Integrable (solvable) particle systems for surface diffusion arising from structure and representation of Hecke algebras

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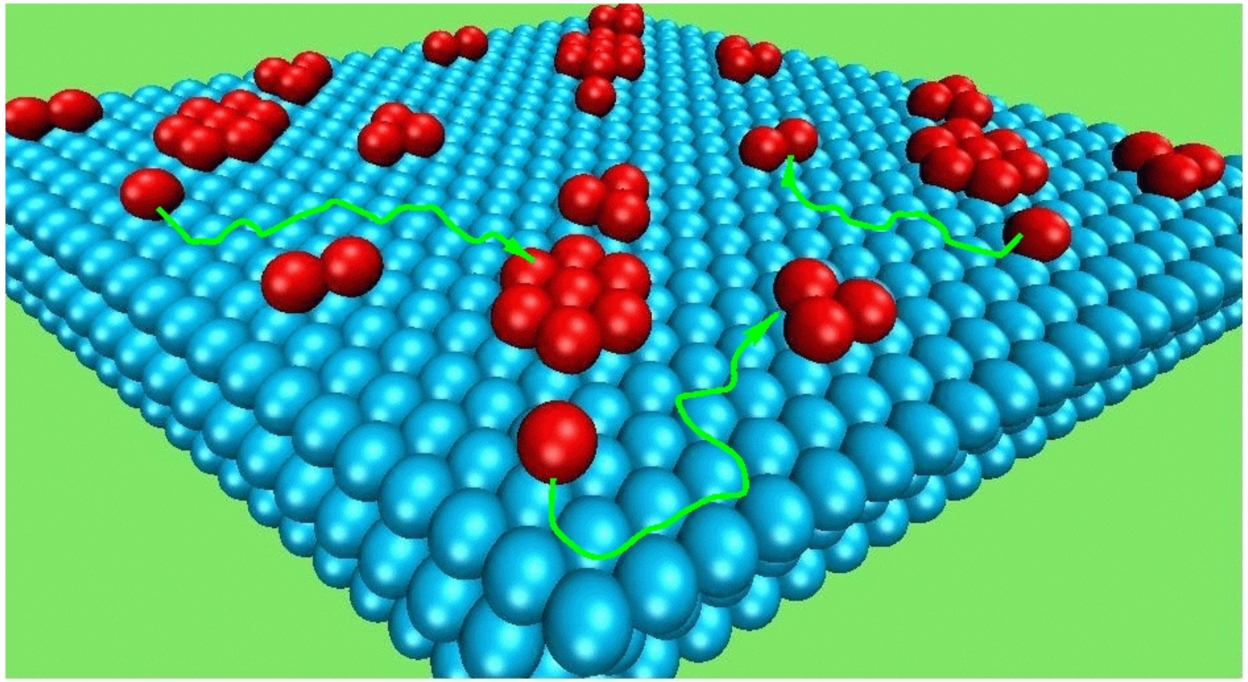
Physical phenomenon: surface diffusion

Surface diffusion involves the movement of particles at solid material surfaces. Hopping or jumping is the most basic mechanism for diffusion of particles. The total number of particles is conserved.



Picture shows a surface with a 2-dimensional substrate, but we will focus on surfaces with **1-dimensional** substrate.

Types of models for surface diffusion

- Continuous models: diffusion processes, stochastic differential systems
- Discrete models: Markov jump processes

A continuous model for surface diffusion and its scaling limit (Savu, 2006)

Kawasaki dynamics

Quadratic form: $$N^4 D_N(f) = \frac{N^4}{2}\int_{\mathbb{R}^N}\sum_{i=1}^{N} a_i(x)\left(\partial_{i+1}f - 2\partial_i f - \partial_{i-1}f\right)^2(x)\frac{e^{-H_N(x)}}{Z^N}dx.$$

Generator: $$N^4 L_N(f) = \frac{N^4}{2}\sum_{i=1}^{N} a_i(x)\left(\partial_{i+1} - 2\partial_i + \partial_{i-1}\right)^2 f + w_i(x)\left(\partial_{i+1} - 2\partial_i + \partial_{i-1}\right)f,$$

SDS: $$dx_i = \frac{N^4}{2}(w_{i-1} - 2w_i + w_{i+1})dt + N^2\left(\sqrt{a_{i-1}}dB_{i-1} - 2\sqrt{a_i}dB_i + \sqrt{a_{i+1}}dB_{i+1}\right), \quad i = 1,\cdots,N$$

Hydrodynamic scaling limit of continuum solid-on-solid model

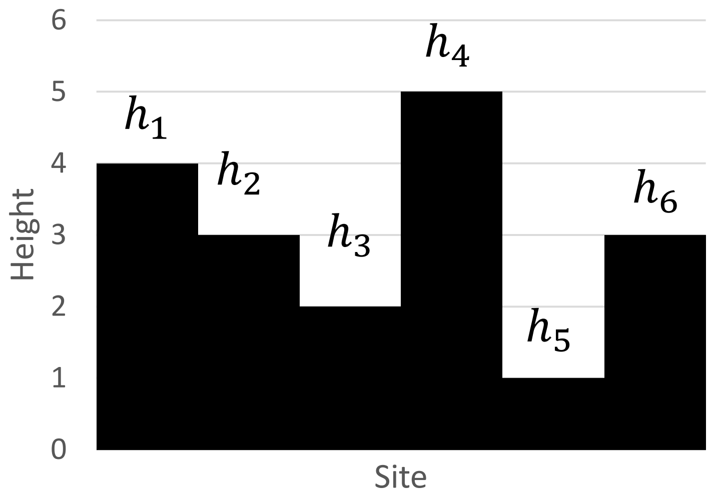
$$\frac{1}{N}\left(x_1(t)\delta_{\frac{1}{N}} + \cdots + x_N(t)\delta_{\frac{N}{N}}\right) \xrightarrow[N\to\infty]{} m(t,\theta)d\theta \quad \forall\theta\in\mathbb{T}\ \forall t\geq 0$$

$$\partial_t m = -\frac{1}{2}\partial_\theta^2(\hat{a}(m)\partial_\theta^2 m)$$

$$m(0,\theta) = m_0(\theta)$$

h_i height of the surface at site i

$x_i = h_{i+1} - h_i$



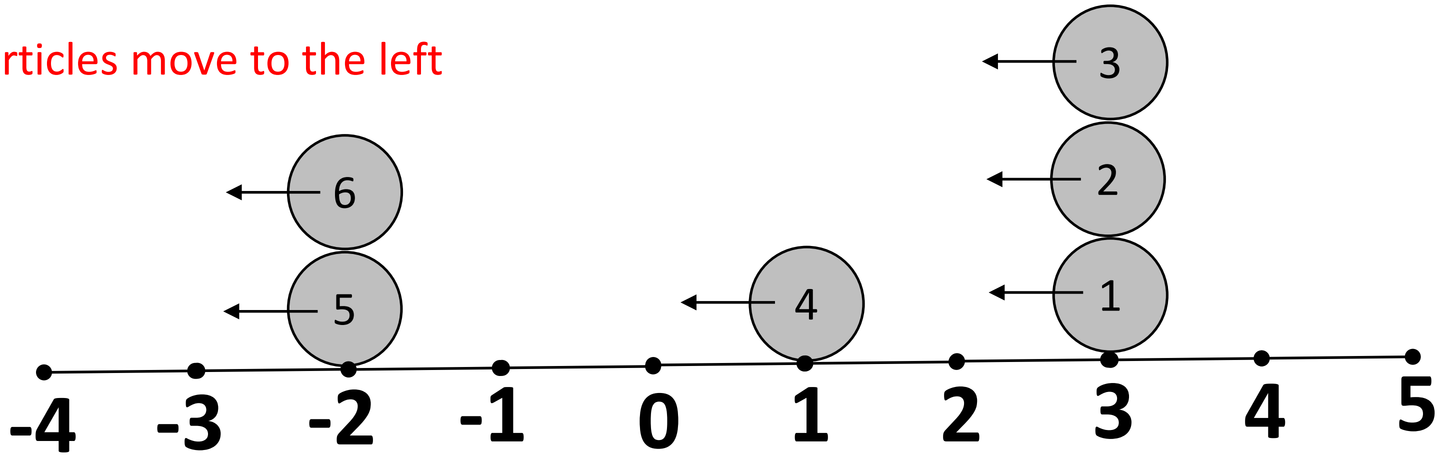
Discrete models for surface diffusion

- Takeyama, Y., A deformation of affine Hecke algebra and integrable stochastic particle systems, *Journal of Physics A: and Mathematical and Theoretical* 47(46) (2014).
- Takeyama, Y., A discrete analogue of periodic delta Bose gas and affine Hecke algebra, *Funkcialaj Ekvacioj*, 57 (2014), 107–118.
- Sasamoto, T., and Wadati, M., Exact results for one-dimensional totally asymmetric diffusion models, *J. Phys. A: Math. Gen.* 31 (1998), 6057-6071.

Pros: integrable system

several particles can leave a site simultaneously

Cons: all particles move to the left



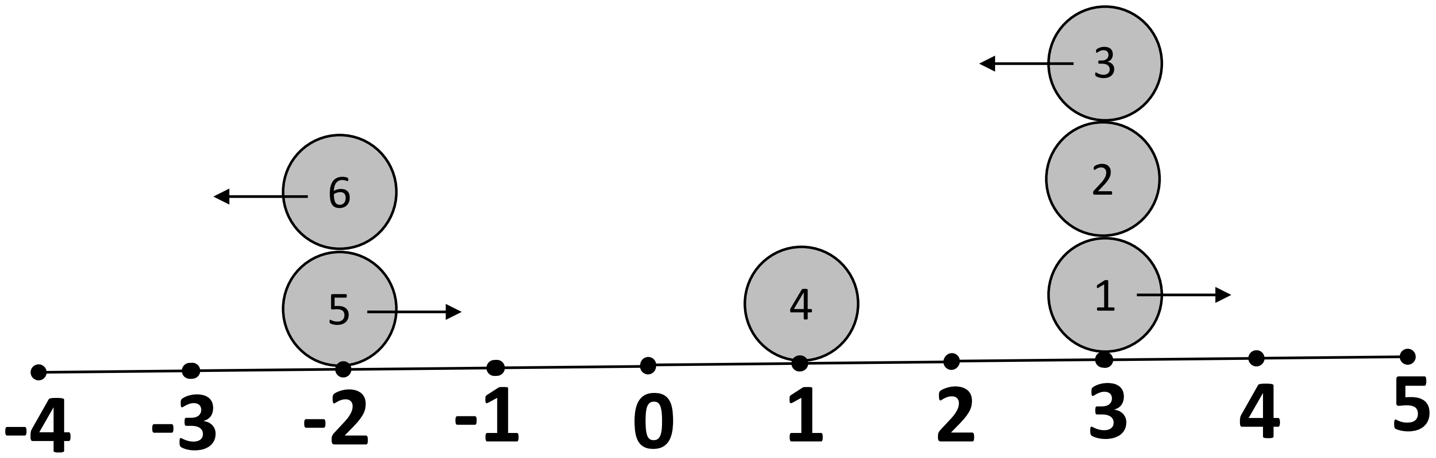
Allowed transitions

- Particle 3 moves left
- P3 left + P2 left
- P3 left + P2 left +P1 left
- Particle 4 moves left
- Particle 6 moves left
- P6 left + P5 left

All particles move left

Can we construct a discrete model for surface diffusion that incorporates left and right particle movement?

Are such models integrable?



Allowed transitions

- P1 right
- P3 left
- P3 left + P1 right
- P5 right
- P6 left
- P6 left + P5 right

Particles move left and right. Two particles move at the same time.

Basic facts on Markov jump processes with continuous time on a countable configuration space, *S*

- The infinitesimal generator / generator matrix / Hamiltonian is an operator $H\colon F(S) \to F(S)$ that is stochastic ($\sum_j H_{ij} = 0, i \in S$).

- The Markov semigroup of transition probabilities $\big(P(t)\big)_{t\geq 0}$ satisfies the Chapman-Kolmogorov: $P(t+s) = P(t)P(s)$ and the Kolmogorov backward / Fokker-Planck equation: $P'(t) = HP(t)$.

- If the eigenfunctions of the Hamiltonian can be calculated then the system is referred to as completely integrable / exactly solvable / or solvable.

Transition probabilities and steady states can be calculated for solvable systems

handout_Lecture10_2019.pdf (nyu.edu) 0903.3097.pdf (arxiv.org)

Basic facts on Markov jump processes with continuous time on a countable configuration space, *S*

Interacting particle system

Markov process on a countable space S

$(X(t))_{t\geq 0}$

$P_{ij}(t) = P(X(t) = j | X(0) = i)$

Markov semigroup of transition probabilities

$(P(t))_{t\geq 0}$

Chapman-Kolmogorov equation: $P(t+s) = P(t)P(s)$

$H_{ij} = P'_{ij}(0)$

Infinitesimal generator / generator matrix / Hamiltonian

H

stochastic matrix: $\sum_j H_{ij} = 0, i \in S$

Kolmogorov backward / Fokker-Plank equation: $P'(t) = HP(t)$

The semigroup of transition probabilities satisfies the Chapman-Kolmogorov and the Kolmogorov backward / Fokker-Planck equations.

Hamiltonian is an operator $H: F(S) \to F(S)$ that is stochastic.

If the eigenfunctions of the Hamiltonian can be calculated then the system is referred to as completely integrable / exactly solvable / or solvable.

One reason is that the transition probabilities can be calculated exactly.

Hamiltonian (Y. Takeyama 2014)

$$(H_{1/2}\, f)(x) = -a(x)f(x) + \sum_{r=1}^{k} \sum_{1 \le j_1 \le \cdots \le j_r \le k} b(x, j_1, \cdots, j_r)\left(\hat{X}_{j_1} \cdots \hat{X}_{j_r} f\right)(x)$$

$$\left(\hat{X}_{j_1} \cdots \hat{X}_{j_r} f\right)(x) = f\left(x - (v_{j_1} + \cdots + v_{j_r})\right)$$

f is a symmetric function defined on the entire lattice L

- $a(x) = \sum_{r=1}^{k} a(x, r) = \alpha\gamma \sum_{r=1}^{k} \frac{[d_r^+(x)]}{1+\beta\gamma[d_r^+(x)]}$
  - $[n] = \frac{1-q^n}{1-q}$
  - $d_r^+(x)$ count of r with the properties $r < p \le k$ and $x_p = x_r$

$$q = 1 + \beta\gamma - \alpha\delta$$

- $b(x, j_1, \cdots, j_r) = \frac{(-\beta\gamma)^{r-1}[r-1]!\, q^{-\frac{r(r-1)}{2}} q^{d_{j_1}^-(x)+\cdots+d_{j_r}^-(x)}}{\prod_{p=0}^{r-1}\left(1+\beta\gamma\left[d_{j_p}^+(x)+d_{j_p}^-(x)-p\right]\right)} \delta_{j_1,\cdots,j_r}$
  - $\delta_{j_1,\cdots,j_r} = 1$ if $x_{j_1} = \cdots = x_{j_r}$, otherwise $= 0$
  - $d_r^-(x)$ count of r with the properties $1 \le p < r$ and $x_p = x_r$

Reminder: simple facts on matrices

- Let A, G, and Δ be n by n matrices such that G is invertible, and Δ admits n linearly independent eigenvectors. Then the following are equivalent
  - $A = G\Delta G^{-1}$
  - $AG = G\Delta$
  - If v is an eigenvector of Δ with eigenvalue μ then Gv is an eigenvector of A with eigenvalue μ.

- A square matrix (A) is diagonalizable if it is similar to a diagonal matrix (Δ). There exists an invertible matrix G such that

$$A = G\Delta G^{-1} \text{ or equivalently } AG = G\Delta.$$

A method to construct solvable systems

Takeyama's method (generalization of matrix diagonalization to operator setting)

- Identify a "simple/diagonal" operator (Δ) such that the eigenvectors of Δ can be calculated
- Identify a propagator operators G
- Identify the Hamiltonian H that solves the equations $HG = G\Delta$
- Transform the eigenvectors of Δ into eigenvectors of H

$$
\begin{array}{ccccc}
 & F(S) & \overset{\boldsymbol{H}}{\rightarrow} & F(S) & \\
\boldsymbol{G} & \uparrow & comm & \uparrow & \boldsymbol{G} \\
 & F(S) & \overset{\Delta}{\rightarrow} & F(S) &
\end{array}
$$

Mathematical encoding of the particle configuration

- **"common-sense"**

For each site record the height of the surface, h_i. Encode each surface by the sequence of heights $(\cdots, h_{-2}, h_{-1}, h_0, h_1, h_2, \cdots)$. Heights are non-negative integers. If the number of particles is fixed only finitely many of the heights are non-zero and the sum of the heights is constant.

$$\left\{ h: Z \to \{0,1,2,\ldots\} \middle| \sum_{i \in Z} h_i = C \right\}$$

Mathematical encoding of the particle configuration

- "transformed"

  From right to left number the particles, $1, 2, 3, \cdots$ and record the positions of the particles: x_i is the position of the i^{th} particle. Sites are integers. If we have k particles then we have k sites: $(x_1, \cdots, x_k)$ and $x_1 \geq \cdots \geq x_k$.

  - fundamental chamber of the lattice generated by k orthonormal vectors

    $$L_+ = \{x_1 v_1 + \cdots + x_k v_k | x_1 \in Z, \cdots, x_k \in Z,\ x_1 \geq \cdots \geq x_k\}$$

  - clusters of a set of k elements

    $$C = \{(c_1, \cdots, c_M) \in \{1, \cdots, k\}^M | \, 1 \leq M \leq k, c_1 + \cdots + c_M = k\}$$

Mathematical encoding of the particle configuration

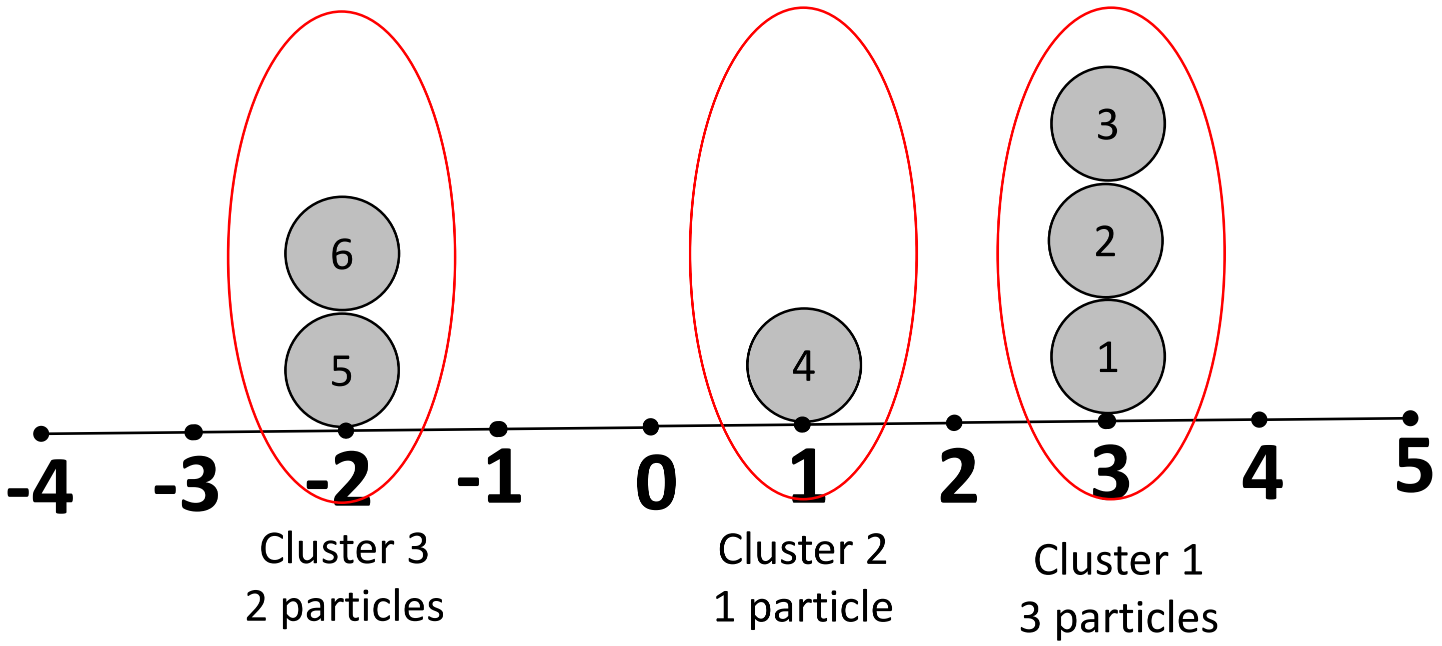
- "transformed, amenable to computations"

From right to left number the particles, $1, 2, 3, \cdots$ and record the positions of the particles: x_i is the position of the i^{th} particle. Sites are integers. If we have k particles then we have k sites: $(x_1, \cdots, x_k)$ and $x_1 \geq \cdots \geq x_k$. The orbit of $(x_1, \cdots, x_k)$ of the action of the symmetric group in the entire lattice .

- Lattice generated by k orthonormal vectors

$$L = \{x_1 v_1 + \cdots + x_k v_k | x_1 \in Z, \cdots, x_k \in Z \quad \}$$

Mathematical encoding of the particle configuration



Sequence of heights: $\dots, 0, 0, 0, \dots, 3, 0, 1, 0, 0, 2, 0, 0, 0, 0, \dots$

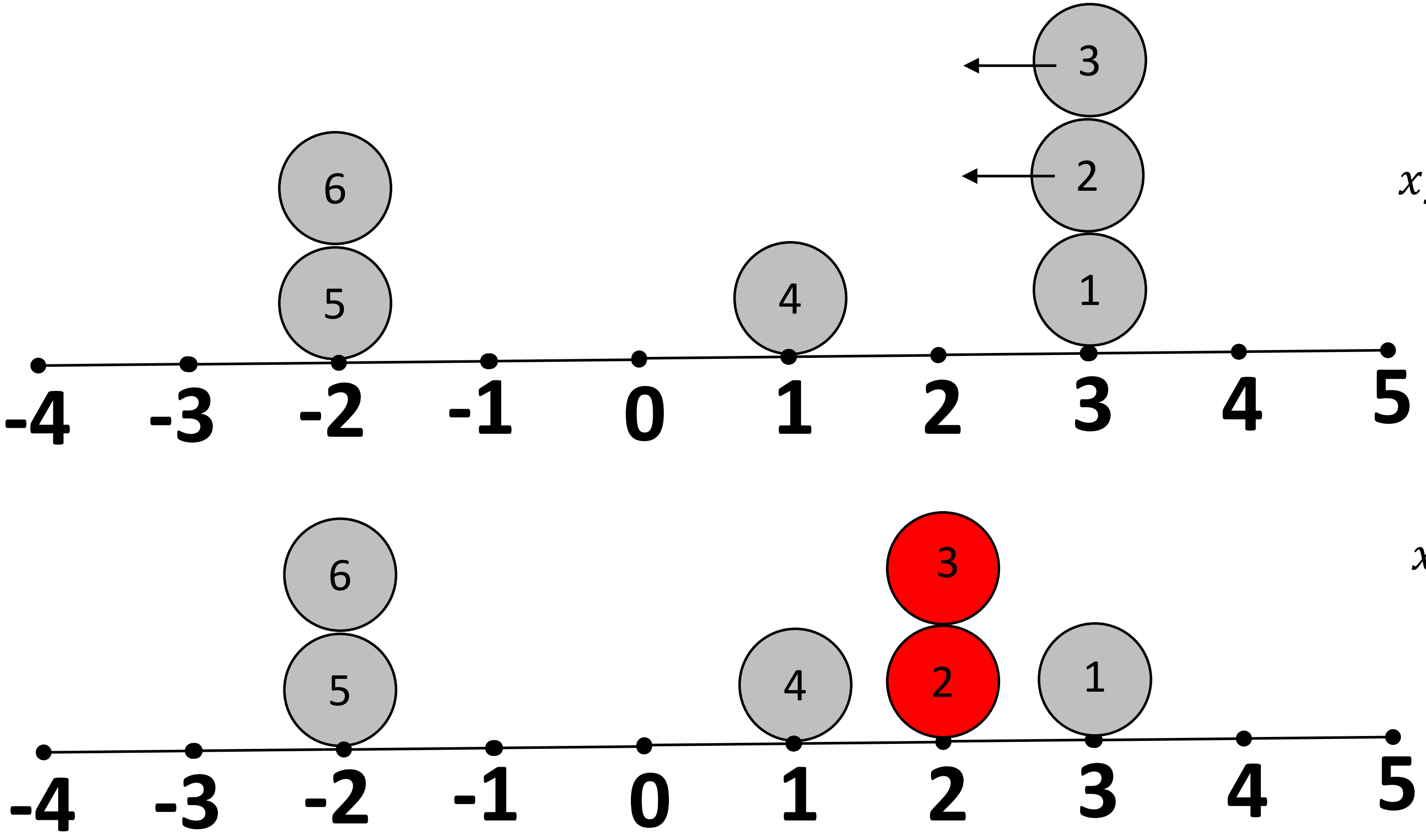
Sequence of sites: 3,3,3,1,-2,-2

Lattice point in the fundamental chamber: $3v_1 + 3v_2 + 3v_3 + v_4 - 2v_5 - 2v_6$

Cluster sizes: $c_1 = 3, c_2 = 1, c_3 = 2$ $\quad c_1 + c_2 + c_3 = 6$

Orbit in the lattice due to symmetric group action: (3,3,3,1,-2,-2) (1,3,-2,3,3,-2) ….. 6! points

Mathematical encoding of the dynamics



$$x_{start} = 3v_1 + 3v_2 + 3v_3 + v_4 - 2v_5 - 2v_6$$

$$x_{end} = 3v_1 + \mathbf{2}v_2 + \mathbf{2}v_3 + v_4 - 2v_5 - 2v_6$$
$$= x_{start} - v_2 - v_3$$

Dynamics (particle i moves one site to the left) is described by operators: $\left(\widehat{X}_i f\right)(x) = f(x - v_i),\ 1 \le i \le k$

Reminder: Time-independent Schrodinger equation and eigenfunction problem for the Laplacian (n=1)

- $-\varphi''(x) + 2c\,\delta(x)\varphi(x) = E\varphi(x)$ Schrodinger equation

φ is continuous on R, φ', φ'' are continuous on $R \setminus\{0\}$, φ' has a jump of size $c\varphi(0)$ at 0

δ: delta function E: negative constant, energy level

- $-u''(x) = Eu(x)$ eigenfunction problem for the Laplacian

$$\varphi(x) = \begin{cases} \varphi_L(x) = A_1 e^{kx} + A_2 e^{-kx}, & x \leq 0 \\ \varphi_R(x) = B_1 e^{kx} + B_2 e^{-kx}, & x \geq 0 \end{cases} \qquad k = \sqrt{-E}$$

φ is continuous: $\varphi_R(0) = \varphi_L(0),\ \ A_1 + A_2 = B_1 + B_2$

φ' has a jump at 0: $-\big(\varphi_R'(0) - \varphi_L'(0)\big) + 2c\varphi(0) = 0,\ \ -k(A_1 - A_2 - B_1 + A_2) + 2c(A_1 + A_2) = 0$

Reminder: Time-independent Schrodinger equation and eigenfunction problem for the Laplacian (n=1)

$$\varphi(x) = Ae^{-kx} + Be^{kx} + (A+B)\frac{e^{kx} - e^{-kx}}{k}\begin{cases} -c, & x \le 0 \\ 0, & x \ge 0 \end{cases}$$

$$\varphi(x) = u(x) + \left(\int_{-x}^{x} u(t)dt\right)\begin{cases} -c, & x \le 0 \\ 0, & x \ge 0 \end{cases}$$

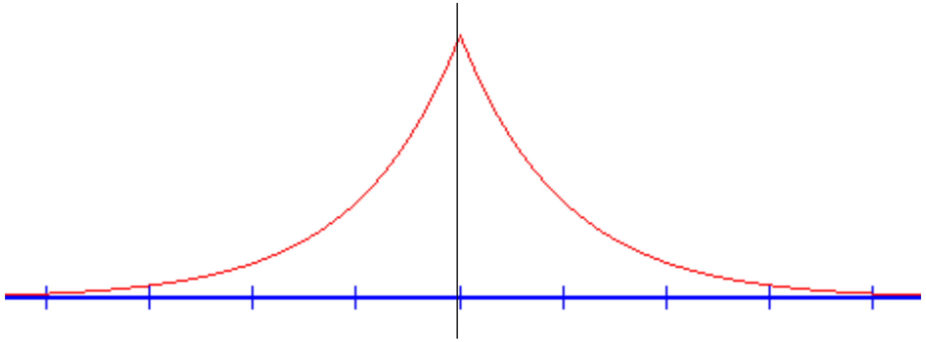
$$\varphi = Pu = u + c\begin{cases} Iu \\ 0 \end{cases}$$

Bounded solution for $c = -k = \sqrt{-E}$

$$\varphi(x) = P\left(Ae^{-kx}\right) = \begin{cases} Ae^{kx}, & x \le 0 \\ Ae^{-kx}, & x \ge 0 \end{cases}$$

After reflecting the eigenfunction of the Laplacian in the wall $x = 0$ we obtain an eigenfunction of the delta potential.

Start with an eigenfunction of the Laplacian on $x > 0$, this function is uniquely propagated on $x < 0$ so that the resulting function is continuous at 0 and its derivative has a jump of $2c\varphi(0)$ at 0.



Lascoux and Schutzenberger operators on a polynomial ring

- Ring of Laurent polynomials in k variables, with complex coefficients $\mathbb{C}\left[\frac{1}{z_1}, z_1, \cdots, \frac{1}{z_k}, z_k\right]$

  Let P be a Laurent polynomial

$$P\check{X}_i = \frac{1}{z_i} P, \qquad i = 1, \cdots, k$$

$$P\check{T}_i = P.s_i + \frac{(\alpha z_i + \beta)(\gamma z_{i+1} + \delta)}{z_i - z_{i+1}} (P - P.s_i), \qquad i = 1, \cdots, k-1$$

$$P\left(\cdots, \frac{1}{z_i}, z_i, \frac{1}{z_{i+1}}, z_{i+1}, \cdots\right).s_i = P\left(\cdots, \frac{1}{z_{i+1}}, z_{i+1}, \frac{1}{z_i}, z_i, \cdots\right)$$

Example: Lascoux and Schutzenberger operators case: exponent of z1 > exponent of z2

- Ring of Laurent polynomials in 2 variables, with complex coefficients $\mathrm{C}\left[\frac{1}{z_1}, z_1, \frac{1}{z_2}, z_2\right]$

$$(z_1^6 z_2^2)\check{X}_1 = z_1^5 z_2^2$$
$$(z_1^6 z_2^2)\check{X}_2 = z_1^6 z_2^1$$
$$(z_1^6 z_2^2).\, s_1 = z_1^2 z_2^6$$
$$(z_1^6 z_2^2)\check{T}_1 = z_1^2 z_2^6 + \frac{(\alpha z_1 + \beta)(\gamma z_2 + \delta)}{z_1 - z_2}(z_1^6 z_2^2 - z_1^2 z_2^6)$$

$$= z_1^2 z_2^6 + z_1^2 z_2^2(\alpha z_1 + \beta)(\gamma z_2 + \delta)\frac{z_1^4 - z_2^4}{z_1 - z_2}$$
$$= z_1^2 z_2^6 + (\alpha\gamma z_1^3 z_2^3 + \alpha\delta z_1^3 z_2^2 + \beta\gamma z_1^2 z_2^3 + \beta\delta z_1^2 z_2^2)(z_1^3 + z_1^2 z_2^1 + z_1^1 z_2^2 + z_2^3)$$

Example: Lascoux and Schutzenberger operators case: exponent of z1 > exponent of z2

$$(z_1^6 z_2^2)\check{T}_1 =$$

$$+\alpha\gamma\left(z_1^6 z_2^3 + z_1^5 z_2^4 + z_1^4 z_2^5 + z_1^3 z_2^6\right)$$

9th line 6+3=9

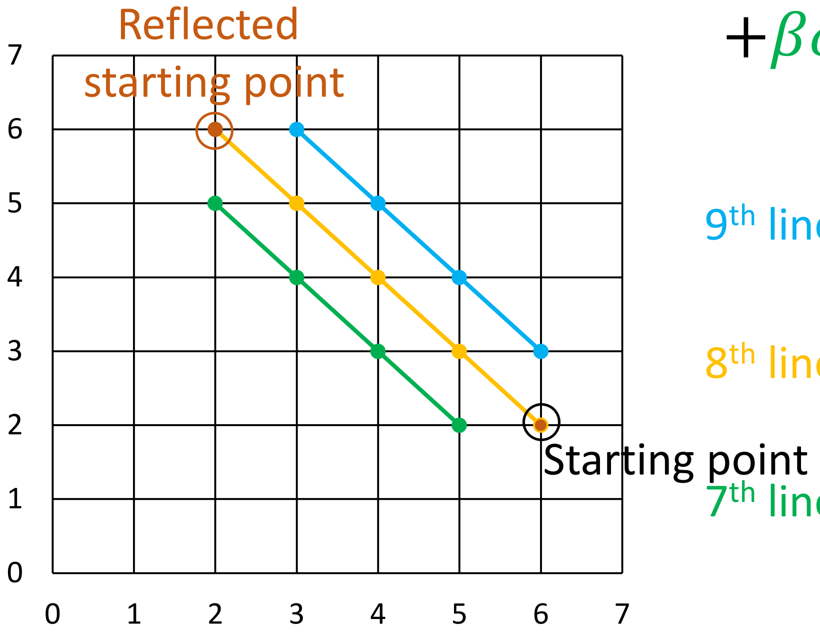
$$+z_1^2 z_2^6 + \alpha\delta\left(z_1^2 z_2^6 + z_1^5 z_2^3 + z_1^4 z_2^4 + z_1^3 z_2^5\right)$$

8th line 6+2=8

$$+\beta\gamma\left(z_1^5 z_2^3 + z_1^4 z_2^4 + z_1^3 z_2^5 + z_1^6 z_2^2\right)$$

$$+\beta\delta\left(z_1^5 z_2^2 + z_1^4 z_2^3 + z_1^3 z_2^4 + z_1^2 z_2^5\right)$$

7th line 5+2=7



9th line, coefficients are $\alpha\gamma$

8th line, coefficients are $\alpha\delta + \beta\gamma$, except for starting point and reflection of starting point

7th line, coefficients are $\beta\delta$

Example: Lascoux and Schutzenberger operators
case: exponent of z1 > exponent of z2

$$(z_1^6 z_2^2)\check{T}_1 =$$

$$+\alpha\gamma\left(z_1^6 z_2^3 + z_1^5 z_2^4 + z_1^4 z_2^5 + z_1^3 z_2^6\right)$$

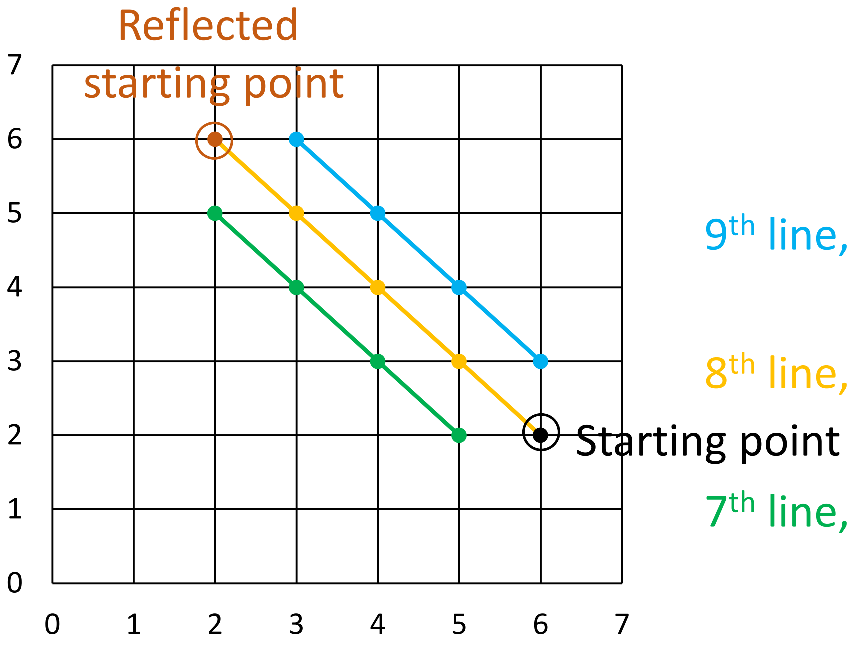
9th line 6+3=9

$$+\beta\gamma(z_1^6 z_2^2) + (\alpha\delta + \beta\gamma)\left(z_1^5 z_2^3 + z_1^4 z_2^4 + z_1^3 z_2^5\right) + (1 - \alpha\delta) z_1^2 z_2^6$$

8th line 6+2=8

$$+(-\beta\delta)\left(z_1^5 z_2^2 + z_1^4 z_2^3 + z_1^3 z_2^4 + z_1^2 z_2^5\right)$$

7th line 5+2=7



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Example: Lascoux and Schutzenberger operators

Case: exponent of z1 < exponent of z2

- Ring of Laurent polynomials in 2 variables, with complex coefficients $\mathrm{C}\left[\frac{1}{z_1}, z_1, \frac{1}{z_2}, z_2\right]$

$$(z_1^2 z_2^6)\check{X}_1 = z_1^1 z_2^6$$

$$(z_1^2 z_2^6)\check{X}_2 = z_1^2 z_2^5$$

$$(z_1^2 z_2^6).s_1 = z_1^6 z_2^2$$

$$(z_1^2 z_2^6)\check{T}_1 = z_1^6 z_2^2 + \frac{(\alpha z_1 + \beta)(\gamma z_2 + \delta)}{z_1 - z_2}(z_1^2 z_2^6 - z_1^6 z_2^2)$$

$$= z_1^6 z_2^2 + z_1^2 z_2^2 (\alpha z_1 + \beta)(\gamma z_2 + \delta)\frac{z_2^4 - z_1^4}{z_1 - z_2}$$

$$= z_1^6 z_2^2 + (\alpha\gamma z_1^3 z_2^3 + \alpha\delta z_1^3 z_2^2 + \beta\gamma z_1^2 z_2^3 + \beta\delta z_1^2 z_2^2)(-z_1^3 - z_1^2 z_2^1 - z_1^1 z_2^2 - z_2^3)$$

Example of Lascoux and Schutzenberger operators

Case: exponent of z1 < exponent of z2

$$(z_1^2 z_2^6)\check{T}_1 =$$

$$+(-\alpha\gamma)(z_1^6 z_2^3 + z_1^5 z_2^4 + z_1^4 z_2^5 + z_1^3 z_2^6)$$

9th line 6+3=9

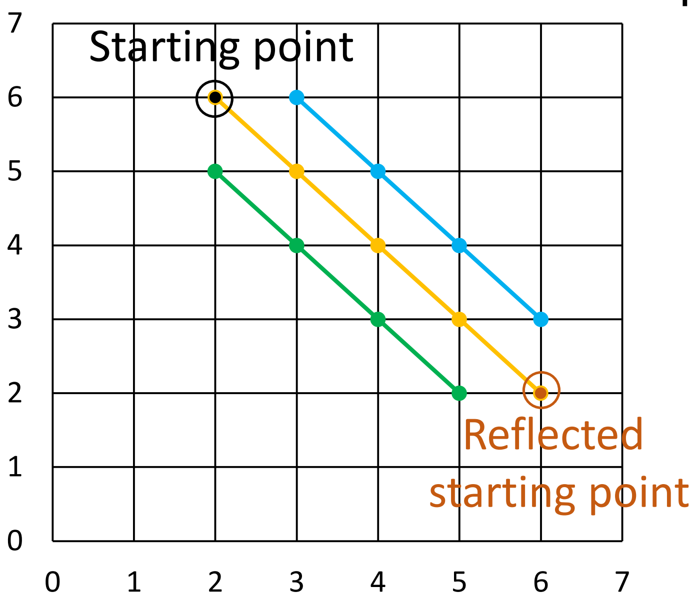
$$+z_1^6 z_2^2 + (-\alpha\delta)(z_1^6 z_2^2 + z_1^5 z_2^3 + z_1^4 z_2^4 + z_1^3 z_2^5)$$

$$+(-\beta\gamma)(z_1^5 z_2^3 + z_1^4 z_2^4 + z_1^3 z_2^5 + z_1^2 z_2^6)$$

8th line 6+2=8

$$+(-\beta\delta)(z_1^5 z_2^2 + z_1^4 z_2^3 + z_1^3 z_2^4 + z_1^2 z_2^5)$$

7th line 5+2=7



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Example of Lascoux and Schutzenberger operators

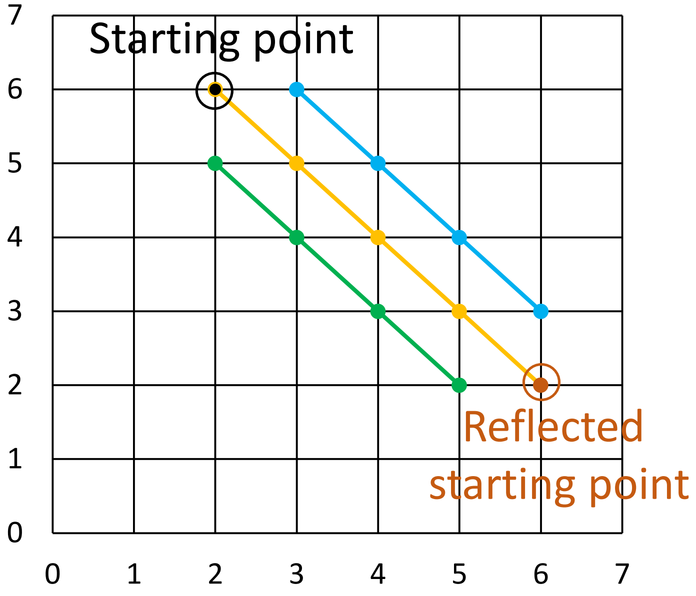
Case: exponent of z1 < exponent of z2

$$(z_1^2 z_2^6)\check{T}_1 =$$

$$+(-\alpha\gamma)\left(z_1^6 z_2^3 + z_1^5 z_2^4 + z_1^4 z_2^5 + z_1^3 z_2^6\right)$$ 9th line 6+3=9

$$+(-\beta\gamma)(z_1^2 z_2^6) + (-\alpha\delta - \beta\gamma)\left(z_1^5 z_2^3 + z_1^4 z_2^4 + z_1^3 z_2^5\right) + (1 - \alpha\delta) z_1^6 z_2^2$$ 8th line 6+2=8

$$+(-\beta\delta)\left(z_1^5 z_2^2 + z_1^4 z_2^3 + z_1^3 z_2^4 + z_1^2 z_2^5\right)$$ 7th line 5+2=7



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Integral reflection operators on function space

- Vector space, $F(L)$, of complex-valued functions defined on the k-dimensional, orthogonal lattice $L = \{x_1 v_1 + \cdots + x_k v_k | x_1 \in Z, \cdots, x_k \in Z\}$. Let f $\in F(L)$, $f: L \to C$. Define the operators

$$\left(\hat{X}_i f\right)(x) = f(x - v_i), \qquad i = 1, \cdots, k$$

$$\left(\hat{T}_i f\right)(x) = \left(\hat{T}_i f\right)(x_1 v_1 + \cdots + x_k v_k), \qquad i = 1, \cdots, k-1$$

is a linear combination of the values of f at the exponents of those monomials identified in $(z^{x_1} \cdots z^{x_k})\check{T}_i$

$$(s_i f)(x_1 v_1 + \cdots + x_i\, v_i + x_{i+1} v_{i+1} + \cdots + x_k v_k)$$
$$= f(x_1 v_1 + \cdots + x_{i+1}\, v_i + x_i v_{i+1} + \cdots + x_k v_k), \qquad i = 1, \cdots, k-1$$

$$(\hat{T}_i f)(x) =$$

$$\alpha\gamma \sum_{j=1}^{x_i - x_{i+1}} f(s_i x + j(v_i - v_{i+1}) + v_{i+1})$$ $|x|_1 + 1^{st}$ line

$$\alpha\delta f(x) + (\alpha\delta + \beta\gamma) \sum_{j=1}^{x_i - x_{i+1} - 1} f\big(s_i x + j(v_i - v_{i+1})\big) + (1 + \beta\gamma) f(s_i x)$$ $|x|_1{}^{th}$ line

$$+\beta\delta \sum_{j=1}^{x_i - x_{i+1} - 1} f(s_i x + j(v_i - v_{i+1}) - v_{i+1})$$ $|x|_1 - 1^{th}$ line

if $x_i - x_{i+1} > 0,$

$s_i x$ permutes the i^th and i+1^th coordinates of x

$$|x|_1 = x_1 + \cdots x_k$$

Deformed affine Hecke algebra of type A_{k-1} (Y. Takeyama 2014)

Hecke algebra of type A_{k-1}

deformation of the group algebra of the symmetric group S_k

- Eigenvalue relations: $(T_i - 1)(T_i + q) = 0$, $q = 1 + \beta\gamma - \alpha\delta$
- Braid relations: $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$ and $T_i T_j = T_j T_i$ if $|i-j| > 1$

affine Hecke algebra of type A_{k-1}

universal, unital, associative $\mathbb{C}$-algebra generated by the elements $T_1, \cdots, T_{k-1}, X_1, \cdots, X_k, X_1^{-1}, \cdots, X_k^{-1}$ subject to eigenvalue relations, braid relations and

- Laurent relations: $X_i X_j = X_j X_i$ and $X_i X_i^{-1} = X_i^{-1} X_i = 1$
- Action relations: $T_i X_i T_i = q X_{i+1}$ and $T_i X_j = X_j T_i$ if $i \neq j, j-1$

deformed affine Hecke algebra of type A_{k-1}

universal, unital, associative $\mathbb{C}$-algebra generated by the elements $T_1, \cdots, T_{k-1}, X_1, \cdots, X_k, X_1^{-1}, \cdots, X_k^{-1}$ subject to eigenvalue relations, braid relations, Laurent relations and

- Action relations: $X_{i+1} T_i - T_i X_i = T_i X_{i+1} - X_i T_i = (\alpha + \beta X_i)(\gamma + \delta X_i)$ and $T_i X_j = X_j T_i$ if $i \neq j, j-1$

If $\beta = \gamma = 0$ then deformed affine Hecke algebra = affine Hecke algebra

Representations of deformed affine Hecke algebra

deformed affine Hecke algebra of type A_{k-1}

Right representation on the ring of Laurent polynomials in k variables, with complex coefficients

$$C\left[\frac{1}{z_1}, z_1, \cdots, \frac{1}{z_k}, z_k\right]$$

Left representation on the vector space, $F(L)$, of complex-valued functions defined on the k-dimensional, orthogonal lattice L

Propagation operator

$$G(f)(x) = \left(\hat{T}_{w_x} f\right)(w_x x)$$

For $x \in L$ there exists the shortest sequence $s_{i_1}, \cdots, s_{i_l}$ selected from permutations $s_1 = (12), s_2 = (23), \cdots, s_{k-1} = (k-1k)$ such that $s_{i_1} \circ \cdots \circ s_{i_l}(x)$ belongs to the fundamental chamber $L_+ = \{x_1 v_1 + \cdots + x_k v_k | x_1 \in Z, \cdots, x_k \in Z,\ x_1 \geq \cdots \geq x_k\}$ of the orthogonal lattice. Then

$$\hat{T}_{w_x} = \hat{T}_{i_1} \circ \cdots \circ \hat{T}_{i_l}$$

Propagation operator

$\boldsymbol{x}$	$\boldsymbol{\sigma_1}$	…	$\boldsymbol{\sigma_j}$	…	$\boldsymbol{\sigma_{k!}}$
$\boldsymbol{\hat{T}_{\sigma_1}}$	$\hat{T}_{\sigma_1} f(\sigma_1 x)$		$\hat{T}_{\sigma_1} f(\sigma_j x)$		$\hat{T}_{\sigma_1} f(\sigma_{k!} x)$
⋮					
$\boldsymbol{\hat{T}_{\sigma_j}}$	$\hat{T}_{\sigma_j} f(\sigma_1 x)$		$\boldsymbol{\hat{T}_{\sigma_j} f(\sigma_j x)}$		$\hat{T}_{\sigma_j} f(\sigma_{k!} x)$
⋮					
$\boldsymbol{\hat{T}_{\sigma_{k!}}}$	$\hat{T}_{\sigma_{k!}} f(\sigma_1 x)$		$\hat{T}_{\sigma_{k!}} f(\sigma_j x)$		$\hat{T}_{\sigma_{k!}} f(\sigma_{k!} x)$

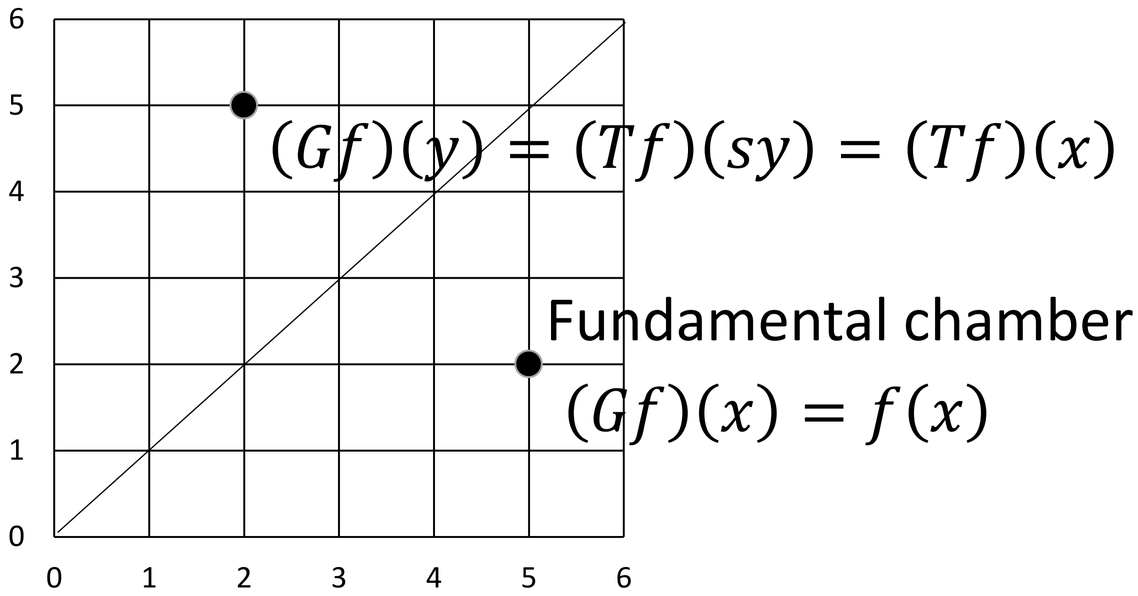
$\hat{T}_\sigma = \hat{T}_{s_{i_1}} \circ \cdots \circ \hat{T}_{s_{i_l}}$ with $\sigma = s_{i_1} \circ \cdots \circ s_{i_l}$ being shortest path in the symmetric group

For $x \in L$ identify $\sigma_j \in S_k$ such that $\sigma_j x$ belongs to the fundamental chamber. Then

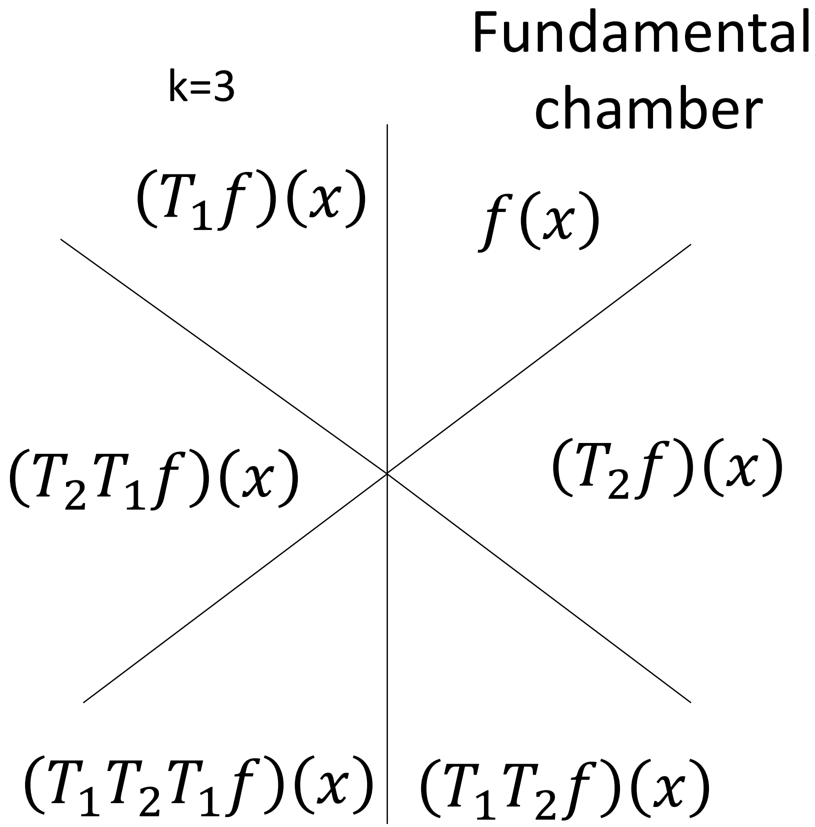
$$G(x) = \hat{T}_{\sigma_j} f(\sigma_j x)$$

Propagation operator

k=2



k=3

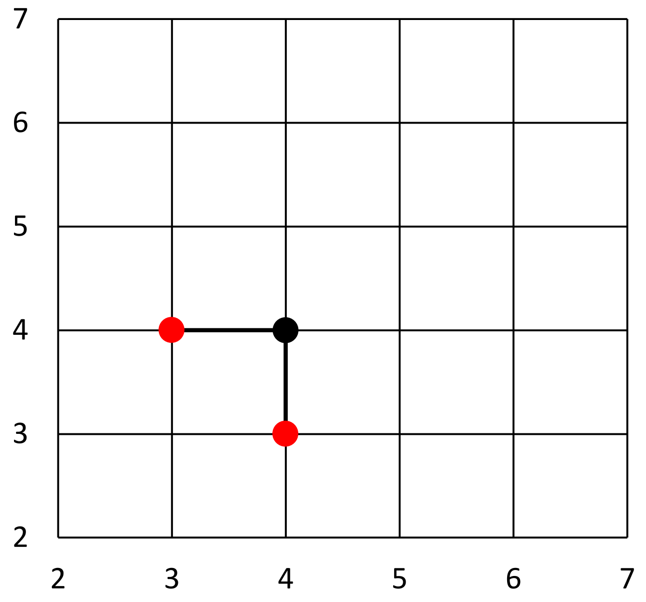


Fundamental chamber of the lattice generated by k orthonormal vectors

$$L_+ = \{x_1 v_1 + \cdots + x_k v_k | x_1 \in Z, \cdots, x_k \in Z,\ x_1 \geq \cdots \geq x_k\}$$

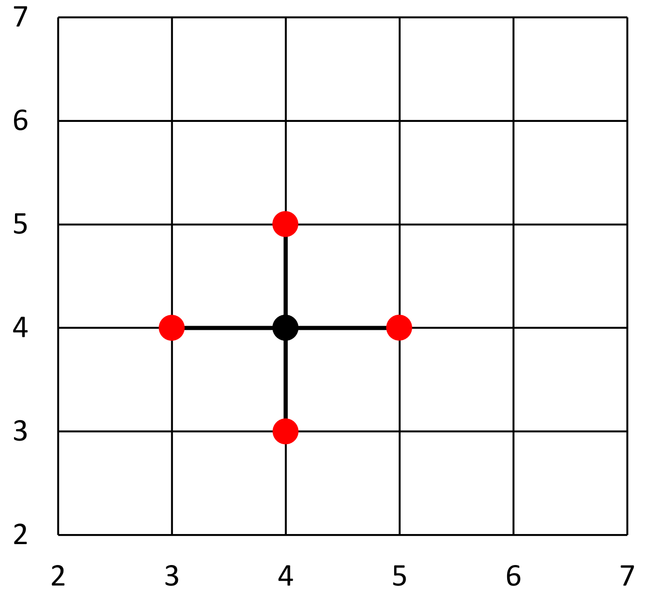
Half Laplacian

- $(\Delta_{1/2} f)(x) \quad = \left((\hat{X}_1 + \cdots + \hat{X}_k) f\right)(x)$
  $= f(x - v_1) + \cdots + f(x - v_k)$
- Eigenvectors of half Laplacian are $f(x) = \sum_{\sigma \in S_k} p_{\sigma(1)}^{x_1} \cdots p_{\sigma(k)}^{x_k}$
  with $p_1 > 0, \cdots, p_k > 0$



Laplacian

- $(\Delta_1 f)(x) \quad = \left((\hat{X}_1 + \hat{X}_1^{-1} + \cdots + \hat{X}_k + \hat{X}_k^{-1}) f\right)(x)$
  $= f(x - v_1) + f(x + v_1) + \cdots + f(x - v_k) + f(x + v_k)$
- Eigenvectors of Laplacian are $f(x) = \sum_{\sigma \in S_k} p_{\sigma(1)}^{x_1} \cdots p_{\sigma(k)}^{x_k}$
  with $p_1 > 0, \cdots, p_k > 0$



Hamiltonian (Y. Takeyama 2014)

$$(H_{1/2}\, f)(x) = -a(x)f(x) + \sum_{r=1}^{k} \sum_{1\le j_1\le\cdots\le j_r\le k} b(x, j_1, \cdots, j_r)\left(\hat{X}_{j_1} \cdots \hat{X}_{j_r} f\right)(x)$$

$$= f(x - (v_{j_1} + \cdots + v_{j_r}))$$

$$q = 1 + \beta\gamma - \alpha\delta$$

f is a symmetric function defined on the entire lattice L

- $a(x) = \sum_{r=1}^{k} a(x, r) = \alpha\gamma \sum_{r=1}^{k} \frac{[d_r^+(x)]}{1+\beta\gamma[d_r^+(x)]}$
  - $[n] = \frac{1-q^n}{1-q}$
  - $d_r^+(x)$ count of p with the properties $r < p \le k$ and $x_p = x_r$
- $b(x, j_1, \cdots, j_r) = \frac{(-\beta\gamma)^{r-1}[r-1]!\, q^{-\frac{r(r-1)}{2}} q^{d_{j_1}^-(x)+\cdots+d_{j_r}^-(x)}}{\prod_{p=0}^{r-1}\left(1+\beta\gamma\left[d_{j_p}^+(x)+d_{j_p}^-(x)-p\right]\right)} \delta_{j_1,\cdots,j_r}$
  - $\delta_{j_1,\cdots,j_r} = 1$ if $x_{j_1} = \cdots = x_{j_r}$, otherwise $= 0$
  - $d_r^-(x)$ count of p with the properties $1 \le p < r$ and $x_p = x_r$

Takeyama model for surface diffusion depends on 2 parameters (s,q)

r particles move one site to the left from a cluster with c particles at a rate given by

$$\frac{s^{r-1}}{[r]}\prod_{p=0}^{r-1}\frac{[c-p]}{1+s[c-1-p]}$$

$$[n]=\frac{1-q^n}{1-q}$$

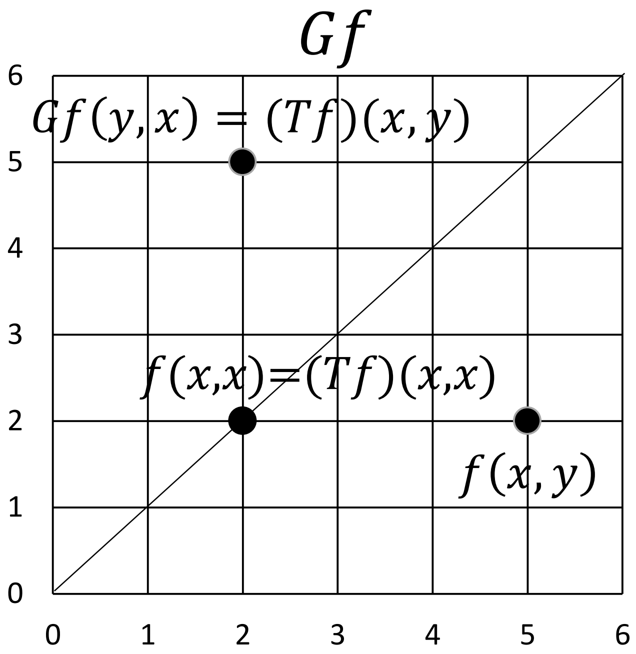
Properties of Takeyama's Hamiltonian

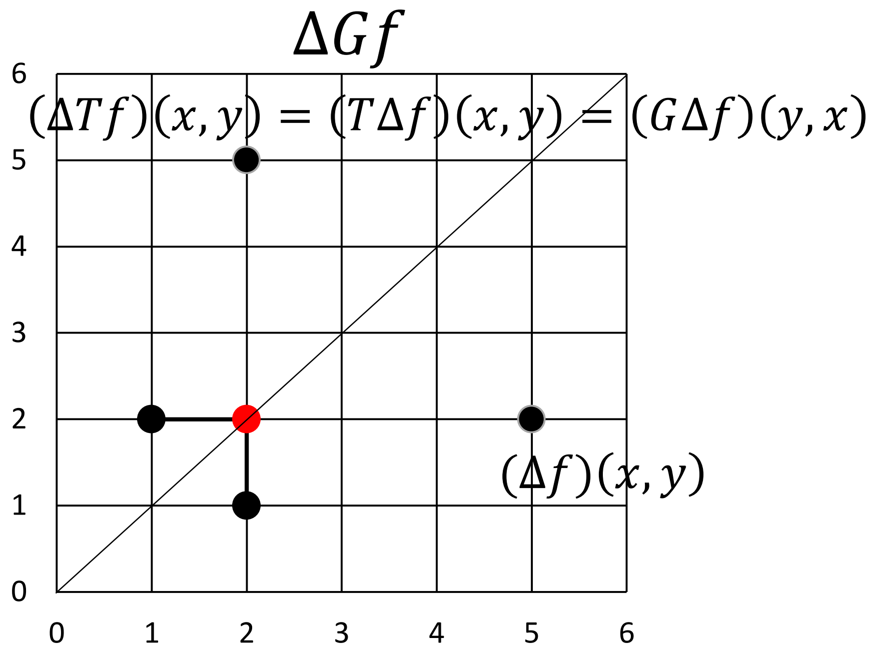
- $H_{1/2}(F(L)^W)$ is included in $F(L)^W$
- Commutation relation of $H_{1/2}$ and propagation operator

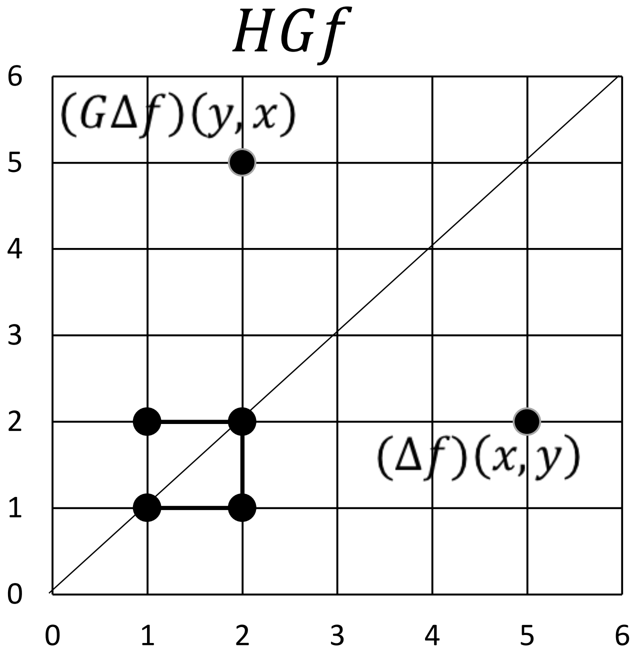
$$H_{1/2}G = G\left(\sum_{i=1}^{k} \hat{X}_i\right) = G\Delta_{1/2}$$

- Some eigenvectors of $H_{1/2}$ can be found. The Bethe wave functions, the eigenfunctions of $\Delta_{1/2}$ transformed under the propagation operator G to become eigenfunctions of H
- Hamiltonian $H_{1/2} - \theta I$ is stochastic if $(\alpha + \beta)(\gamma + \delta) = 0$.

How can we construct Takeyama's Hamiltonian? Case k=2







$$f(x,x-1)+(Tf)(x,x-1)=((X_1+X_2T)f)(x,x)\neq((X_1+X_2)f)(x,x)$$

$$af(x,x-1)+b(Tf)(x,x-1)+cf(x,x)+df(x-1,x-1)=\big((aX_1+bX_2T+c1+dX_1X_2)f\big)(x,x)=((X_1+X_2)f)(x,x)$$

$\Delta Gf = G\Delta f$ except on $x_1 = x_2$

Choose the coefficients a, b, c, d such that $HGf = G\Delta f$

Use action relation

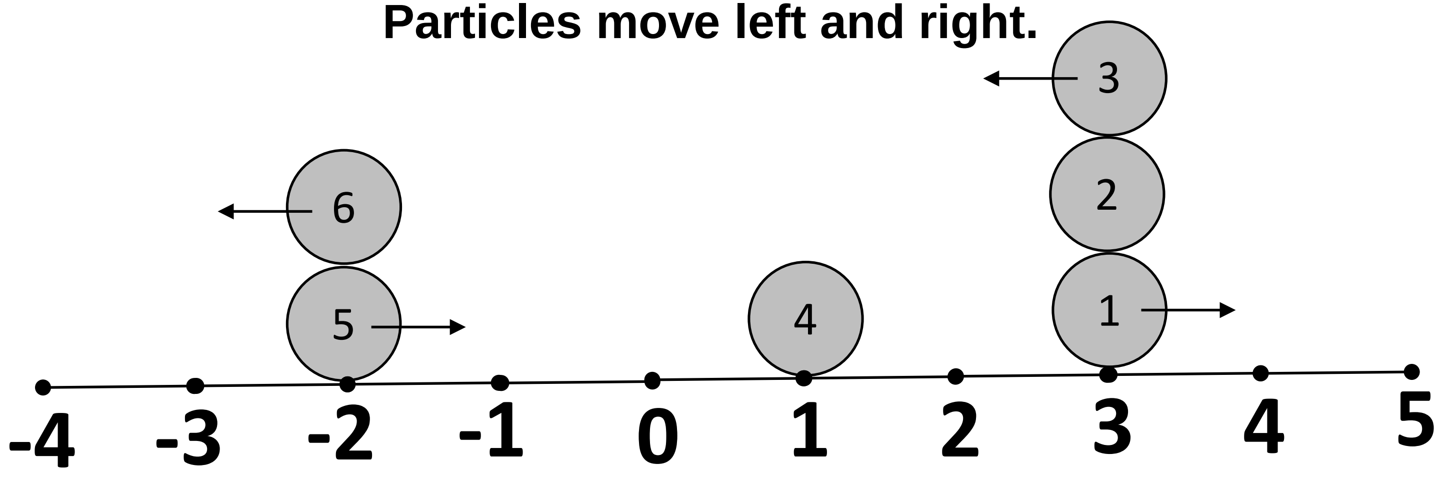
$$X_2T = TX_1 + (\alpha+\beta X_1)(\gamma+\delta X_1)$$

Can we construct a discrete model for surface diffusion that incorporates left and right particle movement?

Are such models integrable?

Are there other Hamiltonians that are integrable?

Particles move left and right.



Result for k=2 (two particles)

Let L be the 2-dimensional lattice. There exits an operator $H_1 \colon F(L) \to F(L)$ such that

- $H_1 G = G\Delta_1$ (commutation relation involving the propagator operator and the full Laplacian)
- has $F(L)^W$ as an invariant subspace
- has the form

$$
(H_1 f)(x,y) = \begin{cases} \left(\left(\hat{X}_1 + \hat{X}_2 + \hat{X}_1^{-1} + \hat{X}_2^{-1}\right) f\right)(x,y) & if\ x \neq y \\ \left(\left(a_0 + a_1\hat{X}_1 + a_2\hat{X}_2 + a_3\hat{X}_1\hat{X}_2 + b_1\hat{X}_1^{-1} + b_2\hat{X}_2^{-1} + b_3\hat{X}_1^{-1}\hat{X}_2^{-1}\right) f\right)(x,y) & if\ x = y \end{cases}
$$

Proof. We identify the coefficients a and b such that $H_1 G = G\Delta_1$

$$
\begin{cases} a_0 + a_1\alpha\gamma + b_2\beta\delta & = 0 \\ a_1(1+\beta\gamma) & = 1 \\ a_2 + a_1\alpha\delta & = 1 \\ a_3 + a_1\beta\delta & = 0 \\ b_1 + b_2\alpha\delta & = 1 \\ b_2(1+\beta\gamma) & = 1 \\ b_3 + b_2\alpha\gamma & = 0 \end{cases}
\qquad
\begin{cases} a_0 & = -\frac{\alpha\gamma+\beta\delta}{1+\beta\gamma} \\ a_1 & = \frac{1}{1+\beta\gamma} \\ a_2 & = 1 - \frac{\alpha\delta}{1+\beta\gamma} = \frac{q}{1+\beta\gamma} \\ a_3 & = -\frac{\beta\delta}{1+\beta\gamma} \\ b_1 & = 1 - \frac{\alpha\delta}{1+\beta\gamma} = \frac{q}{1+\beta\gamma} \\ b_2 & = \frac{1}{1+\beta\gamma} \\ b_3 & = -\frac{\alpha\gamma}{1+\beta\gamma} \end{cases}
$$

$a_1 + a_2$ rate for one particle to leave a cluster of two particles

a_3 rate for two particle to leave a cluster of two particles

Result for k=2 (two particles)

- An equivalent form for the Hamiltonian H_1 is

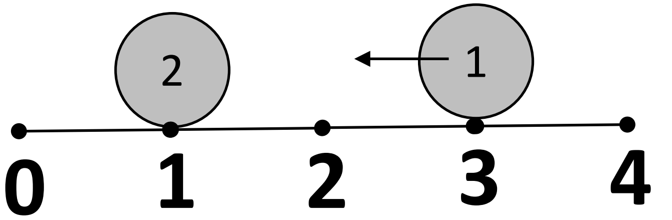
$$H_1 = H_{1/2} + H_{1/2}\hat{X}_1^{-1}\hat{X}_2^{-1}$$

Proof. Let $g(x) = f(x + e_1 + e_2) = (\hat{X}_1^{-1}\hat{X}_2^{-1}f)(x)$, then $\Delta_1 f = (X_1 +$

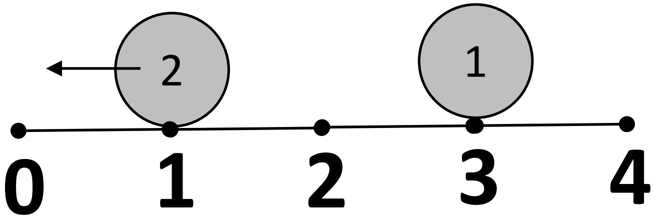
Propagator and symmetric polynomials in $\hat{X}_1, \hat{X}_2$ communte.

Dynamics of 2 particles

$$q = 1 + \beta\gamma - \alpha\delta$$
$$(\alpha + \beta)(\gamma + \delta) = 0$$



Rate=1



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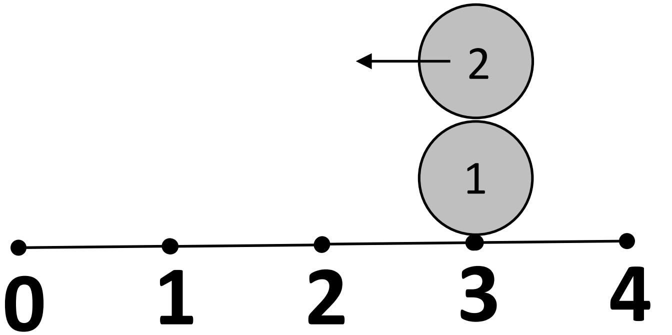
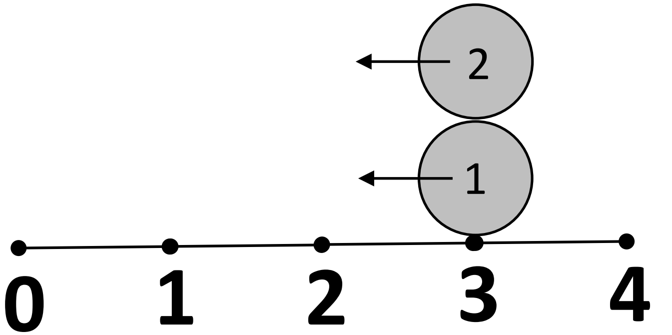
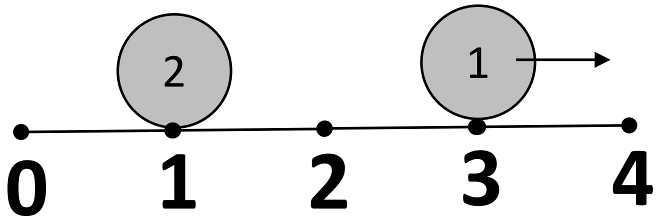


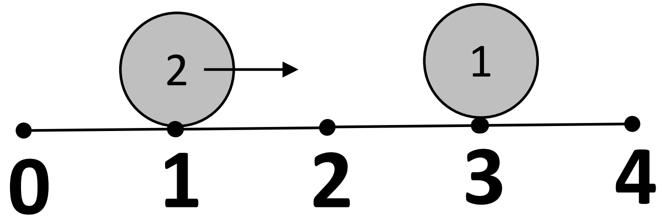
Rate=$\frac{1+q}{1+\beta\gamma} = 1 + \frac{1-\alpha\delta}{1+\beta\gamma}$



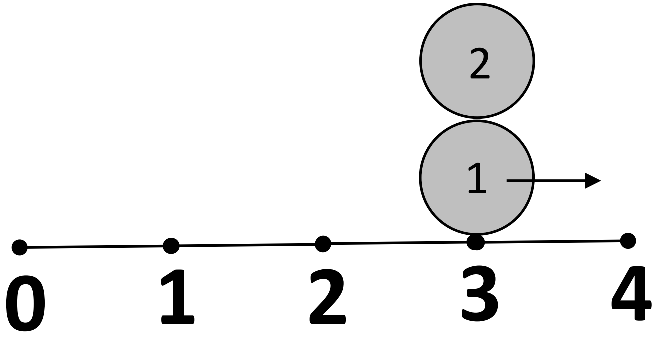
Rate=$\frac{-\beta\delta}{1+\beta\gamma}$



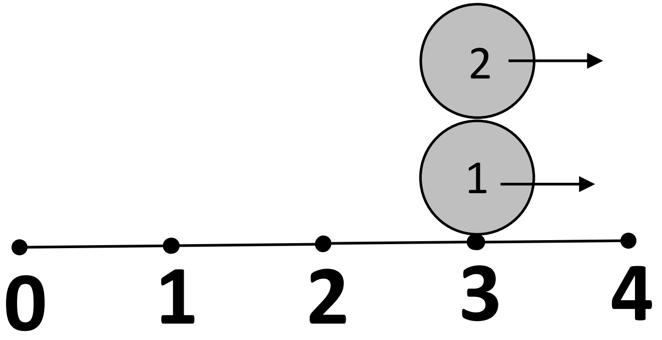
Rate=1



Rate=1



Rate=$\frac{1+q}{1+\beta\gamma}$



Rate=$\frac{-\alpha\gamma}{1+\beta\gamma}$

Dynamics of 2 particles

Rate		**Case 1**	**Case 2**	**Case 3**
Parametrization	$(\alpha+\beta)(\gamma+\delta)=0$ $q=1+\beta\gamma-\alpha\delta$	$\alpha+\beta=0$ $\gamma+\delta=0$ $s=\beta\gamma=-\alpha\gamma$ $q=1$	$\alpha+\beta=0$ $s=\beta\gamma$ $u=-\beta\delta$ $q=1+\beta(\gamma+\delta)$	$\gamma+\delta=0$ $s=\beta\gamma$ $u=-\alpha\gamma$ $q=1-\delta(\alpha+\beta)$
Parameters		One: s	Three: s, q, u	Three: s, q, u
1 particle / 1 cluster	1	1	1	1
1 particle / 2 cluster	$\frac{1+q}{1+\beta\gamma}$	$\frac{2}{1+s}$	$\frac{1+q}{1+s}$	$\frac{1+q}{1+s}$
2 particle / 2 cluster / left	$\frac{-\beta\delta}{1+\beta\gamma}$	$\frac{s}{1+s}$	$\frac{u}{1+s}$	$\frac{s}{1+s}$
2 particle / 2 cluster / right	$\frac{-\alpha\gamma}{1+\beta\gamma}$	$\frac{s}{1+s}$	$\frac{s}{1+s}$	$\frac{u}{1+s}$

Eigenfunctions of the Laplacian can be propagated into eigenfunctions of the Hamiltonian

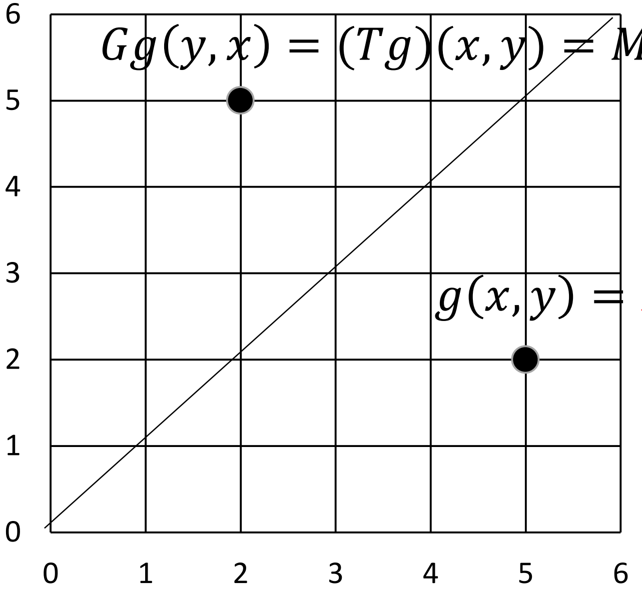
Bethe ansatz (educated guess to find the eigenfuctions)

$$f_{a,b}(x,y) = a^{-x} b^{-y}$$
$$\left(\Delta_{1/2} f_{a,b}\right) = (a+b) f_{a,b}$$
$$\left(\Delta_1 f_{a,b}\right) = (a + a^{-1} + b + b^{-1}) f_{a,b}$$

$$f_{b,a}(x,y) = b^{-x} a^{-y}$$
$$\left(\Delta_{1/2} f_{b,a}\right) = (a+b) f_{b,a}$$
$$\left(\Delta_1 f_{b,a}\right) = (a + a^{-1} + b + b^{-1}) f_{b,a}$$

$$T f_{a,b} = C f_{a,b} + D f_{b,a} = -\frac{(\alpha+\beta a)(\gamma+\delta b)}{a-b} f_{a,b} + \left(1 + \frac{(\alpha+\beta a)(\gamma+\delta b)}{a-b}\right) f_{b,a}$$

$$Gg(y,x) = (Tg)(x,y) = M(Tf_{a,b})(x,y) + N(Tf_{b,a})(x,y)$$



$$g(x,y) = M f_{a,b}(x,y) + N f_{b,a}(x,y)$$

Choose M and N such that $(Tg)(x,y) = g(x,y)$

$$M = 1 + \frac{(\alpha+\beta b)(\gamma+\delta a)}{b-a}$$
$$N = 1 + \frac{(\alpha+\beta a)(\gamma+\delta b)}{a-b}$$

$$\Downarrow$$

Gg is symmetric and an eigenfunction of $H_{1/2}$ and H_1

Proposition (work in progress)

- $H^+ = H_{1/2}$ Takeyama's Hamiltonian,
- H^- reverse of Takeyama's Hamiltonian
- $(H^- f)(x)$

$$= -a(x)f(x) + \sum_{r=1}^{k} \sum_{1 \le j_1 \le \cdots \le j_r \le k} b(x, j_1, \cdots, j_r)\left(\hat{X}_{j_1}^{-1} \cdots \hat{X}_{j_r}^{-1} f\right)(x)$$

- $X_2 T - T X_1 = (\alpha + \beta X_1)(\gamma + \delta X_2)$ $\quad X_1^{-1} T - T X_2^{-1} = (\delta + \gamma X_2^{-1})(\beta + \alpha X_1^{-1})$
- $T X_2 - X_1 T = (\alpha + \beta X_1)(\gamma + \delta X_2)$ $\quad T X_1^{-1} - X_2^{-1} T = (\delta + \gamma X_2^{-1})(\beta + \alpha X_1^{-1})$

$$H^+ = H(\alpha, \beta, \gamma, \delta) \qquad H^- = H(\delta, \gamma, \beta, \alpha)$$

Replace

- X_1 by X_2^{-1}
- X_2 by X_1^{-1}
- $\alpha\gamma$ by $\beta\delta$
- $\beta\delta$ by $\alpha\gamma$

$$(Hf)(x)$$

$$= -a(x)f(x)$$

$$+ \sum_{r=1}^{k} \sum_{\substack{1 \le j_1 \le \cdots \le j_r \le k \\ 1 \le l_1 \le \cdots \le l_r \le k}} b(x, j_1, \cdots, j_r, l_1, \cdots, l_r)\left(\hat{X}_{j_1}^{-1} \cdots \hat{X}_{j_r}^{-1} \hat{X}_{l_1} \cdots \hat{X}_{l_r} f\right)(x)$$

Questions ?

- Check that $H_1 = H_{1/2} + H^- = H(\alpha, \beta, \gamma, \delta) + H(\delta, \gamma, \beta, \alpha)$

satisfies $H_1 G = G\Delta_1$ for any n .

- To incorporate simultaneous left and right movement the Laplacian operator needs to be replaced.
- Can we find all eigenfunctions of $H_{1/2}$?
- Can we calculate transition probabilities of $H_{1/2}$?
- Can we establish asymptotic results for $H_{1/2}$?

References

1. A deformation of affine Hecke algebra and integrable stochastic particle systems, Y. Takeyama, Journal of Physics A: and Mathematical and Theoretical 47(46) 2014
2. A discrete analogue of periodic delta bose gas and affine Hecke algebra, Y. Takeyama, 2014
3. Completely integrable systems and groups generated by reflections, E. Gutkin AND B. Sutherland, Vol. 76, No. 12, pp. 6057-6059, December 1979
4. Exact results for one-dimensional totally asymmetric diffusion models, T. Sasamoto and M. Wadati, Journal of Physics A: 31(28) 1998, 6057-6071
5. On the integrability of zero-range chipping models with factorized steady states, Povolotsky, A. M., J. Phys. A 46 (2013), no. 46.
6. Affine Hecke algebras of type A and their cyclotomic quotient, S. Venkatesh http://math.mit.edu/~sidnv/Cyclotomic_AHA.pdf
7. Introduction to representation theory, Pavel Etingof, Oleg Golberg, Sebastian Hensel, Tiankai Liu, Alex Schwendner, Dmitry Vaintrob, and Elena Yudovina replect.dvi (mit.edu)
8. Hydrodynamic scaling limit of continuum solid-on-solid model, A. Savu, Journal of Applied Mathematics Volume 2006, Article ID 69101, Pages 1–37
9. Closed and exact functions in the context of Ginzburg-Landau models, A. Savu, Canadian Journal of Mathematics, http://arxiv.org/abs/math/0506002