Shuffle algebra, Macdonald operators and lattice models

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Overview

– Basic symmetric functions, Macdonald functions and operators
– Heisenberg algebra and Fock space
– Shuffle algebra
– The isomorphism
– Applications

Basic symmetric polynomials

- The *power sums* $p_r(x) = p_r(x_1, \ldots, x_n)$

$$p_r(x) = x_1^r + \cdots + x_n^r$$

- The *elementary* and *homogeneous* symmetric polynomials are generated by

$$\mathcal{E}(z) = \sum_{k\geq 0} e_k(x) z^k = \prod_{i=1}^{n}(1 + x_i z) = \exp \sum_{r>0} \frac{(-1)^{r+1}}{r} p_r(x) z^r$$

$$\mathcal{H}(z) = \sum_{k\geq 0} h_k(x) z^k = \prod_{i=1}^{n}(1 - x_i z)^{-1} = \exp \sum_{r>0} \frac{1}{r} p_r(x) z^r$$

more explicitly

$$e_k(x) = \sum_{1\leq i_1<\cdots<i_k\leq n} x_{i_1} \ldots x_{i_k}, \qquad h_k(x) = \sum_{1\leq i_1\leq\cdots\leq i_k\leq n} x_{i_1} \ldots x_{i_k}$$

- For a partition $\lambda = (\lambda_1, \ldots, \lambda_N)$ we have

$$p_\lambda(x) = \prod_{j=1}^{N} p_{\lambda_j}(x), \qquad e_\lambda(x) = \prod_{j=1}^{N} e_{\lambda_j}(x), \qquad h_\lambda(x) = \prod_{j=1}^{N} h_{\lambda_j}(x)$$

- The *monomial symmetric polynomials*

$$m_\lambda(x) = \sum_{\sigma\in\mathcal{S}_n/\mathcal{S}_\lambda} x_{\sigma(1)}^{\lambda_1} \cdots x_{\sigma(n)}^{\lambda_n}$$

Symmetric polynomials with two parameters

Symmetric polynomials $p_\lambda(x), e_\lambda(x), h_\lambda(x)$ and $m_\lambda(x)$ form bases in the ring $\Lambda^n = \mathbb{Z}[x_1, \ldots, x_n]^{\mathcal{S}_n}$. We use the degree of polynomials to decompose Λ^n

$$\Lambda^n = \bigoplus_{d\geq 0} \Lambda^{n,d}$$

Consider the ring $\Lambda^n_{\mathbb{F}} = \Lambda^n \otimes \mathbb{F}$ with $\mathbb{F} = \mathbb{Q}(q,t)$.

- With modified generating functions we build elements in $\Lambda^n_{\mathbb{F}}$, e.g.:

$$\mathcal{G}(z) = \sum_{n\geq 0} g_k(x;q,t) z^k = \exp \sum_{r>0} \frac{1}{r} \frac{1-t^r}{1-q^r} p_r(x) z^r$$

and one can write products $g_\lambda(x;q,t) = \prod_{j=1}^N g_{\lambda_j}(x;q,t)$.

- Alternatively one modifies $m_\lambda(x)$

$$P_\lambda(x;q,t) = \sum_{\mu} v_{\lambda,\mu}(q,t) m_\mu(x)$$

for some $v_{\lambda,\mu}(q,t) \in \mathbb{F}$. An important choice is when $v_{\lambda,\mu}(q,t) = 0$ for partitions $\mu > \lambda$ with the *dominance ordering*

$$\mu \geq \lambda \quad \text{when} \quad \sum_{i=1}^k \mu_i \geq \sum_{i=1}^k \lambda_i, \qquad k \geq 0$$

Scalar product and Macdonald polynomials

- Define the scalar product on Λ

$$\langle p_\lambda | p_\mu \rangle = \delta_{\lambda,\mu} c_\lambda, \qquad c_\lambda := \prod_{i=1}^{N} \lambda_i \prod_{i=1}^{\max(\lambda)} m_i(\lambda)!$$

were $m_i(\lambda)$ is the multiplicity of i in λ.

- Define the scalar product (*Macdonald scalar product*) on $\Lambda_{\mathbb{F}}$

$$\langle p_\lambda | p_\mu \rangle_{q,t} = \delta_{\lambda,\mu} c_\lambda \prod_{i=1}^{N} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}$$

For this scalar product there exists a basis $P_\lambda(x;q,t)$ s.t.[1]:

$$\langle P_\lambda | P_\mu \rangle_{q,t} = 0, \quad \lambda \neq \mu,$$

$$\text{and} \qquad P_\lambda(x;q,t) = m_\lambda(x) + \sum_{\mu<\lambda} v_{\lambda,\mu}(q,t) m_\mu(x)$$

Example

$$P_{(2,1)}(x;q,t) = m_{(2,1)}(x) + \frac{(1-t)(2+q+t+2qt)}{1-qt^2} m_{(1,1,1)}(x)$$

[1] Macdonald, *Symmetric functions and Hall polynomials*

Difference operators

Introduce shift operators T_{q,x_i}

$$T_{q,x_i} f(x_1, \ldots, x_i, \ldots, x_n) = f(x_1, \ldots, qx_i, \ldots, x_n)$$

Consider the operator D_k (Macdonald operator)

$$D_k := t^{k(k-1)/2} \sum_{\substack{I \subseteq [n] \\ \#I = k}} \prod_{i \in I} \prod_{j \in I^c} \frac{tx_i - x_j}{x_i - x_j} \cdot \prod_{i=1}^{n} T_{q,x_i}$$

D_k's are diagonal in the basis $P_\lambda(x; q, t)$

$$D_k P_\lambda(x; q, t) = e_k(q^{\lambda_1} t^{n-1}, \ldots, q^{\lambda_n}) P_\lambda(x; q, t)$$

Consider another operator H_k (Noumi operator)

$$H_k := \sum_{\substack{\nu \subseteq \mathbb{N}^n \\ |\nu| = k}} \prod_{1 \le i < j \le n} \frac{q^{\nu_i} x_i - q^{\nu_j} x_j}{x_i - x_j} \prod_{i,j=1}^{n} \frac{(tx_i/x_j; q)_{\nu_i}}{(qx_i/x_j; q)_{\nu_i}} \cdot \prod_{i=1}^{n} T_{q,x_i}^{\nu_i}$$

H_k's are diagonal in the basis $P_\lambda(x; q, t)$

$$H_k P_\lambda(x; q, t) = g_k(q^{\lambda_1} t^{n-1}, \ldots, q^{\lambda_n}; q, t) P_\lambda(x; q, t)$$

Symmetric functions and Macdonald operators

From now on we consider an infinite alphabet

$$p_r(x) = x_1^r + x_2^r + \dots$$

All functions are modified accordingly. The operators are modified as well:

$$E_r := \sum_{k=0}^{r} \frac{(-1)^{r+k} t^{-nr}}{(t,t)_{r-k}} D_k$$

$$G_r := \sum_{k=0}^{r} \frac{q^{r+k} t^{-nr}}{(q^{-1},q^{-1})_{r-k}} H_k$$

The operators E_k and G_k act on $P_\lambda(x;q,t) \in \Lambda_{\mathbb{F}}^{\infty}$ as follows

$$E_k P_\lambda(x;q,t) = e_k(q^{\lambda_1} t^{-1}, q^{\lambda_2} t^{-2}, \dots) P_\lambda(x;q,t)$$

$$G_k P_\lambda(x;q,t) = g_k(q^{\lambda_1} t^{-1}, q^{\lambda_2} t^{-2}, \dots; q,t) P_\lambda(x;q,t)$$

Let $\ell(\lambda) = \#\lambda_i$, s.t. $\lambda_i > 0$. For evaluating the symmetric functions one may use

$$p_r(q^{\lambda_1} t^{-1}, q^{\lambda_2} t^{-2}, \dots) = (1-q^{-r}) \sum_{i=1}^{\ell(\lambda)} \sum_{j=1}^{\lambda_i} q^{jr} t^{-ir} - \frac{1}{(1-t^r)}$$

Heisenberg algebra and Fock space

We will rephrase the symmetric functions and operators using the Heisenberg algebra:

$$H = \{a_n, a_{-n}\}_{n>0}, \qquad [a_m, a_n] = \delta_{m,-n}\, m \frac{1-q^{|m|}}{1-t^{|m|}}$$

Note $[a_n, a_m] = 0$, $m, n < 0$ or $m, n > 0$, then define

$$a_\lambda = a_{\lambda_1} \dots a_{\lambda_{\ell(\lambda)}}.$$

This algebra acts on the vector space $\mathcal{F}$ with the lowest vector $|\varnothing\rangle$

$$a_n\,|\varnothing\rangle = 0, \qquad a_{-n}\,|\varnothing\rangle = \text{higher states}, \qquad n > 0$$

The Fock space is spanned by $|a_\lambda\rangle = a_{-\lambda}\,|\varnothing\rangle$ and the action of H is:

$$a_{-\nu}\,|a_\mu\rangle = |a_{\mu\cup\nu}\rangle\,, \qquad a_\nu\,|a_\mu\rangle = c_\nu \begin{bmatrix} m(\mu) \\ m(\nu) \end{bmatrix} \prod_{r\in\nu} \frac{1-q^r}{1-t^r}\,|a_{\mu/\nu}\rangle$$

Analogously one defines the dual Fock space $\mathcal{F}^*$ spanned by $\langle a_\lambda|$, then

$$\langle a_\lambda | a_\mu\rangle = \delta_{\lambda,\mu} c_\lambda \prod_{i=1}^{\ell(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}$$

Symmetric functions and Fock vectors

Comparing $\mathcal{F}$ and $\Lambda_{\mathbb{F}}$ with the basis $p_\lambda(x)$ we conclude

$$\iota : \mathcal{F} \to \Lambda_{\mathbb{F}}, \qquad \iota : |a_\lambda\rangle \mapsto |p_\lambda\rangle$$

Through this isomorphism we get an H module on $\Lambda_{\mathbb{F}}$

$$a_{-r} f(x) = p_r(x) f(x), \qquad a_r f(x) = r \frac{1-q^r}{1-t^r} \frac{\partial}{\partial p_r} f(x), \qquad f(x) \in \Lambda_{\mathbb{F}}$$

The Macdonald operators in the language of H are produced using the operator[2]

$$\eta(z) := \exp\left(-\sum_r \frac{1-t^r}{r} t^{-r} a_{-r} z^r \right) \exp\left(-\sum_r \frac{1-t^r}{r} a_r z^{-r} \right)$$

This is a generator of operators

$$\eta(z) = \sum_{n\in\mathbb{Z}} \eta_n z^{-n}, \qquad \text{e.g:} \qquad \eta_0 = \sum_{\substack{\lambda,\mu \\ |\lambda|=|\mu|}} c_\lambda c_\mu t^{-|\lambda|} \prod_{r\in\lambda\cup\mu} (t^r - 1) \cdot a_{-\lambda} a_\mu$$

Note: for fixed α, β only finitely many terms $\langle a_\alpha |\, a_{-\lambda} a_\mu \,| a_\beta\rangle$ are non-zero.

[2] Shiraishi '06

Vertex operator $\eta(z)$

Define the normal ordering $\quad : a_{-\lambda}a_{\mu} :=: a_{\mu}a_{-\lambda} := a_{-\lambda}a_{\mu}$

The normal ordering rule for products of $\eta(z)$ is

$$\eta(z)\eta(w) = \frac{(1-w/z)(1-qt^{-1}w/z)}{(1-qw/z)(1-t^{-1}w/z)} : \eta(z)\eta(w) :$$

Comparison of $\eta(z)\eta(w)$ with $\eta(w)\eta(z)$ gives

$$\eta(z)\eta(w) = \frac{-g(w,z)}{g(z,w)}\eta(w)\eta(z), \qquad g(w,z) := (w-tz)(w-q^{-1}z)(w-qt^{-1}z)$$

Realization of the first Macdonald operator:

The mode η_0 is diagonal in the basis $P_\lambda(x;q,t)$

$$\eta_0 = (t-1)E_1$$

and $\qquad \eta_0\, P_\lambda(x;q,t) = (t-1)\, e_1(q^{\lambda_1}t^{-1}, q^{\lambda_2}t^{-2}, \dots)P_\lambda(x;q,t)$

Proof: by direct comparison of the action of η_0 and E_1 in the basis $p_\lambda(x)$.[3]

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[3] Shiraishi '06

Higher Macdonald operators via $\eta(z)$

The operator η_0 can be represented as

$$\eta_0 = \oint \frac{\mathrm{d}\, z}{z} \eta(z)$$

Introduce a rational function $\epsilon_n(z) = \epsilon_n(z_1, \dots, z_n)$

$$\epsilon_n(z) = \prod_{i<j} \frac{(z_i - tz_j)(z_i - t^{-1}z_j)}{(z_i - z_j)^2}$$

With this define the operators

$$\widehat{E}_n := \frac{1}{(t-1)^n n!} \oint \prod_{i=1}^{n} \frac{\mathrm{d}\, z_1 \cdots \mathrm{d}\, z_n}{z_1 \cdots z_n} \frac{1}{\epsilon_n(z)} : \eta(z_1) \dots \eta(z_n) :$$

The action of $\widehat{E}_n$ in the Macdonald basis is determined by

$$\widehat{E}_n = E_n$$

$$\widehat{E}_n P_\lambda(x; q, t) = e_n(q^{\lambda_1} t^{-1}, q^{\lambda_2} t^{-2}, \dots) P_\lambda(x; q, t)$$

Proof: Use the connection between η_0 and E_1 and prove the statement recursively.[4]

[4] Shiraishi '06; Feigin, Hashizume, Hoshino, Shiraishi and Yanagida '09

The shuffle algebra[5] $\mathcal{A}_n^+$

We need three parameters q_1, q_2, q_3 with $q_1 q_2 q_3 = 1$ and

$$q_1 = q^{-1}, \qquad q_3 = t$$

- The elements of $\mathcal{A}_n^+ = \mathcal{A}_n^+(q_1, q_2, q_3)$ are of the form

$$F(z_1, \ldots, z_n) = \frac{f(z_1, \ldots, z_n)}{\prod_{1 \le i < j \le n}(z_i - z_j)^2} \qquad f(z_1, \ldots, z_n) \in \mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]^{\mathcal{S}_n}$$

- The elements of $\mathcal{A}_n^+$ satisfy the *wheel condition*

$$f(z_1, \ldots, z_n) = 0 \quad \text{if} \quad (z_i, z_j, z_k) = (z, q_1 z, q_1 q_2 z) \quad \text{and} \quad (z_i, z_j, z_k) = (z, q_2 z, q_1 q_2 z)$$

- Examples: $\mathcal{A}_0^+ = \mathbb{F}$,

$$z_1^j \in \mathcal{A}_1^+, \qquad j \in \mathbb{Z}$$

$$\frac{(z_1 - q_i z_2)(z_1 - q_i^{-1} z_2)}{(z_1 - z_2)^2} \in \mathcal{A}_2^+, \qquad \text{for} \quad i = 1, 2, 3.$$

[5] Feigin and Odesskii '97; Feigin and Tsymbaliuk '11; Schiffmann and Vasserot '13; Negut '12

The shuffle product

Set

$$\omega(x,y) := \frac{(x-q_1y)(x-q_2y)(x-q_3y)}{(x-y)^3} = \frac{g(x,y)}{(x-y)^3}$$

For $F \in \mathcal{A}_k^+$ and $G \in \mathcal{A}_l^+$ we have $F * G \in \mathcal{A}_{k+l}^+$

$$(F*G)(z_1,\ldots,z_{k+l}) = \operatorname{Sym} F(z_1,\ldots,z_k)G(z_{k+1},\ldots,z_{k+l}) \prod_{\substack{i\in\{1,\ldots,k\}\\ j\in\{k+1,\ldots,k+l\}}} \omega(z_j,z_i)$$

where

$$\operatorname{Sym} f(z_1,\ldots,z_n) = \frac{1}{n!}\sum_{\sigma\in\mathcal{S}_n} f(z_{\sigma(1)},\ldots,z_{\sigma(n)}),$$

Example:

$$z_1^i * z_1^j = \frac{z_1^i z_2^j\,(z_2-q_1z_1)\,(z_2-q_2z_1)\,(z_2-q_3z_1)}{(z_2-z_1)^3} + \frac{z_2^i z_1^j\,(z_1-q_1z_2)\,(z_1-q_2z_2)\,(z_1-q_3z_2)}{(z_1-z_2)^3}$$

The shuffle algebras $\mathcal{A}^+$ and the subalgebra $\mathcal{A}$

- The full shuffle algebra: $\mathcal{A}^+ = \bigoplus_{n\geq 0} \mathcal{A}_n^+$.

$\mathcal{A}^+$ is generated by $z_1^j \in \mathcal{A}_1^+$ for $j \in \mathbb{Z}$ [6], and

$$\delta(u/z_1) * \delta(v/z_1) = \frac{-g(v,u)}{g(u,v)}\delta(v/z_1) * \delta(u/z_1), \qquad \text{where} \quad \delta(w/z_1) := \sum_{j\in\mathbb{Z}} z_1^j w^{-j}$$

- Consider a subalgebra $\mathcal{A}_n \subset \mathcal{A}_n^+$ of degree 0 elements F which satisfy

$$\left(\lim_{\xi\to 0} - \lim_{\xi\to\infty}\right) F(z_1, \ldots, z_{n-k}, \xi z_{n-k+1}, \xi z_{n-k+2} \ldots, \xi z_n) = 0$$

For $n = 0, 1$ we have $\mathcal{A}_0 = \mathbb{F}$, $\mathcal{A}_1 = \mathbb{F}z_1^0$. For $n > 1$ distinguished elements are

$$\epsilon_n(z; q_k) := \prod_{1\leq i<j\leq n} \frac{(z_i - q_k z_j)(z_i - q_k^{-1} z_j)}{(z_i - z_j)^2}, \qquad k = 1, 2, 3$$

In the following we focus only on the subalgebra $\mathcal{A}$.

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[6] Schiffmann and Vasserot '13; Negut '12

Properties[7] *of* $\mathcal{A}$

The dimension of $\mathcal{A}_n$ equals to the number of partitions of n

Proof: for a partition λ we need to show that the following elements form bases

$$\epsilon_\lambda(z;q_k) = \epsilon_{\lambda_1}(z;q_k)\cdots\epsilon_{\lambda_{\ell(\lambda)}}(z;q_k), \qquad k=1,2,3$$

The algebra $\mathcal{A}$ is commutative

Proof: using various specializations of z_i show that

$$\epsilon_n(z,q_1) * \epsilon_m(z,q_1) = \epsilon_m(z,q_1) * \epsilon_n(z,q_1)$$

Recall the vertex operator $\eta(z)$ and define the operators

$$\mathcal{O}(f) = \oint \frac{\mathrm{d}z_1\cdots\mathrm{d}z_n}{z_1\cdots z_n}\frac{f(z_1,\dots,z_n)}{\epsilon_n(z;q_1)\epsilon_n(z;q_3)} : \eta(z_1)\dots\eta(z_n) :, \qquad f\in\mathcal{A}_n$$

The operators $\mathcal{O}(f)$ for $f\in\mathcal{A}$ form a commutative ring

$$\mathcal{O}(f*g) = \mathcal{O}(f)\mathcal{O}(g)$$

[7] Feigin, Hashizume, Hoshino, Shiraishi and Yanagida '09

The map $\mathcal{O}$ defines a ring isomorphism $\Lambda_{\mathbb{F}} \to \mathcal{A}$

This follows by noting that

$$\widehat{E}_n = \mathcal{O}(\epsilon_n(z; q_1))$$

Similarly one constructs more Macdonald operators

$$\widehat{G}_n = \mathcal{O}(\epsilon_n(z; q_3))$$
$$\widehat{F}_n = \mathcal{O}(\epsilon_n(z; q_2))$$

and computes their eigenvalues using identities for symmetric functions.

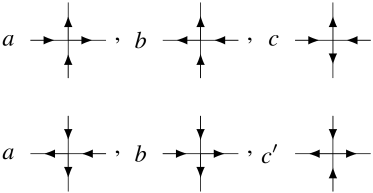
We can state three relations

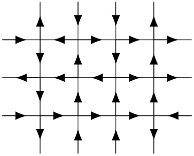
$$\epsilon_\lambda(z; q_1) \mapsto e_\lambda(x)$$
$$\epsilon_\lambda(z; q_3) \mapsto g_\lambda(x; q, t)$$
$$\epsilon_\lambda(z; q_2) \mapsto f_\lambda(x; q, t)$$

where $f_\lambda = f_{\lambda_1} \cdots f_{\lambda_{\ell(\lambda)}}$ is defined by another generating function

$$\mathcal{F}(z) = \sum_{k \geq 0} f_k(x; q, t) z^k = \exp \sum_{r > 0} \frac{1}{r} \frac{q^r - t^r}{1 - q^r} p_r(x) z^r$$

A lattice model





This is the *six vertex model* associated to $U_t(\widehat{sl}_2)$. The states are encoded by

$$R = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & c' & 0 \\ 0 & c & b & 0 \\ 0 & 0 & 0 & a \end{bmatrix} \qquad \textit{or} \qquad R(x) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{t(1-x)}{1-t^2x} & \frac{(1-t^2)x}{1-t^2x} & 0 \\ 0 & \frac{1-t^2}{1-t^2x} & \frac{t(1-x)}{1-t^2x} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and the *Yang–Baxter equation* is satisfied

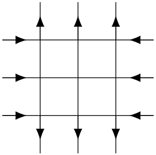
$$R_{1,2}(x_2/x_1)R_{1,3}(x_3/x_1)R_{2,3}(x_3/x_2) = R_{2,3}(x_3/x_2)R_{1,3}(x_3/x_1)R_{1,2}(x_2/x_1)$$

The six vertex partition function on a domain $\mathcal{C}$ is given by

$$\mathcal{Z} = \prod_{(i,j)\in\mathcal{C}} R_{i,j}$$

Partition functions and shuffle elements

Consider the partition functions Z_{DW}



This is known as the (Korepin's) domain wall partition function and it is equal to a determinant (Izergin's determinant)

$$Z_{DW}(x;y) \propto \det_{1\le i,j\le n} \frac{1}{(y_i - x_j)(y_i - tx_j)}$$

Consider the case $y_i = qx_i$ and set $\chi_N(z) := Z_{DW}(z;qz)$

$$\chi_n(z) = \frac{\prod_{i<j}(qz_i - z_j)(qz_i - tz_j)}{\prod_{i<j}(z_i - z_j)^4} \det_{1\le i,j\le n} \frac{1}{(qz_i - z_j)(qz_i - tz_j)}$$

Claim: $\chi_n(z) \in \mathcal{A}$ and

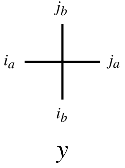
$$\mathcal{O}(\chi_n(z))P_\lambda(x;q,t) = h_n(q^{\lambda_1}t^{-1}, q^{\lambda_2}t^{-2}, \dots)P_\lambda(x;q,t)$$

$$\chi_\lambda(z) \mapsto h_\lambda(x)$$

Coloured vertex models

Generalization the previous model by colouring lattice paths

$$R^{j_a,\,j_b}_{i_a,\,i_b}(x/y) = \; x$$



where $\{i_a, i_b\} = \{j_a, j_b\}$ as sets. This R-matrix is associated to $U_t(\widehat{gl}_n)$.
The weights are

$$R^{j_a,\,j_b}_{i_a,\,i_b}(x) = t^{\theta(j_a<i_b)} x^{\theta(j_a<i_a)} (1 - t^{\theta(j_a=i_b)} x^{\theta(j_a=i_a)})$$

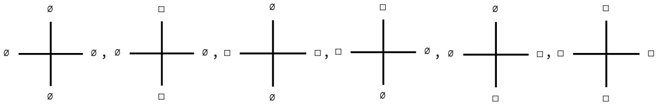
where $\theta(\mathrm{True}) = 1$ and $\theta(\mathrm{False}) = 0$.

Claim: square domain coloured partition functions with the specialized weights $y_i = qx_i$ produce more elements of $\mathcal{A}$ with a similar action as previous operators.[8]

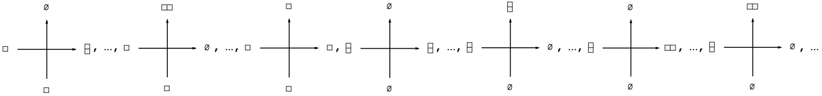
[8] AG and P. Zinn-Justin, in preparation

Vertices of a more general model

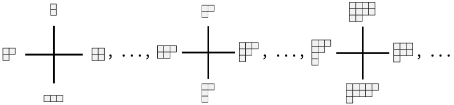
- Start with the six vertex model description using particles (boxes)



- Add higher edge states



- Allow arbitrary Young diagrams such that the number of boxes is conserved



A vertex model of this type can be constructed using the Fock rep. of $U_{q,t}(\ddot{gl}_1)$.

The quantum group $U_{q,t}(\ddot{gl}_1)$

This quantum group $U_{q,t}(\ddot{gl}_1)$ is generated by four currents, two central elements $c, c^{\perp}$ and depends on two parameters q and t

$$e(z), \quad f(z), \quad \psi^{+}(z), \quad \psi^{-}(z)$$

The defining relations are written using the same function $g(z,w)$

$$e(w)e(z) = \frac{-g(z,w)}{g(w,z)} e(z)e(w)$$
$$e(w)\psi^{+}(q^{-c}z) = \frac{-g(z,w)}{g(w,z)} \psi^{\pm}(q^{-c}z)e(w), \quad \dots$$

The currents $\psi^{\pm}(z)$ are e.g.f. for some Heisenberg elements $h_{\pm r}$

$$\psi^{\pm}(z) = q^{\mp c^{\perp}} \exp \pm \sum_{r=1}^{\infty} (q^{r} - q^{-r}) h_{\pm r} z^{\mp r}$$

The coproduct is

$$\Delta(e(z)) = e(zq^{-c_2}) \otimes \psi^{+}(zq^{-c_2}) + 1 \otimes e(z)$$
$$\Delta(h_r) = h_r \otimes 1 + q^{-c_1 r} 1 \otimes h_r, \dots$$

The Fock representation and the R-matrix

The Fock module $\mathcal{F}_u$ is a representation[9] of $U_{q,t}(\ddot{gl}_1)$: $h_r \mapsto a_r$ and

$$f(z) = \frac{u^{-1}}{(1-t^{-1})(1-q)}\eta(z), \qquad e(z) = \frac{u}{(1-t)(1-q^{-1})}\xi(z)$$

where $\xi(z)$ is a vertex operator similar to $\eta(z)$.

$U_{q,t}(\ddot{gl}_1)$ is a quantum double, a consequence is:

$$R(u) = \exp\left(\sum_{r\geq 1}\kappa_r(q,t)a_r\otimes a_{-r}\right)(q/t)^{-\frac{1}{2}(d_1+d_2)}\bar{R}(u)$$

$\bar{R}(u)$ equals to a generator of a modified operator[10] $\mathcal{O}_2(\epsilon_n(z;q_2))$

$$\bar{R}(u) = \sum_{n\geq 0}\frac{u^{-n}}{n!}\mathcal{O}_2(\epsilon_n(z;q_2))$$

In the Macdonald basis it has the form

$$\bar{R}(u) = \sum_{\rho,\sigma}\widehat{P}_\rho(X;q,t)\widehat{P}^*_\sigma(Y;q,t)\sum_{(i,j)\in\rho\cap\sigma}\frac{1}{1-uq^{i+j}t^{i-j}}f^b_{\rho,\sigma}(q,t)$$

[9] Feigin, Kojima, Shiraishi and Watanabe '07; Feigin and Tsimbalyuk 09; Schiffmann and Vasserot '13

[10] AG and A. Negut, in preparation