Connecting q-Whittaker and periodic Schur measures

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(Joint work with T. Imamura, M. Mucciconi)

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Summary

- Identity bet. marginals of q-Whittaker & periodic Schur meas.
- PNG and TASEP are related to Schur measure, determinantal point process (DPP) of free fermion at zero temperature.
- Most KPZ models are related to q-Whittaker measure. By standard approach, one arrives at Fredholm determinant formula after long calculations using Bethe ansatz. Asymptotics are rather involved. Half space is difficult.
- Our identity provides a new approach to study KPZ models through DPP of free fermions at finite temperature. Asymptotics is standard. Half space models can be studied.
- A proof is bijective through a generalization of RSK, which shows similar behaviors to box-ball systems.

Plan

1. Our new identity connecting the two measures
2. PNG, TASEP and Schur measure

   Last passage percolation
3. KPZ equation and $T > 0$ polymer

   Fredholm det with $T > 0$ kernel
4. Discrete KPZ models and q-Whittaker measure

   Periodic Schur measure
5. Bijection
6. A new approach to KPZ models

1. Introduction

Cauchy identities for three polynomials

$\boldsymbol{a} = (a_1, \ldots, a_N), \boldsymbol{b} = (b_1, \ldots, b_M)$, $\mathcal{P}$: set of partitions

Schur

$$\sum_{\lambda\in\mathcal{P}} s_\lambda(a)s_\lambda(b) = \prod_{i=1}^{N}\prod_{j=1}^{M}\frac{1}{1-a_ib_j}$$

q-Whittaker

$$\sum_{\mu\in\mathcal{P}} P_\mu(a)Q_\mu(b) = \prod_{i=1}^{N}\prod_{j=1}^{M}\frac{1}{(a_ib_j;q)_\infty} \quad (=: Z_{q\mathrm{W}})$$

Skew Schur

$$\sum_{\substack{\lambda,\rho\in\mathcal{P}\\ \rho\subset\lambda}} q^{|\rho|}s_{\lambda/\rho}(a)s_{\lambda/\rho}(b) = \frac{1}{(q;q)_\infty}\prod_{i=1}^{N}\prod_{j=1}^{M}\frac{1}{(a_ib_j;q)_\infty} \quad (=: Z_{\mathrm{pS}})$$

Our new identity relating q-Whittaker and skew Schur

Combining the last two identities and writing $\dfrac{1}{(q;q)_\infty} = \sum_{\nu\in\mathcal{P}} q^{|\nu|}$

$$\sum_{\mu,\nu\in\mathcal{P}} q^{|\nu|} P_\mu(a) Q_\mu(b) = \sum_{\substack{\lambda,\rho\in\mathcal{P}\\ \rho\subset\lambda}} q^{|\rho|} s_{\lambda/\rho}(a) s_{\lambda/\rho}(b)$$

Our new identity is the following refinement.

Theorem:

$$\sum_{\substack{\mu,\nu\in\mathcal{P}\\ \mu_1+\nu_1\le n}} q^{|\nu|} P_\mu(a) Q_\mu(b) = \sum_{\substack{\lambda,\rho\in\mathcal{P}\\ \rho\subset\lambda,\lambda_1\le n}} q^{|\rho|} s_{\lambda/\rho}(a) s_{\lambda/\rho}(b)$$

Two proofs: Matching of Fredholm determinants for both sides and a bijective one.

Today we mainly explain the first one in connection to KPZ.

q-Whittaker and periodic Schur measures

q-Whittaker measure

$$M_{q\mathsf{W}}(\mu) = \frac{1}{Z_{q\mathsf{W}}} P_\mu(a) Q_\mu(b)$$

Periodic Schur measure

$$M_{\mathsf{pS}}(\lambda) = \frac{1}{Z_{\mathsf{pS}}} \sum_{\rho \in \mathcal{P}, \rho(\subset \lambda)} q^{|\rho|} s_{\lambda/\rho}(a) s_{\lambda/\rho}(b)$$

Let us also introduce two independent random variables χ, S:

$$\mathbb{P}(\chi = n) = \frac{q^n}{(q;q)_n}(q;q)_\infty, \qquad n = 0, 1, 2, \ldots$$

$$\mathbb{P}(S = \ell) = \frac{t^\ell q^{\ell^2/2}}{(q;q)_\infty \theta(-tq^{1/2})}, \qquad \ell \in \mathbb{Z}, \text{ for } t > 0$$

with $\theta(x) = (x;q)_\infty (q/x;q)_\infty$

Rewriting of the identity

- The identity can be written as the one for marginals

$$\mathbb{P}\left(\mu_1+\chi\le n\right)=\mathbb{P}\left(\lambda_1\le n\right)$$

- Shift mixed periodic Schur measure is a DPP. Using

$$\mathbb{P}(\chi+S\le n)=\frac{1}{(-tq^{\frac{1}{2}+n};q)_\infty}$$

the identity can be written as

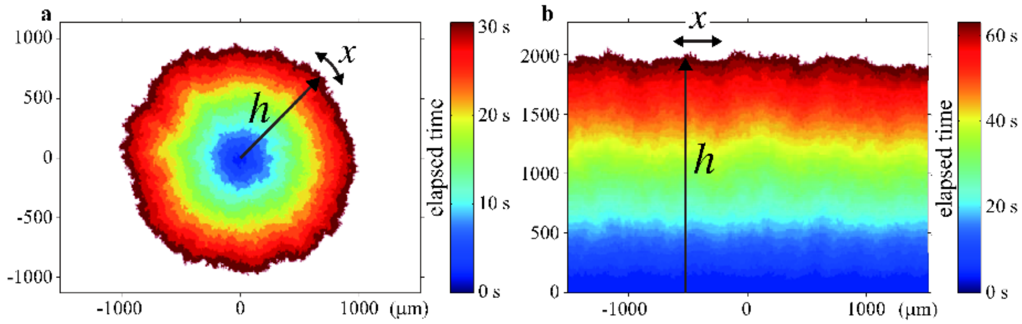
$$\mathbb{E}\left[\frac{1}{(-tq^{\frac{1}{2}+n-\mu_1};q)_\infty}\right]=\mathbb{P}(\lambda_1+S\le n)$$

Both hand sides can be written as Fredholm determinants (RHS: DPP, LHS: Connection to KPZ).

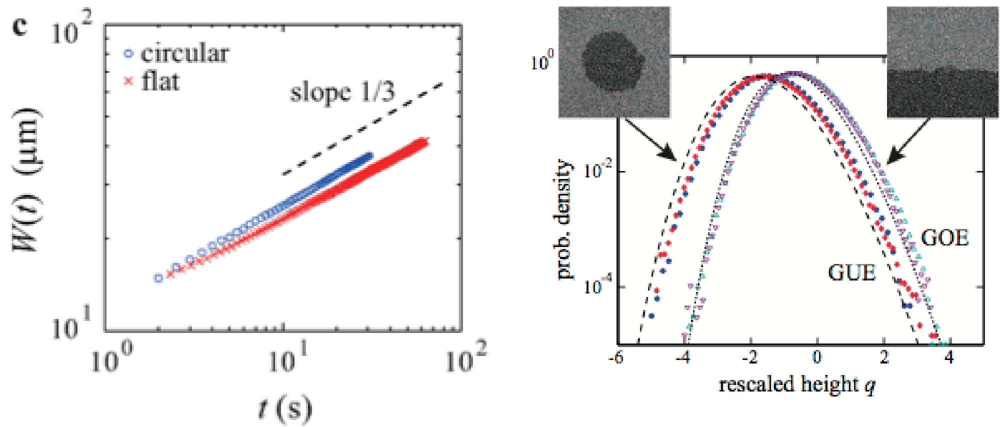
Kardar-Parisi-Zhang (KPZ) universality

Nonequilibrium statistical physics: fluctuations in surface growth.

Experiments by liquid crystal (2010 Takeuchi Sano)

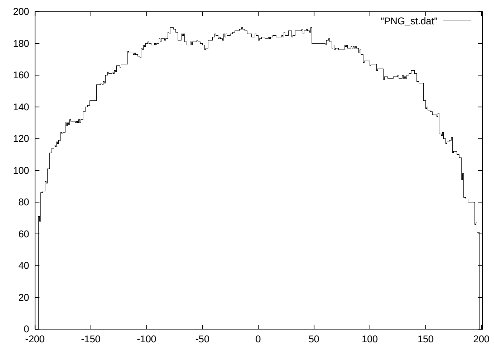


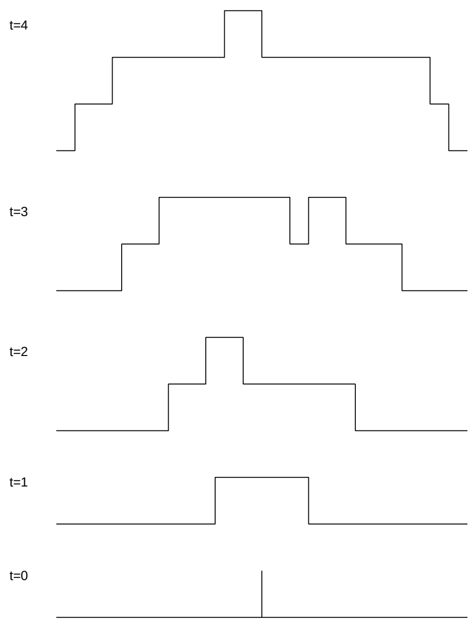
Fluctuations: $O(t^{1/3})$ and scaled distributions

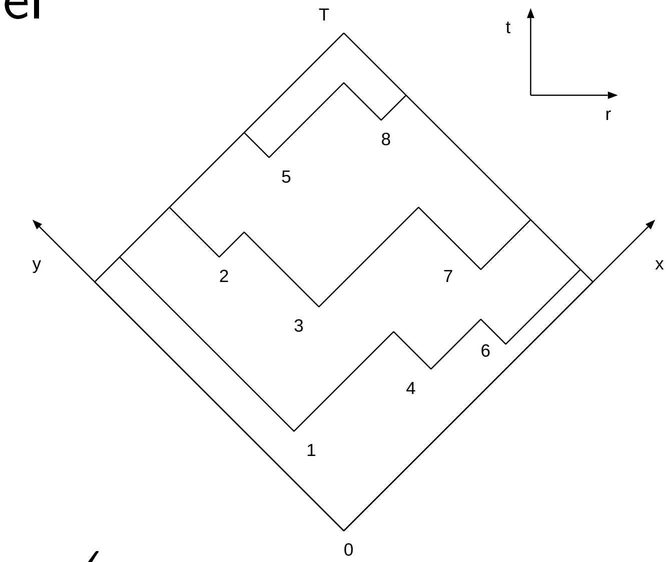


2. PNG, TASEP and Schur measure

PNG (polynuclear growth) model







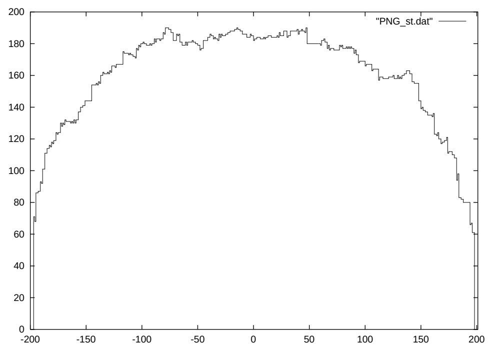
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 4 & 7 & 1 & 3 & 8 & 2 & 5 \end{pmatrix}$$

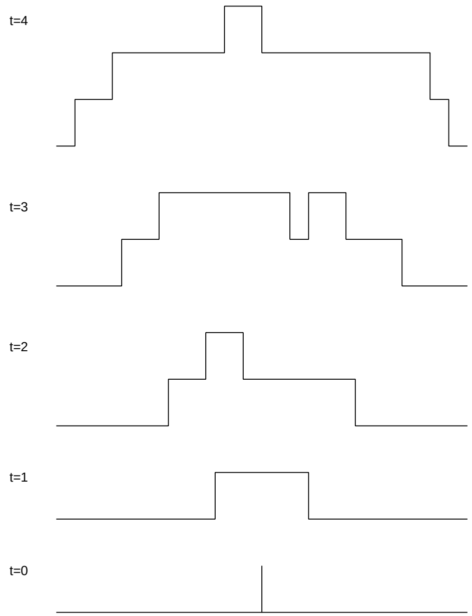
$$P = \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 7 & 8 \\ \hline 4 \\ \cline{1-1} 6 \\ \cline{1-1} \end{array} \qquad Q = \begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 2 & 5 & 8 \\ \hline 4 \\ \cline{1-1} 7 \\ \cline{1-1} \end{array}$$

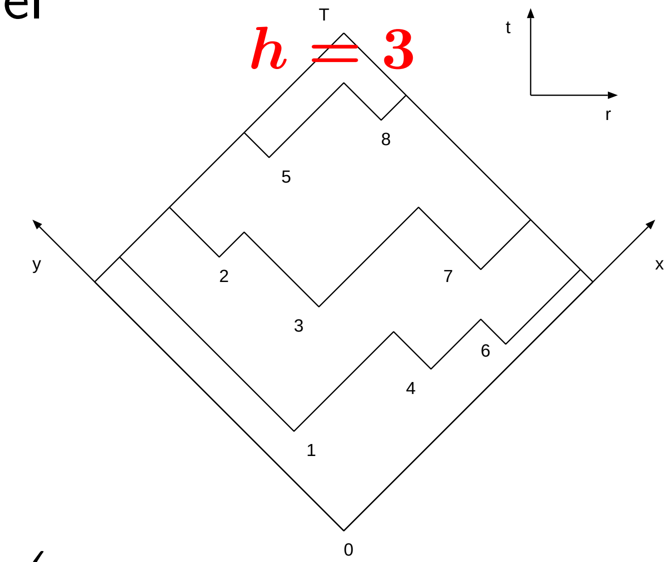
P Q Standard Taubleaux

1. PNG, TASEP and Schur measure

PNG (polynuclear growth) model



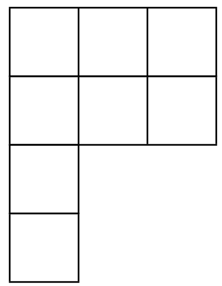




$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 4 & 7 & \color{red}{1} & 3 & 8 & \color{red}{2} & \color{red}{5} \end{pmatrix}$$

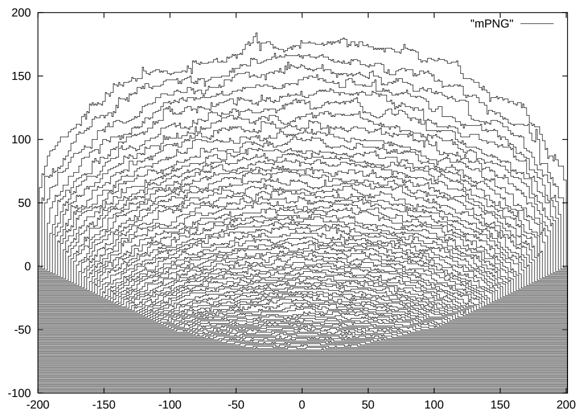
$$P = \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 7 & 8 \\ \hline 4 \\ \cline{1-1} 6 \\ \cline{1-1} \end{array} \qquad Q = \begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 2 & 5 & 8 \\ \hline 4 \\ \cline{1-1} 7 \\ \cline{1-1} \end{array}$$

$\lambda_1 = 3$

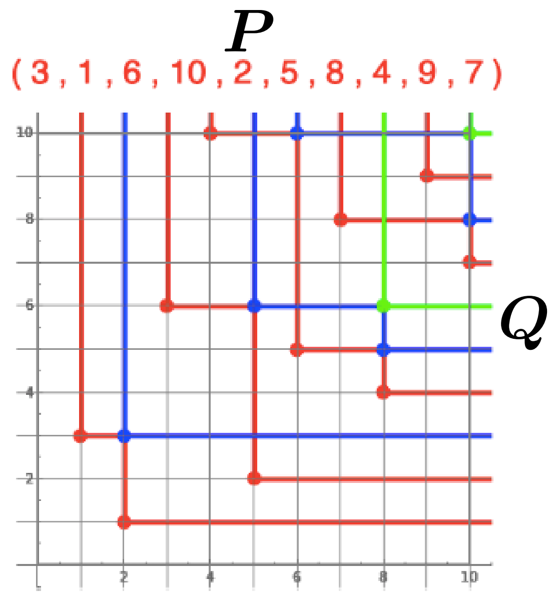
sh $= \lambda =$ 

Zero temperature free fermion for PNG

Multi-layer PNG model



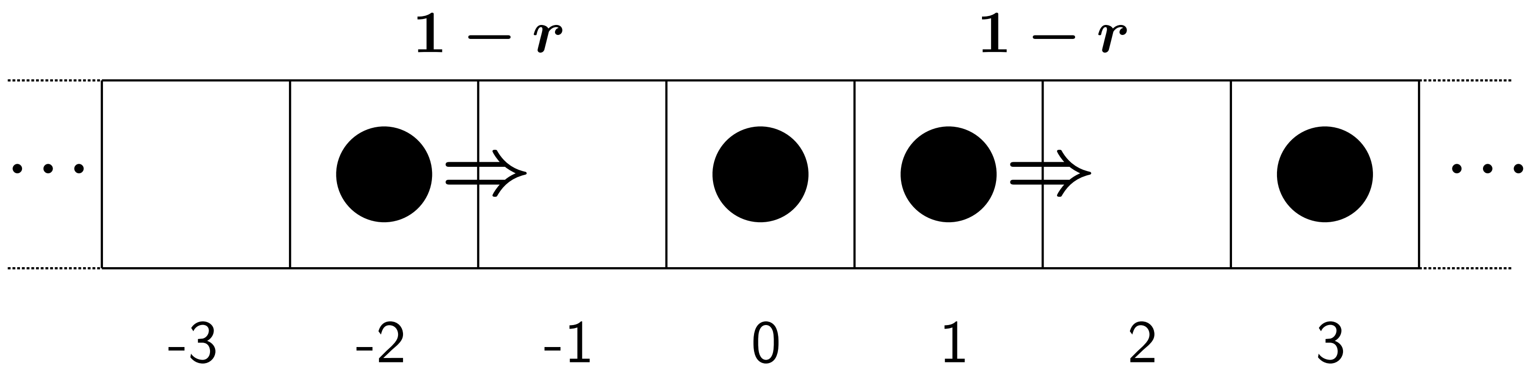
Shadow line construction



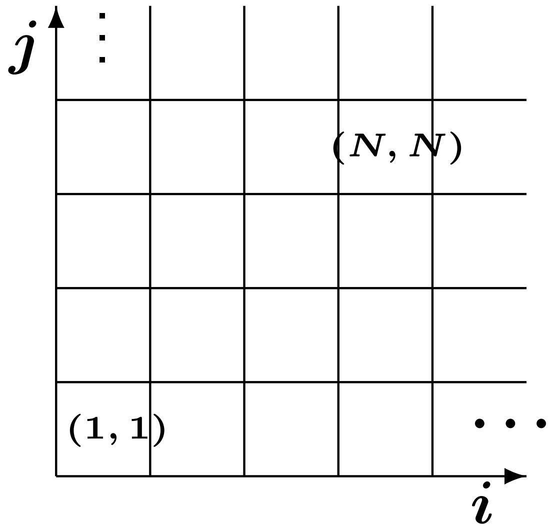
We can see $T = 0$ free fermion before getting formulas.

TASEP and Schur measure

TASEP $=$ totally asymmetric simple exclusion process



$N(t)$: Integrated current at $(0,1)$ upto time t from step i.c.



Waiting time of ith hop of jth particle

w_{ij} on (i,j): geometrically distributed

$$G(N,N) = \max_{\substack{\text{up-right paths from} \\ (1,1)\text{to}(N,N)}} \Bigg(\sum_{\substack{(i,j) \\ \text{on a path}}} w_{i,j} \Bigg)$$

Directed polymer at $T=0$

Time at which Nth particle arrives at the origin

Last passage percolation

- RSK correspondence ⇒ Schur measure, DPP, Fredholm det

$$\mathbb{P}[G(N,N) \le u] = \frac{1}{Z} \sum_{\lambda, \lambda_1 \le u} s_\lambda(a) s_\lambda(b) = \det(1-K)$$

Thanks to Jacobi-Trudi formula, Schur measure is a determinantal point process(DPP), i.e., all correlation functions are determinants with a common kernel K.

- 2000 Johansson

$$\lim_{t\to\infty} \mathbb{P}\left[\frac{N(t) - Jt}{ct^{1/3}} \ge -s\right] = F_2(s) = \det(1-K_2)_{L^2(s,\infty)}$$

where F_2 is GUE Tracy-Widom distribution and kernel K_2 is

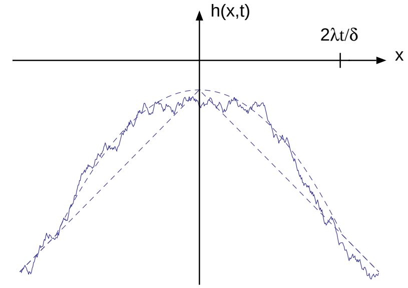
$$K_2(x,y) = \int_{\mathbb{R}_+} \mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda)d\lambda$$

2. KPZ equation and $T > 0$ polymer

- Height function $h = h(x,t), x \in \mathbb{R}, t \in \mathbb{R}_+$,

$$\frac{\partial}{\partial t}h = \frac{1}{2}\frac{\partial^2}{\partial x^2}h + \frac{1}{2}\left(\frac{\partial h}{\partial x}\right)^2 + \eta$$

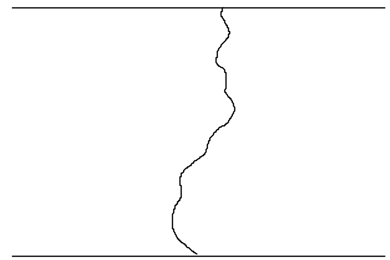
where $\eta = \eta(x,t)$ is the space time white noise.



- Cole-Hopf transformation: $Z = Z(x,t) = e^{h(x,t)}$

$$\frac{\partial}{\partial t}Z = \frac{1}{2}\frac{\partial^2}{\partial x^2}Z + \eta Z$$

Directed polymer at $T > 0$



Fredholm determinant with $T > 0$ kernel

2010 TS-Spohn, Amir-Corwin-Quastel

Laplace transform of Z with i.c. $Z(x,0) = \delta(x)$

$$\mathbb{E}[\exp(-Ze^{\frac{t}{24}-(t/2)^{1/3}s})] = \det(1 - K_t)_{L^2(\mathbb{R}_+)}$$

where the kernel has the Fermi-Dirac factor ("$T > 0$ kernel"),

$$K_t(x,y) = \int_{\mathbb{R}} \frac{\mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda)}{1+e^{(t/2)^{1/3}(s-\lambda)}} d\lambda$$

Easy to take $t \to \infty$ limit to get F_2.

Obtaining the Fredholm determinant formula is nontrivial.

Can we see $T > 0$ free fermion before getting formulas?

3. Discrete KPZ models and q-Whittaker measure

- Discrete KPZ models: ASEP, q-TASEP, sHS6VM, etc.

  2011 Borodin, Corwin

  By the branching rule of q-Whittaker function, these models are related to the q-Whittaker measure.

- q-PushTASEP(2015 Matveev-Petrov): kth particle position

  $$x_k(t+1) = x_k(t) + V_{k,t} + W_{k,t}, \qquad \text{for } k = 1, \ldots, N,$$

  where $V_{k,t} \sim q\mathrm{Po}(a_k b_{t+1})$ and $W_{k,t}$ whose distribution depends on $x_{k-1}(t+1) - x_{k-1}(t), x_{k-1}(t) - x_k(t) - 1$.

  The Nth particle position at time M is related to μ_1 as $X_N(M) \stackrel{d}{=} \mu_1 + N$.

Fredholm determinants for q-Whittaker

- Standard approach (2011- Borowin, Corwin, TS, Petrov, ...)

  Markov duality + Bethe ansatz or by Macdonald operators.

  $$\mathbb{E}\left[\frac{1}{(\zeta q^{-\mu_1}; q)_\infty}\right] = \det(1 - K)$$

  NOT $T > 0$ kernel. Asymptotics is rather involved.

  Generalization to half-space case is difficult.

- Frobenius determinant approach (2019 Imamura, TS)

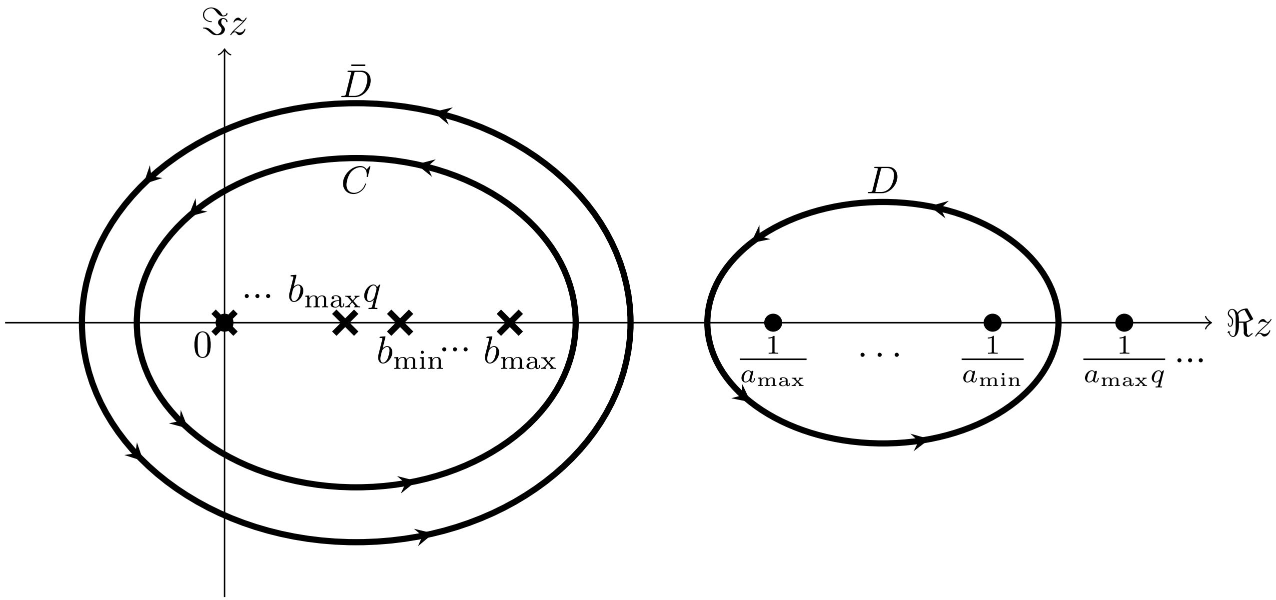
  $$\mathbb{E}\left[\frac{1}{(\zeta q^{-\mu_1}; q)_\infty}\right] = \det\left(1 - f_\zeta K\right)_{\ell^2(\mathbb{Z})}$$

  where $f_\zeta(m) = \frac{-\zeta q^m}{1-\zeta q^m}$ ($T > 0$ kernel!)

and

$$K(m_1,m_2)=\int_C \frac{dz}{z}\int_D \frac{dw}{w} g_{a,b}(z,w;m_1,m_2)$$

$$g_{a,b}(z,w;m_1,m_2)=\frac{w^{m_2}}{z^{m_1}}\frac{w}{z-w}\prod_{i=1}^{N}\frac{(a_i z;q)_\infty}{(a_i w;q)_\infty}\cdot\prod_{j=1}^{N}\frac{(b_j/w;q)_\infty}{(b_j/z;q)_\infty}.$$



Periodic Schur measure

- Periodic Schur measure (2007 Borodin, 2018 Betea-Bouttier)

$$\frac{1}{Z} \sum_{\rho \in \mathcal{P}, \rho (\subset \lambda)} q^{|\rho|} s_{\lambda/\rho}(a) s_{\lambda/\rho}(b)$$

- Its shift mix version is a DPP and hence

$$\mathbb{P}\left(\lambda_1 + S \le k\right) = \det\left(1 + fL\right)_{\ell^2(\mathbb{Z})}$$

where

$$f(m) = f_\zeta(m)|_{\zeta = -tq^{1/2+k}} = \frac{tq^{1/2+k+m}}{1 + tq^{1/2+k+m}}$$

$$L(m_1, m_2) = \int_C \frac{dz}{z} \int_{\bar{D}} \frac{dw}{w} g_{a,b}(z, w, m_1, m_2).$$

Equivalence of the two Fredholm determinants

Combining all the above arguments, our identity can be written as an identity between two Fredholm determinants.

Theorem

$$\det(1 - fK)_{\ell^2(\mathbb{Z})} = \det(1 + fL)_{\ell^2(\mathbb{Z})}$$

The only difference between the kernels K, L is the contours. The equivalence can be proved by shift of contours.

This completes the first proof of our identity.

4. Bijective proof of the identity

Using $Q_\mu = b_\mu P_\mu$ with

$$b_\mu(q) = \prod_{i \geq 1} \frac{1}{(q;q)_{\mu_i - \mu_{i+1}}}$$

the identity can be written as

$$\sum_{\ell=0}^{n} \frac{q^\ell}{(q;q)_\ell} \sum_{\mu:\mu_1 = n-\ell} b_\mu(q) P_\mu(a) P_\mu(b) = \sum_{\lambda,\rho:\lambda_1 = n} q^{|\rho|} s_{\lambda/\rho}(a) s_{\lambda/\rho}(b)$$

Combinatorial formulas

Skew Schur function

$$s_{\lambda/\rho}(x) = \sum_{T \in \mathbf{SST}(\lambda/\rho)} x^T$$

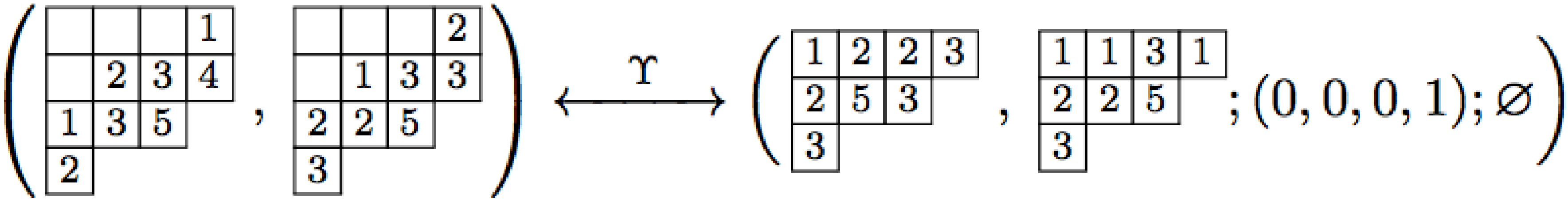
where SST is the set of skew semistandard tableaux.

q-Whittaker function (2000 Sanderson, 2012 Schling-Tingley)

$$P_\mu(x) = \sum_{V \in \mathbf{VST}(\mu)} q^{H(V)} x^V$$

where VST is the set of "vertically strict tableaux" with increasing elements in each column and no condition among columns and H is the energy function.

Bijection $\Upsilon : (P, Q) \leftrightarrow (V, W, \kappa, \nu)$



(P, Q): A pair of skew SSTs with same shape λ/ρ

(V, W): A pair of VSTs with same shape μ

$\kappa \in \mathcal{K}(\mu) = \{\kappa = (\kappa_1, \ldots, \kappa_{\mu_1}) \in \mathbb{N}_0^{\mu_1} : \kappa_i \geq \kappa_{i+1} \text{ if } \mu'_i = \mu'_{i+1}\}$

ν: partition

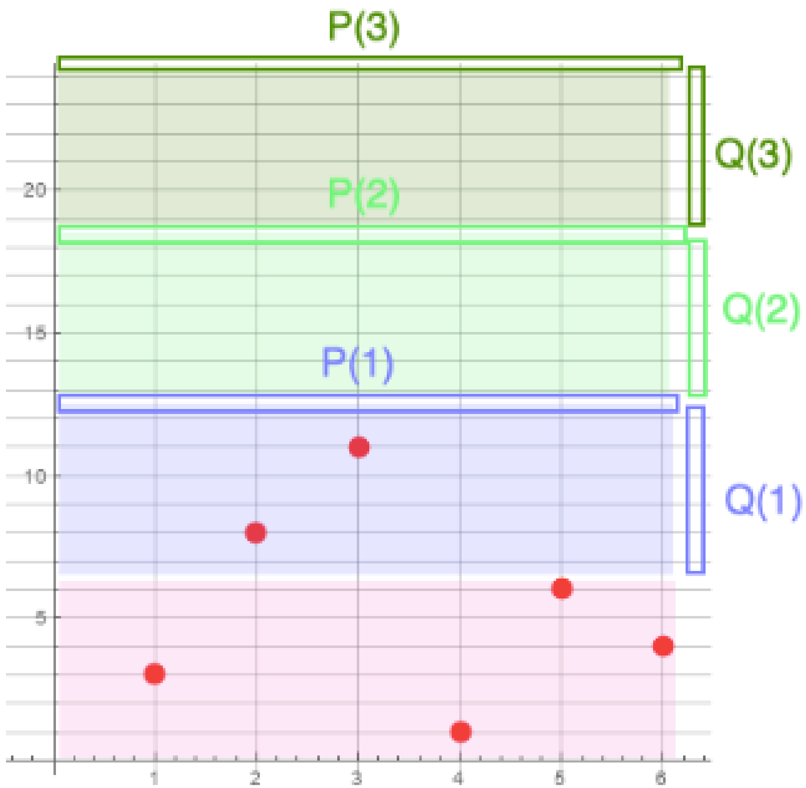
Weight preserving property

$$|\rho| = H(V) + H(W) + |\kappa| + |\nu|$$

Note $\sum_{\kappa \in \mathcal{K}(\mu)} q^{|\kappa|} = b_\mu(q)$ and $\mathbb{P}[\nu_1 = \ell] = \frac{q^\ell}{(q;q)_\ell}(q;q)_\infty$.

Skew RSK dynamics

Iterated skew RSK maps by Sagan-Stanley on a periodic cylinder



$$(P,Q) = \left(\begin{array}{|c|c|c|c|} \hline \ & \ & \ & 1 \\ \hline \ & 2 & 3 & 4 \\ \hline 1 & 3 & 5 & \\ \hline 2 & & & \\ \hline \end{array} \, , \, \begin{array}{|c|c|c|c|} \hline \ & \ & \ & 2 \\ \hline \ & 1 & 3 & 3 \\ \hline 2 & 2 & 5 & \\ \hline 3 & & & \\ \hline \end{array} \right)$$

$$(P_2,Q_2) = \left(\begin{array}{|c|c|c|c|} \hline \ & \ & \ & \ \\ \hline \ & \ & \ & \ \\ \hline \ & \ & \ & 4 \\ \hline \ & 1 & 3 & \\ \hline 1 & 2 & 5 & \\ \hline 2 & 3 & & \\ \hline \end{array} \, , \, \begin{array}{|c|c|c|c|} \hline \ & \ & \ & \ \\ \hline \ & \ & \ & \ \\ \hline \ & \ & \ & 3 \\ \hline \ & 1 & 2 & \\ \hline 2 & 2 & 3 & \\ \hline 3 & 5 & & \\ \hline \end{array} \right)$$

$$(P_{10},Q_{10}) = \left(\begin{array}{c|c|c|c|c|} \vdots & \vdots & \vdots & \vdots & \vdots \\ 12 & \ & \ & \ & 3 \\ \vdots & \vdots & \vdots & \vdots & \\ 22 & \ & \ & 2 & \\ 23 & \ & 2 & 3 & \\ 24 & \ & 5 & & \\ \vdots & \vdots & & & \\ 31 & 1 & & & \\ 32 & 2 & & & \\ 33 & 3 & & & \end{array} \, , \, \begin{array}{c|c|c|c|c|} \vdots & \vdots & \vdots & \vdots & \vdots \\ 12 & \ & \ & \ & 1 \\ \vdots & \vdots & \vdots & \vdots & \\ 22 & \ & \ & 3 & \\ 23 & \ & 1 & 5 & \\ 24 & \ & 2 & & \\ \vdots & \vdots & & & \\ 31 & 1 & & & \\ 32 & 2 & & & \\ 33 & 3 & & & \end{array} \right)$$

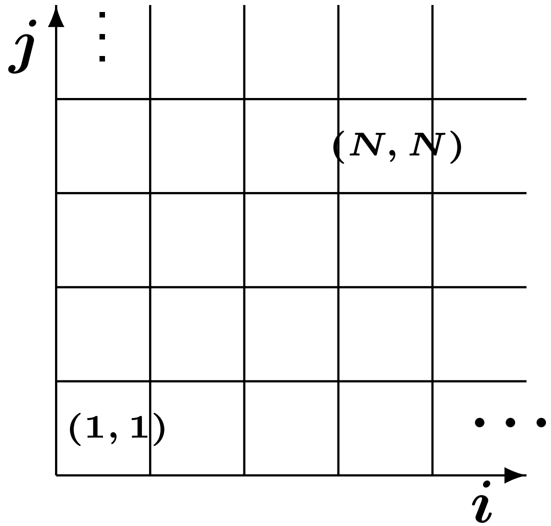
Similar to Box-Ball systems! Crystals

5. New approach to KPZ models

- Standard approach to KPZ models with q has been to apply Markov duality & Bethe ansatz or Macdonald operator
- With our bijection, one does not need to use such methods any more. Once the mapping to (periodic) Schur measure is established, then one can simply apply the methods of DPP to get Fredholm determinants, for which asymptotic analysis by now can be done in a standard way.
- Our approach also works also for half-space models.

Log Gamma polymer

Finite temperature directed polymer model (2009 Seppäläinen)



w_{ij} on (i,j): Gamma distributed with parameter $\alpha_i - \alpha_j$

Prob for a path π from $(1,1)$ to (N,N)

$\frac{1}{Z_N}\prod_{(i,j)\in\pi} w_{i,j}$

- Can be studied by taking $q \to 1$ limit of q-PushTASEP.
- Continuous limit $\Rightarrow$ KPZ equation

Half space case

Half space q-Whittaker measure (2021 Imamura-Mucciconi-TS)

$$\frac{1}{\Phi(a,z;q)} b_\mu(q;z) P_\mu(a,q^2)$$

with $\quad \Phi(a,z;q) = \prod_{i=1}^{n} \frac{1}{(a_i z;q)_\infty} \prod_{1\le i<j\le n} \frac{1}{(a_i a_j;q^2)_\infty}$

Here

$$b_\mu(q;z) = \prod_{i=2,4,6\ldots} \frac{[qz^2+1]_{q^2}^{\mu_i-\mu_{i+1}}}{(q^2;q^2)_{\mu_i-\mu_{i+1}}} \prod_{i=1,3,5,\ldots} \frac{z^{1_{\mu_i>\mu_{i+1}}}}{(q;q)_{\mu_i-\mu_{i+1}}},$$

$$[A+B]_p^k = \sum_{j=0}^{k} A^j B^{k-j} \binom{k}{j}_p, \quad \binom{k}{j}_p = \frac{(p;p)_k}{(p;p)_j (p;p)_{k-j}}.$$

Free boundary Schur measure

- Free boundary Schur measure (2017 B-B-Nejjar-Vuletic)

$$\frac{1}{Z} z_1^{\mathrm{odd}\mu} z_2^{\mathrm{odd}\lambda} q^{|\mu|} s_{\lambda/\mu}(a) \quad \text{(Pfaffian point process)}$$

- Thm:Identity relating HS q-Whittaker & FB Schur measures

$$\sum_{\ell=0}^{k} g_\ell(\gamma, q) \sum_{\mu:\mu_1=k-\ell} b_\mu(q;\gamma) P_\mu(a;q^2) = \sum_{\lambda,\rho:\lambda_1=k} \gamma^{\mathrm{odd}(\lambda')+\mathrm{odd}(\rho')} q^{|\rho|} s_{\lambda/\rho}(a)$$

$$\text{with} \quad g_k(\gamma, q) = \frac{[q\gamma^2 + q^2]_{q^2}^k}{(q^2;q^2)_k}.$$

- Fredholm Pfaffian for half-space q-Whittaker measure

$$\mathbb{E}[1/(-\zeta q^{-\mu_1-\chi};q)_\infty] = \mathrm{Pf}(J-K)$$

where χ is a certain indep. r.v. and J is an anti-sym. kernel.

$\Rightarrow$ Log-Gamma polymer and KPZ equation in half-space

Summary

- We presented a new approach to study KPZ models by mapping them to free fermions at finite temperature.
- Most KPZ models are related to q-Whittaker measure. We have found an identity which relates marginals of q-Whittaker and periodic Schur measures. The latter is related to free fermions at finite temperature and DPP. This allows us to write a quantity of a KPZ model as Fredholm determinant with $T > 0$ kernel, which admits straightforward asymptotics. A bijective proof of the identity will be explained in detail by Matteo Mucciconi next week.
- Our approach works also for half-space models with Pfaffian structures.

Geometric q-PushTASEP

2015 Matveev-Petrov

$x_k(t)$: kth particle position at time t.

$$x_k(t+1) = x_k(t) + V_{k,t} + W_{k,t}, \qquad \text{for } k = 1, \ldots, N,$$

where $V_{k,t} \sim q\mathrm{Po}(a_k b_{t+1})$ with

$$X \sim q\mathrm{Po}(\alpha) \Leftrightarrow \mathbb{P}[X = n] = \frac{\alpha^n}{(q;q)_k}(\alpha;q)_\infty$$

and $W_{k,t} \sim \varphi_{q^{-1},q^{\mathrm{gap}_k(t)}}(\bullet \,|\, x_{k-1}(t+1) - x_{k-1}(t))$ where

$$\mathrm{gap}_k(t) = x_{k-1}(t) - x_k(t) - 1$$

$$\varphi_{q^{-1},q^a}(r|c) = q^{ar}(q^a;q^{-1})_{c-r}\frac{(q^{-1};q^{-1})_c}{(q^{-1};q^{-1})_r(q^{-1};q^{-1})_{c-r}}.$$

Kernel for half-space model

$$K_{1,1}(x,y) = \oint_C \frac{dz}{z^{x+1}} \oint_D \frac{dw}{w^{y+1}} F(z)F(w)\kappa_{1,1}(z,w)$$

$$K_{1,2}(x,y) = -K_{2,1}(y,x) = \oint \frac{dz}{z^{x+1}} \oint \frac{dw}{w^{-y+1}} \frac{F(z)}{F(w)}\kappa_{1,2}(z,w)$$

$$K_{2,2}(x,y) = \oint \frac{dz}{z^{-x+1}} \oint \frac{dw}{w^{-y+1}} \frac{1}{F(z)F(w)}\kappa_{2,2}(z,w)$$

where

$$F(z) = \prod_{i=1}^{n} \frac{(a_i/z;q^2)_\infty}{(a_i z;q^2)_\infty}$$

$\kappa_{i,j}$

$$\kappa_{1,1} = \frac{1}{\zeta z^{1/2} w^{3/2}} \frac{(q^2;q^2)_\infty^2}{(qz, qw, -\frac{1}{z}, -\frac{1}{w}; q)_\infty} \frac{\theta_{q^2}(w/z)}{\theta_{q^2}(q^2 zw)} \frac{\theta_3(\zeta^2 z^2 w^2; q^4)}{\theta_3(\zeta^2, q^4)} g(z) g(w)$$

$$\kappa_{1,2} = \frac{w^{1/2}}{z^{1/2}} \frac{(q^2;q^2)_\infty^2}{(qz, -qw, -1/z, 1/w; q)_\infty} \frac{\theta_{q^2}(q^2 wz)}{\theta_{q^2}(w/z)} \frac{\theta_3(\zeta^2 z^2/w^2; q^4)}{\theta_3(\zeta^2, q^4)} \frac{g(z)}{g(w)}$$

$$\kappa_{2,2} = \frac{\zeta}{z^{1/2} w^{3/2}} \frac{(q^2;q^2)_\infty^2}{(-qz, -qw, \frac{1}{z}, \frac{1}{w}; q)_\infty} \frac{\theta_{q^2}(w/z)}{\theta_{q^2}(q^2 wz)} \frac{\theta_3(\zeta^2/z^2 w^2; q^4)}{\theta_3(\zeta^2, q^4)} \frac{1}{g(z) g(w)}$$

with

$$g(z) = \frac{(qz;q)_\infty}{(1/z;q)_\infty} \frac{(\gamma q/z, \gamma/z; q^2)_\infty}{(\gamma qz, \gamma q^2 z; q^2)_\infty} \frac{1}{(\gamma^2 q; q^2)_\infty (-q;q)_\infty}$$