A correction factor for Kac-Moody groups and t-deformed root multiplicities

Anna Puskás

University of Queensland

Solvable Lattice Models Seminar
25 August 2021

Macdonald's identity (1972, *The Poincaré series of a Coxeter group*):

$$\sum_{w\in W} w\left(\prod_{\alpha\in\Phi^+}\frac{1-te^{\alpha}}{1-e^{\alpha}}\right)=\sum_{w\in W} t^{\ell(w)}$$

W Weyl group, $\ell : W \to \mathbb{Z}_{\geq 0}$ length function, Φ^+ positive roots.

Kac-Moody root systems:

$$\mathfrak{m}\cdot\sum_{w\in W} w\left(\prod_{\alpha\in\Phi^+_{\mathrm{re}}}\frac{1-te^{\alpha}}{1-e^{\alpha}}\right)=\sum_{w\in W} t^{\ell(w)}$$

$$\mathfrak{m}'\cdot\sum_{w\in W} w\left(\prod_{\alpha\in\Phi^+}\left(\frac{1-te^{\alpha}}{1-e^{\alpha}}\right)^{\mathrm{mult}(\alpha)}\right)=\sum_{w\in W} t^{\ell(w)}$$

Joint work with Dinakar Muthiah and Ian Whitehead

We wish to write:

$$\mathfrak{m} = \prod_{\lambda \in Q_{\mathrm{im}}^{+}} \prod_{n \geq 0} (1 - t^n e^{\lambda})^{-m(\lambda, n)}$$

where Q_{im}^{+} positive imaginary root cone; and

$$m_{\lambda}(t) = \sum_{n \geq 0} m(\lambda, n) t^n$$

are polynomials with constant term: $m_{\lambda}(0) = \mathrm{mult}(\lambda)$:

$$\mathfrak{m}|_{t=0} = \prod_{\alpha \in \Phi_{\mathrm{im}}^{+}} (1 - e^{\alpha})^{-\mathrm{mult}(\alpha)}$$

Formulae of p-adic Kac-Moody groups

$$S(\mathbb{1}_{K\pi^\lambda K}) = \frac{1}{\mathfrak{m}} \cdot \frac{t^{\langle\rho,\lambda\rangle}}{P_\lambda(t)} \cdot \sum_{w\in W} w\left(e^\lambda \frac{\Delta_{t,\mathrm{re}}}{\Delta_{\mathrm{re}}}\right)$$

- Macdonald's formula for the spherical function
- Generalizations: Braverman–Kazhdan–Patnaik (affine), Bardy-Panse–Gaussent–Rousseau (Kac–Moody)

Taking a limit in λ, this converges to the Gindikin-Karpelevich formula, $\mathfrak{m}$ persists. (Braverman–Garland–Kazhdan–Patnaik, Hébert, Ali)

Remark. Here $\mathfrak{m}$ (not $\mathfrak{m}'$) appears, factors corresponding to the multiplicities of imaginary roots were included in $\mathfrak{m}$

Formulae of *p*-adic Kac-Moody groups, continued

- Casselman-Shalika formula for the spherical Whittaker function in affine type [Patnaik]

$$\mathcal{W}(\pi^{\lambda}) = t^{-\langle \rho, \lambda \rangle} \mathfrak{m}' \cdot \prod_{\alpha \in \Phi^+} (1 - te^{\alpha}) \chi_{\lambda}$$

- Metaplectic analogue in Kac-Moody type [Patnaik, P.]

$$\mathcal{W}(\pi^{\lambda}) = \mathfrak{m}' \, \Delta \sum_{w \in W} (-1)^{\ell(w)} \left(\prod_{\widetilde{a} \in \Phi(w)} e^{-\widetilde{a}} \right) w \star e^{\lambda}$$

The factor $\mathfrak{m}$ relates Hecke symmetrizers to Weyl symmetrizers.

Relating symmetrizers

$$\sum_{w\in W} \mathcal{T}_w = \mathfrak{m}' \, \Delta_\Phi \sum_{w\in W} (-1)^{\ell(w)} \left(\prod_{\alpha\in\Phi(w)} \mathbf{x}^{-\alpha} \right) w$$

Macdonald's identity:

$$\sum_{w\in W} w \left(\prod_{\alpha\in\Phi^+} \frac{1-te^{\alpha}}{1-e^{\alpha}} \right) = \sum_{w\in W} t^{\ell(w)}$$

For Kac-Moody root systems, the left-hand side has the same factor $\mathfrak{m}$.

From the norm used to define Macdonald polynomials: $\mathfrak{m} = ||1||^2$.

Geometry of flag varieties

Macdonald's second proof of $1 \cdot \sum_{w \in W} w\left(\frac{\Delta_t}{\Delta}\right) = P(t)$ for finite W

Computation of the Betti numbers of a flag variety using Hodge theory.

- Right hand side: counting Schubert cells
- Left hand side: a computation of Dolbeault cohomology using localization at fixed points for the action of the maximal torus.
- The flag variety is smooth and projective Dolbeault cohomology is equal to Betti cohomology by the Hodge theorem.

Failure of $\mathfrak{m} = 1$ beyond finite type $\Rightarrow$ Kac-Moody flag varieties are not smooth; they are homogeneous $\Rightarrow$ everywhere singular.

Fishel-Grojnowski-Teleman explicitly compute the Dolbeault cohomology of the affine flag variety, prove *Strong Macdonald Conjecture*.

Preparations

Recall: we wish to define $\quad \mathfrak{m} \sum_{w \in W} w\left(\frac{\Delta_{t,\mathrm{re}}}{\Delta_{\mathrm{re}}}\right) \stackrel{?}{=} P(t)$

$$\Delta_{\mathrm{re}} = \prod_{\alpha \in \Phi^+_{\mathrm{re}}} (1 - e^{\alpha}), \qquad \Delta_{t,\mathrm{re}} = \prod_{\alpha \in \Phi^+_{\mathrm{re}}} (1 - te^{\alpha}), \qquad P(t) = \sum_{w \in W} t^{\ell(w)}$$

$Q^+ \supseteq Q^+_{\mathrm{im}}$ cones graded by height;

Laurent series units on Q^+ have form $ue^{\lambda_0} \prod_{\lambda \in Q^+ \setminus \{0\}} \prod_{n} (1 - t^n e^{\lambda})^{m(\lambda,n)}$

W acts on a multiplicative subset containing $\frac{\Delta_{t,\mathrm{re}}}{\Delta_{\mathrm{re}}}$

$\sum_{w \in W} w\left(\frac{\Delta_{t,\mathrm{re}}}{\Delta_{\mathrm{re}}}\right)$ unit in $\mathbb{Z}[[t]][t^{-1}][[Q^+]]$, regular at $t = 0$, constant coefficient $P(t)$.

Definition of $\mathfrak{m}$

$\sum_{w \in W} w\left(\frac{\Delta_{t,\mathrm{re}}}{\Delta_{\mathrm{re}}}\right)$ and $P(t)$ unit in $\mathbb{Z}[[t]][t^{-1}][[Q^+]]$, regular at $t = 0$.

We may define $\mathfrak{m}$ by

$$\mathfrak{m} \sum_{w \in W} w\left(\frac{\Delta_{t,\mathrm{re}}}{\Delta_{\mathrm{re}}}\right) = P(t)$$

The factor $\mathfrak{m}$ is Weyl-invariant and therefore supported on Q^+_{im}.

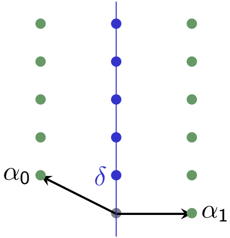
Both $\mathfrak{m}$, $\mathfrak{m}^{-1}$ units in $\mathbb{Z}[t, t^{-1}][[Q^+]]$, regular at $t = 0$, constant coefficient 1.

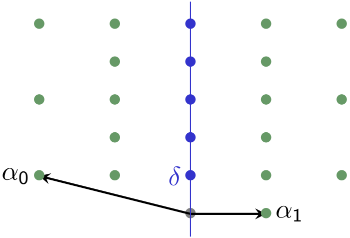
$$\left(\mathfrak{m}^{-1} \frac{\Delta_{\mathrm{re}}}{\Delta_{t,\mathrm{re}}}\right)\Bigg|_{Q^+_{\mathrm{im}}} = 1$$

Support - Affine type

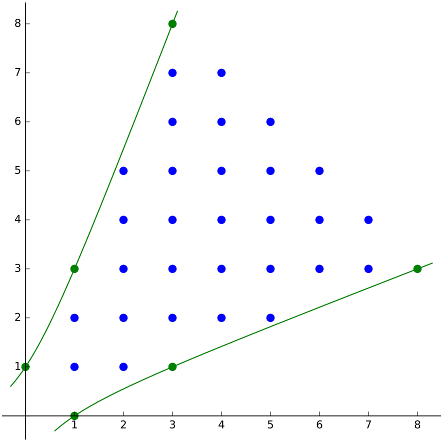
$$\mathfrak{m} \cdot \sum_{w \in W} w \left(\prod_{\alpha \in \Phi^+_{\mathrm{re}}} \frac{1 - te^{\alpha}}{1 - e^{\alpha}} \right) = \sum_{w \in W} t^{\ell(w)}$$

$A_1^{(1)}$ ($\widehat{\mathfrak{sl}}_2$), , $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ $A_2^{(2)}$ , $\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$





Support - Hyperbolic type with Cartan matrix $\begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix}$



Caution!

$$\left(\mathfrak{m}^{-1}\frac{\Delta_{\mathrm{re}}}{\Delta_{t,\mathrm{re}}}\right)\Bigg|_{Q^+_{\mathrm{im}}} = 1.$$

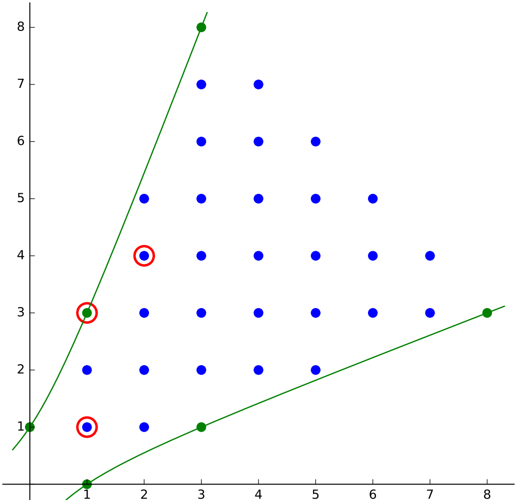
In the affine case, this implies "constant term property"

$$\left(\frac{\Delta_{\mathrm{re}}}{\Delta_{t,\mathrm{re}}}\right)\Bigg|_{Q^+_{\mathrm{im}}} = \mathfrak{m}$$

In the Kac-Moody case, this is **not true**!

$$\mathrm{Supp}(\mathfrak{a}) \subseteq Q^+_{\mathrm{im}} \not\Rightarrow (\mathfrak{a}\cdot\mathfrak{b})|_{Q^+_{\mathrm{im}}} = \mathfrak{a}\cdot(\mathfrak{b})|_{Q^+_{\mathrm{im}}}$$

$$\mathrm{Supp}(\mathfrak{a}) \subseteq Q_{\mathrm{im}}^{+} \not\Rightarrow (\mathfrak{a}\cdot\mathfrak{b})|_{Q_{\mathrm{im}}^{+}} = \mathfrak{a}\cdot(\mathfrak{b})|_{Q_{\mathrm{im}}^{+}}$$



Cherednik's solution of Macdonald's Constant Term Conjecture:
$\mathfrak{m}$ is known for Φ of affine type.

For untwisted, simply laced affine types:

$$\mathfrak{m} = \prod_{i=1}^{\infty} \left(\left(\frac{1 - t \cdot e^{i \cdot \delta}}{1 - e^{i \cdot \delta}} \right)^r \cdot \prod_{j=1}^{r} \frac{1 - t^{m_j} \cdot e^{i \cdot \delta}}{1 - t^{m_j + 1} \cdot e^{i \cdot \delta}} \right)$$

where r is the rank, m_j exponents of underlying finite-dimensional root system, δ the minimal imaginary root.

$$\mathfrak{m} = \prod_{i=1}^{\infty} \prod_{j=1}^{r} \left(\prod_{k=1}^{m_j} \frac{(1 - t^k \cdot e^{i \cdot \delta})^2}{(1 - t^{k-1} e^{i \cdot \delta})(1 - t^{k+1} e^{i \cdot \delta})} \right)$$

$$-m_{i \cdot \delta}(t) = \sum_{j=1}^{r} \left(\sum_{k=1}^{m_j} t^{k-1} \cdot (-1 + 2t - t^2) \right) = -(1-t)^2 \cdot \sum_{j=1}^{r} \frac{t^{m_j} - 1}{t - 1}$$

More generally, by work of Cherednik and Macdonald, for any affine Φ:

$$\mathfrak{m} = \prod_{\lambda \in Q_{\mathrm{im}}^{+}} \left(\prod_{\beta \in S(\lambda)} \frac{(1 - t^{\mathrm{ht}(\beta)} e^{\lambda})^2}{(1 - t^{\mathrm{ht}(\beta)-1} e^{\lambda})(1 - t^{\mathrm{ht}(\beta)+1} e^{\lambda})} \right)$$

where

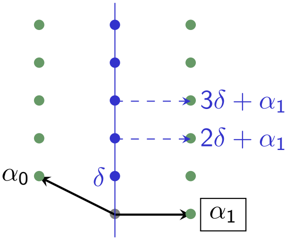
$$S(\lambda) = \{\beta \in Q_{\mathrm{fin}}^{+} \mid \beta + \lambda \in \Phi_{\mathrm{re}}\},$$

Q_{fin}^{+} is a root lattice corresponding to a finite root subsystem $\Phi_{\mathrm{fin}} \subseteq \Phi$ determined by omitting an appropriate simple root.

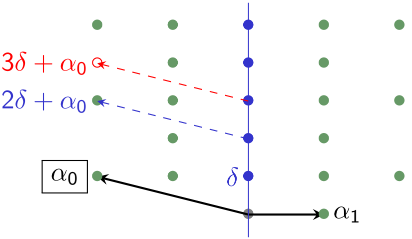
$$m_{\lambda} = (1-t)^2 \cdot \sum_{\beta \in S(\lambda)} t^{\mathrm{ht}(\beta)-1}$$

$$S(\lambda) = \{\beta \in Q_{\mathrm{fin}}^+ \mid \beta + \lambda \in \Phi_{\mathrm{re}}\}$$

$A_1^{(1)}$ $(\widehat{\mathfrak{sl}}_2)$,



$A_2^{(2)}$



Generalized Petersen algorithm

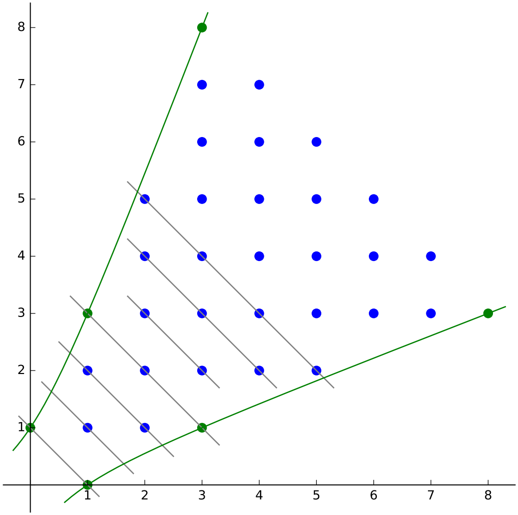
We wish to write

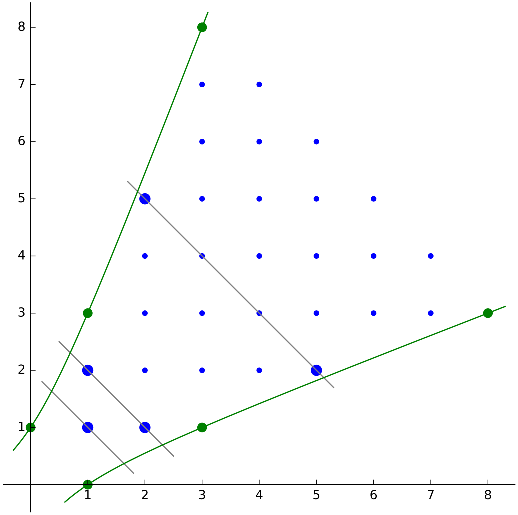
$$\mathfrak{m} = \prod_{\lambda \in Q^+_{\mathrm{im}}} \prod_{n=0}^{N_\lambda} (1 - t^n e^\lambda)^{-m(\lambda,n)}$$

starting from

$$\left. \left(\mathfrak{m}^{-1} \frac{\Delta_{\mathrm{re}}}{\Delta_{t,\mathrm{re}}} \right) \right|_{Q^+_{\mathrm{im}}} = 1$$

- power series inverse with respect to Q^+_{im}
- by induction on height
- algorithm polynomial in height
- generalization of the Petersen algorithm for $\mathrm{mult}(\lambda)$
- suffices to compute for one λ per W-orbit, i.e. on antidominant cone


8
7
6
5
4
3
2
1
1
2
3
4
5
6
7
8


1
2
3
4
5
6
7
8
1
2
3
4
5
6
7
8

Generalized Berman-Moody formula

Theorem [Muthiah-P-Whitehead] For all $\lambda \in Q^+$, we have:

$$m_\lambda(t) = \sum_{\kappa|\lambda} \mu(\lambda/\kappa) \left(\frac{\lambda}{\kappa}\right)^{-1} \sum_{\underline{\kappa} \in \mathrm{Par}(\kappa)} (-1)^{|\underline{\kappa}|} \frac{B(\underline{\kappa})}{|\underline{\kappa}|} \prod_{i=1}^{|\underline{\kappa}|} P_{\kappa_i}(t^{\lambda/\kappa})$$

- For $t = 0$ recovers the Berman-Moody formula for $\mathrm{mult}(\lambda) = m_\lambda(0)$
- $\lambda, \kappa \in Q^+$, $\lambda = k \cdot \kappa$, then $\kappa|\lambda$, $\frac{\lambda}{\kappa} = k \in \mathbb{Z}$, $\mu(\lambda/\kappa)$ Möbius function
- $\mathrm{Par}(\lambda)$ vector partitions of λ, $|\underline{\kappa}|$, $B(\underline{\kappa})$
- $P_{\kappa_i}(t^{\lambda/\kappa}) = P_{\kappa_i}(t^k)$ given in terms of Kostant partitions of $\kappa_i \in Q^+$.
- For any $\mu \in Q^+$, $\mu \neq 0$ $P_\mu(1) = 0$

Properties of $\mathfrak{m}$

Theorem [Muthiah-P-Whitehead] For $\lambda \in Q^+_{\mathrm{im}}$, $m_\lambda(t) \neq 0 \Leftrightarrow \lambda \in \Phi_{\mathrm{im}}$.

- If $\Phi_1 \subseteq \Phi$ root subsystem, $Q_1 \subseteq Q$, $\mathfrak{m}_1$, $\mathfrak{m}$; then $\mathfrak{m}|_{Q_1} = \mathfrak{m}_1$.
- If $\Phi_1, \Phi_2 \subseteq \Phi$, simple roots $\Delta_1 \perp \Delta_2$, then $\mathfrak{m} = \mathfrak{m}_1 \mathfrak{m}_2$.
- If $\lambda \in Q^+_{\mathrm{im}} \setminus \Phi_{\mathrm{im}}$ antidominant, then $\mathrm{Supp}_\Delta \lambda$ disconnected.

Theorem [Muthiah-P-Whitehead] For $\lambda \in Q^+_{\mathrm{im}}$, $(1-t)^2 | m_\lambda(t)$.

- Use Generalized Berman-Moody formula
- Observation: $\lambda \in Q^+_{\mathrm{im}}$ as sum over Φ_{re} has at least two terms.
- For any $\mu \in Q^+$, $\mu \neq 0$: $P_\mu(1) = 0$

$$m_\lambda(t) = \sum_{\kappa | \lambda} \mu(\lambda/\kappa) \left(\frac{\lambda}{\kappa}\right)^{-1} \sum_{\underline{\kappa} \in \mathrm{Par}(\kappa)} (-1)^{|\underline{\kappa}|} \frac{B(\underline{\kappa})}{|\underline{\kappa}|} \prod_{i=1}^{|\underline{\kappa}|} P_{\kappa_i}(t^{\lambda/\kappa})$$

An illustration...

$$\chi_\lambda(t) = \frac{m_\lambda(t)}{(1-t)^2} = \frac{\sum_{i=0}^{N_\lambda} m(\lambda, n) \cdot t^n}{(1-t)^2}$$

Using the Generalized Petersen algorithm, compute this for the hyperbolic root systems with Cartan matrices

$$\begin{bmatrix} 2 & -3 \\ -2 & 2 \end{bmatrix}, \begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

$\chi_\lambda(t)$, Cartan matrix $\begin{bmatrix} 2 & -3 \\ -2 & 2 \end{bmatrix}$

λ	$\chi_\lambda(t)$
$(1,1)$	1
$(2,2)$	$-t+1$
$(3,2)$	t^2+0t+2
$(3,3)$	$-t^3-2t+2$
$(4,3)$	$t^4-t^3+2t^2-3t+3$
$(4,4)$	$-t^5+t^4-2t^3+3t^2-6t+3$
$(5,4)$	$t^6-2t^5+4t^4-6t^3+9t^2-9t+6$
$(5,5)$	$-t^7+t^6-4t^5+6t^4-10t^3+13t^2-13t+7$
$(6,4)$	$t^6-4t^5+5t^4-8t^3+11t^2-13t+6$
$\dots$	$\dots$
$(10,9)$	$t^{16}-7t^{15}+29t^{14}-91t^{13}+248t^{12}-584t^{11}+1197t^{10}-2170t^9+3505t^8-5039t^7+6437t^6-7253t^5+7042t^4-5618t^3+3405t^2-1372t+272$

$\chi_\lambda(t)$, Cartan matrix $\begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix}$

λ	$\chi_\lambda(t)$
$(1,1)$	1
$(2,2)$	$-2t+1$
$(2,3)$	t^2-t+2
$(3,3)$	$-2t^3+3t^2-4t+3$
$(3,4)$	$t^4-3t^3+6t^2-6t+4$
$(4,4)$	$-2t^5+7t^4-12t^3+17t^2-16t+6$
$(4,5)$	$t^6-5t^5+15t^4-26t^3+30t^2-23t+9$
$(4,6)$	$t^6-8t^5+19t^4-31t^3+36t^2-28t+9$
$(5,5)$	$-2t^7+9t^6-30t^5+58t^4-82t^3+77t^2-50t+16$
$\dots$	$\dots$
$(10,9)$	$t^{16}-15t^{15}+135t^{14}-811t^{13}+3535t^{12}-11729t^{11}+30615t^{10}-64282t^9+110096t^8-154852t^7+178868t^6-168420t^5+127110t^4-74539t^3+32094t^2-9070t+1267$

$\chi_\lambda(t)$, Cartan matrix $\begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

λ	$\chi_\lambda(t)$	λ	χ_λ
$(1,1,0)$	1	$(5,5,1)$	$-5t+7$
$(2,2,0)$	1	$(5,5,2)$	$2t^3+2t^2-17t+15$
$(2,2,1)$	2	$(5,6,2)$	$-t^4+3t^3+6t^2-26t+22$
$(3,3,0)$	1	$(5,6,3)$	$-3t^4+6t^3+13t^2-43t+30$
$(3,3,1)$	$-t+3$	$(6,6,0)$	1
$(3,4,2)$	$-2t+5$	$(6,6,1)$	$t^2-8t+11$
$(4,4,0)$	1	$(6,6,2)$	$-2t^4+5t^3+11t^2-43t+30$
$(4,4,1)$	$-2t+5$	$(6,6,3)$	$-6t^4+8t^3+23t^2-65t+42$
$(4,4,2)$	$-t^2-6t+7$	$(6,7,2)$	$-5t^4+6t^3+22t^2-63t+42$
$(4,5,2)$	$t^3+t^2-9t+11$	$(7,7,0)$	1
$(5,5,0)$	1	$(7,7,1)$	$2t^2-15t+15$

Questions

Conjecture The polynomials χ_λ have alternating sign coefficients in rank two hyperbolic type.

Problem Interpret all coefficients of χ_λ in terms of the Kac-Moody Lie algebra.

Problem Give upper bounds for the degree and coefficients of $\chi_\lambda(t)$.

Question Relationship of $m_\lambda(t)$ and Kac polynomials?

Question What is the Dolbeault cohomology of Kac-Moody flag varieties? (A two-parameter generalization of $\mathfrak{m}$.)

Thank you!