dimer coverings

- are dimer coverings;
- Each left boundary vertex is incident to exactly one edge (resp. no edges) if $y > 0$ (resp. $y < 0$).
- Each right boundary vertex is incident to exactly one edge (resp. no edges) if $y < 0$ (resp. $y > 0$).

Partitions

- A partition λ is a non-increasing sequence of non-negative integers:

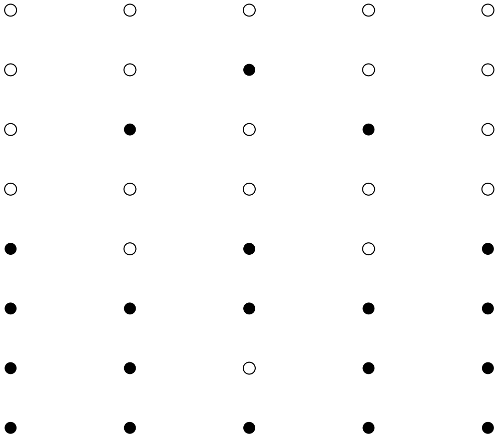
$$\lambda := \lambda_1 \geq \lambda_2 \geq \dots$$

- Counting measure:

$$m(\lambda) = \frac{1}{N} \sum_{i=1}^{N} \delta\left(\frac{\lambda_i + N - i}{N}\right). \tag{1}$$

- Given a dimer covering, associate a particle-hole configuration to each red vertex:
  - If the present edge is on the right (resp. left), associate a hole (resp. particle).
- Associate a partition to each column of red vertices: λ_i is the total number of holes below the ith particle, counting from the top.

Particle-hole configuration and partitions



- $\emptyset \prec (2,0,\ldots) \prec' (3,1,1,\ldots) \succ' (2,0,\ldots) \succ \emptyset.$

Partition function for pure dimer coverings

- Assign diagonal edges adjacent to the ith column of blue vertices weight x_i; and all the other edges weight 1.
- For pure dimer coverings, (Boutillier, Bouttier, Chauy, Corteel, Ramassamy, 2015)

$$Z = \prod_{l \leq i < j \leq r; b_i = +, b_j = -} z_{i,j}$$

where

$$z_{ij} = \begin{cases} 1 + x_i x_j & \text{if } a_i \neq a_j \\ \frac{1}{1 - x_i x_j} & \text{if } a_i = a_j \end{cases}$$

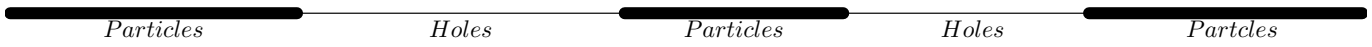
Partition function for dimer coverings on contracting graphs

- Assume the partition on the left (resp. right) boundary is given by λ (resp. $\emptyset$).
- Assume $(a_i, b_i) \neq (R, -)$ for all i.
- Correspondence between partitions on nearest red columns and the blue column in between;
  - $(a_i, b_i) = (L, +)$: left $\prec$ right;
  - $(a_i, b_i) = (L, -)$: left $\succ$ right;
  - $(a_i, b_i) = (R, +)$: left $\prec'$ right;
  - $(a_i, b_i) = (R, -)$: left $\succ'$ right;
- $$Z_{\lambda,\emptyset} = s_\lambda\left(\underline{x}^{(L,-)}\right) \prod_{l \leq i < j \leq r; b_i = +, b_j = -} z_{i,j} \qquad (2)$$
- rational Schur function

$$s_\lambda(u_1, \ldots, u_N) = \frac{\det_{i,j \in 1,\ldots,N}\left(u_i^{\lambda_j + N - j}\right)}{\prod_{1 \leq i < j \leq N}(u_i - u_j)}$$

Overview

- Staircase boundary condition: $\lambda = ((N-1)(\ell-1), (N-2)(\ell-1), \ldots, \ell-1, 0)$; $\ell \in \mathbb{N}^+$ fixed.
- Piecewise boundary condition:



  - $\frac{x_{j,N}}{x_{i,N}} \leq e^{-\alpha N}$, where $i < j$, $(a_i, b_i) = (a_j, b_j) = (L, -)$, $\alpha > 0$.
  - $x_i = 1$, whenever $(a_i, b_i) = (L, -)$.

Staircase left boundary condition and periodic weights

- ℓ: fixed positive integer
- Staircase boundary condition: $\lambda = ((N-1)(\ell-1), (N-2)(\ell-1), \ldots, \ell-1, 0)$.
- $s_\lambda(x_1, \ldots, x_N) = \prod_{1 \le i < j \le N} \frac{x_i^\ell - x_j^\ell}{x_i - x_j}$.
- $x_i = x_{[i \mod n]}$;

Limit shape for uniform bottom boundary condition and periodic weights

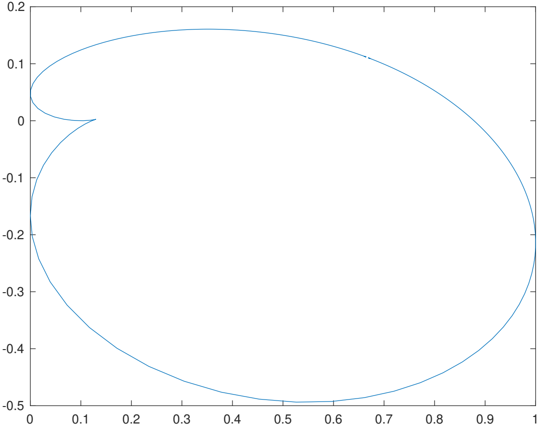


Figure: Frozen boundary: $\ell = 1$, $n = 3$, $(a_1, b_1) = (L, -)$, $(a_2, b_2) = (R, +)$,$(a_3, b_3) = (L, +)$, $x_1 = 3, x_2 = 2, x_3 = 1$

Limit shape for uniform bottom boundary condition and periodic weights

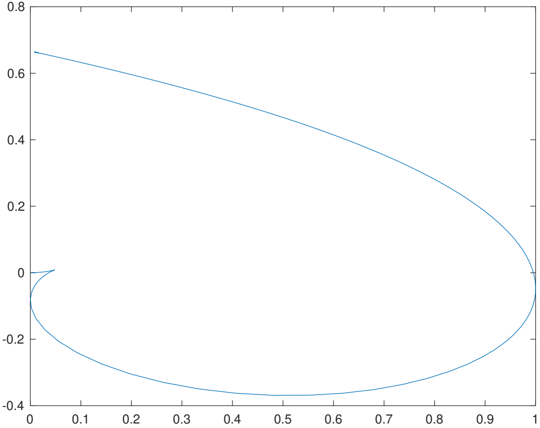


Figure: Frozen boundary: $\ell = 2$, $n = 3$, $(a_1, b_1) = (L, -)$, $(a_2, b_2) = (R, +)$,$(a_3, b_3) = (L, +)$, $x_1 = 3, x_2 = 2, x_3 = 1$

Obtain the limit shape

- ρ^t: probability measure for random partitions corresponding to the dimer configurations on the tth column.
- Schur generating function (definition):

$$\mathcal{S}_{\rho^t, X^{(N-t)}}(u_1, \ldots, u_{N-t}) = \sum_{\lambda \in \mathbb{Y}_{N-t}} \rho^t(\lambda) \frac{s_\lambda(u_1, \ldots, u_{N-t})}{s_\lambda(X^{(N-t)})}.$$

- $V_N = \prod_{1 \leq i < j \leq N} (u_i - u_j)$;
- $\mathcal{D}_p = \frac{1}{V_N} \sum_{i=1}^{N} \left(u_i \frac{\partial}{\partial u_i} \right)^p V_N$;

Proof of the limit shape with staircase left boundary condition

- Schur function is an eigenfunction of the operator $\mathcal{D}_p$.
- $$\mathbb{E}\left(\int_{\mathbb{R}} x^p m_{\rho^t} dx\right)^m = \frac{1}{(N-t)^{m(l+1)}}(\mathcal{D}_p)^m$$
  $$\mathcal{S}_{\rho^t, X^{(N-t)}}(u_1, \ldots, u_{N-t})\Big|_{(u_1,\ldots,u_N)=(x_1,\ldots,x_N)}$$

Proof of the limit shape with uniform bottom boundary condition

- By Schur branching formula,

$$\mathcal{S}_{\rho^t,\underline{x}^{(L,-,>t)}}\left(\underline{u}^{(L,-,>t)}\right) = \frac{s_\lambda\left(\underline{u}^{(L,-,>t)}, x^{(L,-,\leq t)}\right)}{s_\lambda\left(\underline{x}^{(L,-)}\right)} \prod_{i\in[l..t],b_i=+;\, j\in[t+1..r],a_j=L,b_j=-.} \frac{\xi_{ij}}{z_{ij}}.$$

-

$$w_i = \begin{cases} u_i & \text{if } i \in [t+1..r], a_i = L, b_i = - \\ x_i & \text{otherwise} \end{cases}$$

and

$$\xi_{ij} = \begin{cases} 1 + w_i w_j & \text{if } a_i \neq a_j \\ \frac{1}{1-w_i w_j} & \text{if } a_i = a_j \end{cases}.$$

- Analyzing the leading terms,

$$\mathbb{E}\left(\int_{\mathbb{R}} x^p m\left[\lambda_{\rho^t}\right] dx\right) \approx I_p$$

$$\mathbb{E}\left(\int_{\mathbb{R}} x^p m\left[\lambda_{\rho^t}\right] dx\right)^2 \approx I_p^2$$

A formula for Schur function

- $\lambda(N) \in \mathbb{Y}^N$
- Σ_N: symmetric group of N-elements.
- $\sigma \in \Sigma_N$.
- $\Sigma_N^X = \{\sigma \in \Sigma_N : x_i = x_{\sigma(i)}\}$.
- $[\Sigma_N/\Sigma_N^X]^r$: all the right cosets of Σ_N^X in Σ_N
- $\eta_j^\sigma(N) = |\{k : k > j, x_{\sigma(k)} \neq x_{\sigma(j)}\}|$.
- $\Phi^{(i,\sigma)}(N) = \{\lambda_j(N) + \eta_j^\sigma(N), x_{\sigma(j)} = x_i\}$.
- $\phi^{(i,\sigma)}(N)$: the partition obtained by decreasingly ordering elements in $\Phi^{(i,\sigma)}(N)$.

A formula for Schur function

Theorem

(Li 2018) $\lambda \in \mathbb{Y}_N$

$$s_\lambda(x_1, \ldots, x_N) = \sum_{\overline{\sigma} \in [\Sigma_N / \Sigma_N^X]^r} \left(\prod_{i=1}^{n} x_i^{|\phi^{(i,\sigma)}(N)|} \right) \left(\prod_{i=1}^{n} s_{\phi^{(i,\sigma)}(N)}(1, \ldots, 1) \right)$$

$$\times \left(\prod_{i<j, x_{\sigma(i)} \neq x_{\sigma(j)}} \frac{1}{x_{\sigma(i)} - x_{\sigma(j)}} \right)$$

Assumptions on edge weights

- $x_{1,N} > x_{2,N} > \ldots > x_{n,N} > 0$;
- $\frac{N}{n}$ is a positive integer;
- $x_{i,N} = x_{j,N}$ if $[i \mod n] = [j \mod n]$;
- $$\liminf_{N\to\infty} \frac{\log\left(\min_{1\le i<j\le n} \frac{x_{i,N}}{x_{j,N}}\right)}{N} \geq \alpha > 0,$$
- α: a sufficiently large positive constant independent of N.

Limit shape with piecewise boundary condition

- $\overline{\sigma}_0 \in [\Sigma_N/\Sigma_N^X]^r$, such that $x_{\sigma_0(1)} \geq x_{\sigma_0(2)} \geq \ldots \geq x_{\sigma_0(N)}$.
- $\mathbf{m}_i$: the limit of the counting measure for $\phi^{(i,\sigma_0)}$.
- $$s_\lambda(x_1,\ldots,x_N) \approx \left(\prod_{i=1}^{n} x_i^{|\phi^{(i,\sigma_0)}(N)|}\right)\left(\prod_{i=1}^{n} s_{\phi^{(i,\sigma_0)}(N)}(1,\ldots,1)\right) \times \left(\prod_{i<j, x_{\sigma_0(i)}\neq x_{\sigma_0(j)}} \frac{1}{x_{\sigma_0(i)} - x_{\sigma_0(j)}}\right)$$

Frozen Boundary

- The density f_μ of a probability measure μ is given by

$$f_\mu(x) = -\lim_{\epsilon \to 0+} \frac{1}{\pi} \Im(\mathrm{St}_\mu(x + \mathbf{i}\epsilon))$$

- If the density of the limit counting measure of partitions is 0 or 1 at a point, then that point is in the frozen region; otherwise it is in the liquid region.
- Frozen boundary: the boundary of frozen region.
- $\mathrm{St}_{\mathbf{m}}(x) = \sum_{j=0}^{\infty} t^{-j-1} \int_{\mathbb{R}} x^j d\mathbf{m}$

Limit shape with piecewise boundary condition

Figure: Frozen boundary for a rail-yard graph (rotated by 90 degrees) when $n = 2$, $(a_1, b_1) = (a_2, b_2) = (L, -)$, represented by the union of the red curve and the blue curve.

Figure: Frozen boundary for a tower graph lattice with $n = 2$, $|I_2 \cap \{1, 2\}| = 1$ when $(r_1, r_2, r_3, r_4) = (12, 8, 5, 2)$, $c_r = \frac{1}{2}$, represented by the union of the red curve and the blue curve.

Pure Dimer Coverings and Macdonald Processes

- 1. If $(a_i, b_i) = (L, -)$, $\lambda^{(M,i+1)} \prec \lambda^{(M,i)}$;
  2. If $(a_i, b_i) = (L, +)$, $\lambda^{(M,i+1)} \succ \lambda^{(M,i)}$;
  3. If $(a_i, b_i) = (R, -)$, $[\lambda^{(M,i+1)}]' \prec [\lambda^{(M,i)}]'$;
  4. If $(a_i, b_i) = (R, +)$, $[\lambda^{(M,i+1)}]' \succ [\lambda^{(M,i)}]'$
-

$$\Pr(M|\lambda^{(l)}, \lambda^{(r)}) := \frac{1}{Z_{\lambda^{(l)},\lambda^{(r+1)}}(G, \underline{x})} \prod_{\substack{i\in[l..r] \\ (a_i,b_i)=(L,-)}} s_{\lambda^{(M,i)}/\lambda^{(M,i+1)}}(x_i) \prod_{\substack{j\in[l..r] \\ (a_i,b_i)=(L,+)}} s_{\lambda^{(M,j+1)}/\lambda^{(M,j)}}(x_j) \prod_{\substack{i\in[l..r] \\ (a_i,b_i)=(R,-)}} s_{[\lambda^{(M,i)}]'/[\lambda^{(M,i+1)}]'}(x_i) \prod_{\substack{j\in[l..r] \\ (a_i,b_i)=(R,+)}} s_{[\lambda^{(M,j+1)}]'/[\lambda^{(M,j)}]'}(x_j) \quad (3)$$

Macdonald processes when $q = t$: schur process.

Pure Dimer Coverings: Moment formula

- $\lambda^{(l)} = \lambda^{(r)} = \emptyset$.
- $t \in (0,1)$

$$\int_{-\infty}^{\infty} h(x,y) t^{ky} dy = \frac{2}{(k \log t)^2} \times \left[t^{-kl(\lambda^{(x)})} + (1 - t^{-k}) \sum_{i=1}^{l(\lambda^{(x)}])} t^{k(\lambda_i^{(x)} - i + 1)} \right] := \gamma_k(\lambda^{(x)}; t, t)$$

- (Li and Vuletic 2021) Moment formula for $\mathbb{E}_{\Pr} \left[\prod_{j \in I} \gamma_{l_j}(\lambda^{(M,i_j)}; t, t) \right]$, where $i_1 \leq i_2 \leq \ldots \leq i_m \in [l+1..r]$ and $l_1, \ldots, l_m > 0$ are integers.

Pure Dimer Coverings: Asymptotics

- (Li and Vuletic 2021) the rescaled height function converges to a deterministic function

$$\int_{-\infty}^{\infty} e^{-n\alpha\kappa}\mathcal{H}(\chi,\kappa)d\kappa = \lim_{\epsilon\to 0}\int_{-\infty}^{\infty} e^{-n\alpha\kappa}\epsilon h_M\left(\frac{\chi}{\epsilon},\frac{\kappa}{\epsilon}\right)d\kappa \qquad (4)$$

$$= \frac{1}{n^2\alpha^2\pi\mathbf{i}}\oint_{\mathcal{C}}[\mathcal{G}_\chi(w)]^\alpha\frac{dw}{w}, \qquad (5)$$

- (Li and Vuletic 2021) the fluctuations of the unrescaled height function converges to GFF
  - Gaussian fluctuation obtained by verifying the Wick's theorem for Gaussian distribution
  - The liquid region is mapped to $\mathbb{H}$ by an explicit homeomorphism; the height fluctuation is the pull-back of GFF in $\mathbb{H}$.

Example: pyramid partitions

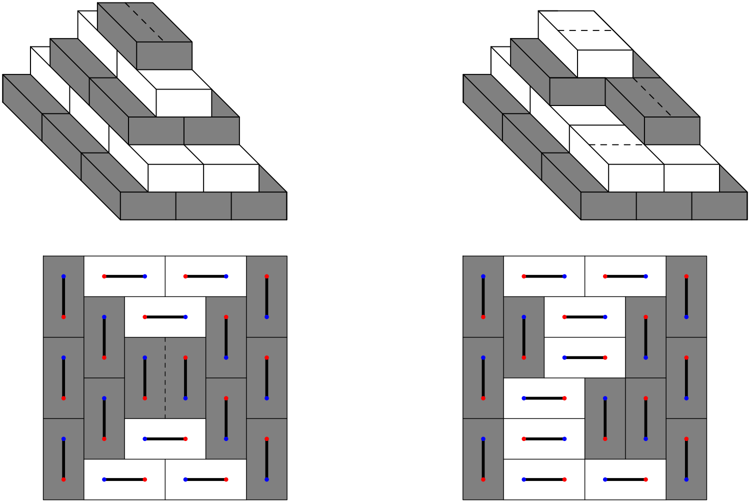


Figure:

Pyramid partition: corresponding rail-yard graph

There is a one-to-one correspondence between pyramid partitions in Λ_s and pure dimer coverings on the rail-yard graph such that for $i \in [-s..s-1]$

$$a_i = \begin{cases} L & i \text{ is odd} \\ R & i \text{ is even} \end{cases} \quad \textit{and} \quad b_i = \begin{cases} + & i < 0 \\ - & i \geq 0. \end{cases}$$

Equivalently, there is a bijection between pyramid partitions in Λ_s and sequences of partitions $(\lambda^{(-s)}, \lambda^{(-s+1)}, \dots, \lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(s)})$ such that

$$\emptyset = \lambda^{(-s)} \prec \lambda^{(-s+1)} \prec' \lambda^{(-s+2)} \dots \prec \lambda^{(0)} \succ' \lambda^{(1)} \succ \lambda^{(2)} \dots \succ' \lambda^{(s)} = \emptyset.$$

Example: limit shape of pyramid partitions

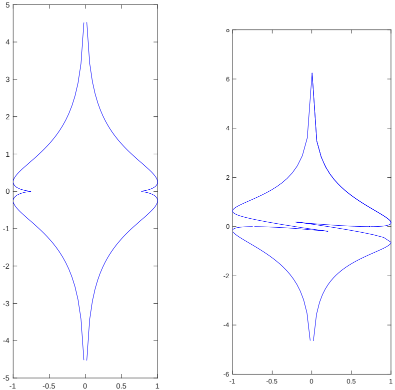


Figure: Frozen boundary of pyramid partitions with parameters $V_0 = -1$, $V_1 = 0$, $V_2 = 1$. The left graph has $\tau_1 = \tau_2 = 1$, the right graph has $\tau_1 = 10, \tau_2 = 1/10$.

Gaussian Unitary Ensemble (GUE)

- GUE: a random Hermitian matrix whose eigenvalues $\epsilon_1 \geq \epsilon_2 \geq \ldots \epsilon_k$ have a distribution $\mathbb{P}_{\mathbb{GUE}_k}$ on $\mathbb{R}^k$ with a density with respect to the Lebesgue measure on $\mathbb{R}^k$ proportional to:

$$\prod_{1 \leq i < j \leq k} (\epsilon_i - \epsilon_j)^2 \exp\left(- \sum_{i=1}^{k} \epsilon_i^2 \right),$$

Dimers near the top and GUE

- $x_i = 1$; if $(a_i, b_i) = (L, -)$
- $x_i = x_{[i \mod n]}$.
- $\lambda^k(N)$ be the signature corresponding to the dimer configuration incident to the $(N-k+1)$th row of white vertices in $\mathcal{R}(\Omega(N), \check{a})$, and for $1 \leq l \leq k$,
- $b_{kl}^N = \lambda_l^k(N) + N - l$.
- $\psi_1 = \int_{\mathbb{R}} x d\mathbf{m}^1$; $\psi_2 = \int_{\mathbb{R}} x^2 d\mathbf{m}^1$
  where $\mathbf{m}^1$ is the limit counting measure of signatures on the top of $\mathcal{R}(\Omega(N), \check{a})$.
-

$$\tilde{b}_{kl}^{(N)} = \frac{\frac{b_{kl}^{(N)}}{\sqrt{N}} - \sqrt{N}\left(\psi_1 - \frac{1}{2} + \frac{1}{n}\sum_{i \in I_2 \cap \{1,\ldots,n\}} \frac{y_i}{1+y_i}\right)}{\psi_2 - \psi_1^2 - \frac{1}{12} + \frac{1}{n}\sum_{i \in I_2 \cap \{1,2,\ldots,n\}} \frac{y_i}{(1+y_i)^2}}, \quad 1 \leq l \leq k.$$

Theorem

(Boutillier and Li 2017) For any fixed k, the distribution of $\left(\tilde{b}_{kl}^{(N)}\right)_{l=1}^{k}$ converges weakly to $\mathbb{P}_{\mathbb{GUE}_k}$ as $N \to \infty$.

- $(q_1, \ldots, q_k) \in \mathbb{R}^k$ be a random vector with distribution $\mathbb{P}$
- $Q = \mathrm{diag}[q_1, \ldots, q_k]$.
- $\mathbb{P}$ is $\mathbb{P}_{\mathbb{GUE}_k}$ if and only if for any diagonal matrix P,

$$\mathbb{E} \int_{U(k)} \exp[\mathrm{Tr}(PUQU^*)] dU = \exp\left(\frac{1}{2}\mathrm{Tr} P^2\right).$$

Thank you!