Transition probabilities and asymptotics for integrable two-species stochastic processes

Jan de Gier

The University of Melbourne

21 July 2021, Solvable Lattice Models Seminar, Stanford

In collaboration with:

Zeying Chen
Iori Hiki
William Mead
Masato Usui
Michael Wheeler
Tomohiro Sasamoto

Motivation

Universal behaviour in stochastic processes (KPZ)

Accessible via solvable lattice models

1.2 Measurement Techniques 5

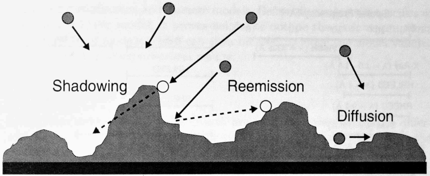


Fig. 1.3. Diagram of growth effects including diffusion, shadowing, and reemission that may affect surface morphology during thin film growth. The incident particle flux may arrive at the surface with a wide angular distribution depending on the deposition methods and parameters.

1+1D growth

KPZ growth

- Stochastic growth normal to the surface
- Kardar-Parisi-Zhang (KPZ) 1986
- Basic object: (random) height function $h(x, t)$

KPZ equation (nonlinear stochastic PDE):

KPZ equation

$$\partial_t h(t,x) = \frac{1}{2}\partial_x^2 h(t,x) + \frac{1}{2}\left(\partial_x h(t,x)\right)^2 + \xi(t,x)$$

$\xi(t,x)$: space-time Gaussian white noise

Realisation in liquid crystal growth

KPZ growth

Theorem (Non-Gaussian fluctuations)

$$h \sim vt + ct^{1/3}, \quad t \to \infty$$

Transformation to Stochastic Heat Equation (SHE):

$$h(t,x) := \log z(t,x).$$

SHE equation

$$\partial_t z(t,x) = \frac{1}{2}\partial_x^2 z(t,x) + \xi(t,x) z(t,x)$$

Fluctuations

The Laplace transform formula for $z(t,x)$ can be written as a Fredholm determinant

Theorem (Laplace transform of SHE)

$$\mathbb{E}[e^{-\zeta z(t,0)}] = \det(I - K_\zeta)_{L^2(\mathbb{R}_+)}$$

$$K_\zeta(\eta,\eta') = \int_{\mathbb{R}} f(\zeta,\xi,t)\mathrm{Ai}(\xi+\eta)\mathrm{Ai}(\xi+\eta')\mathrm{d}\xi.$$

Theorem (Fluctuations of SHE)

$$\lim_{t\to\infty} P\left(\frac{h(t,0)-t}{t^{1/3}} < s\right) = F_{\mathrm{GUE}}(s).$$

$F_{\mathrm{GUE}}(s)$ is the Tracy-Widom distribution of the Gaussian Unitary Ensemble of random matrix theory.

Connection to Painlevé II

The cumulative Tracy-Widom distribution is explicitly given by

$$F_{\mathrm{GUE}}(s) = \exp\left(-\int_s^\infty (x-s)u(x)^2 \,\mathrm{d}\, x\right),$$

and $u(x)$ is a solution of the Painlevé (PII) equation with Airy boundary condition, that is

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\, x^2} = 2u^3 + xu, \qquad u(x) \sim \mathrm{Ai}(x) \quad \text{as} \quad x \to \infty.$$

Universality

The Tracy-Widom distribution also appears in the asymmetric exclusion process on $\mathbb{Z}$.

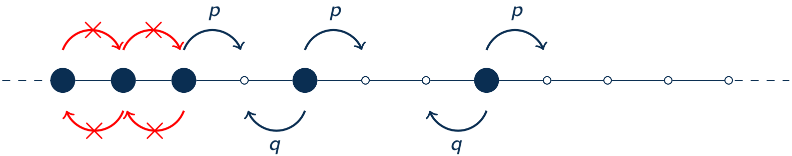


Figure: Configuration of particles and hopping rates in the ASEP on $\mathbb{Z}$

Let $N_y(t)$ be the net number of particles to have crossed a given site y after time t.

Let $Q_y(t) = \tau^{N_y(t)}$ with $\tau = \frac{p}{q}$ and

$$\widetilde{Q}_y(t) = \frac{Q_y(t) - Q_{y-1}(t)}{\tau - 1}.$$

Fundamental solution of ASEP

Theorem (Fundamental solution of ASEP)

$$\mathbb{E}[\widetilde{Q}_{x_1}(t)\cdots\widetilde{Q}_{x_k}(t)] = \oint\cdots\oint \prod_{1\le i<j\le k} \frac{z_i - z_j}{z_i - \tau z_j} \prod_{j=1}^{k} \mathrm{e}^{-\lambda(z_j)t} \left(\frac{1+z_j/\tau}{1+z_j}\right)^{x_j+1} \mathrm{d}z_j$$

Fluctuations of particle flow across the origin follow KPZ statistics given by the Tracy-Widom distribution:

Theorem (Fluctuations of ASEP)

$$\lim_{t\to\infty} P\left(\frac{N_0(t) - vt}{t^{1/3}} > -s\right) = F_{\mathrm{GUE}}(s).$$

Summary

Ingredients:

- Integrable stochastic lattice model
- Observable expressed in terms of k-fold integral
- Asympotics for large k (Fredholm determinant)
- Saddle point analysis
- Exact universal probability distribution functions

New results:

- Rank two models
- Dynamic poles in integral (from nested Bethe ansatz)
- Combination of Gaussian and GUE modes

Two-species TASEP

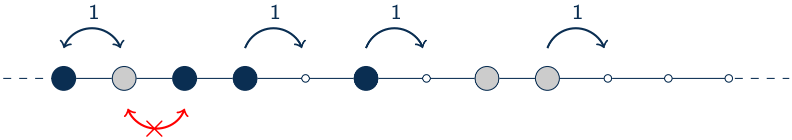


Figure: Configuration of particles and hopping rates in the 2-TASEP on $\mathbb{Z}$

Transition events and rates:

$$(1,0) \mapsto (0,1) \text{ at rate } 1,$$
$$(2,0) \mapsto (0,2) \text{ at rate } 1,$$
$$(2,1) \mapsto (1,2) \text{ at rate } 1.$$

Transition probability (Green's function)

Coordinates of all particles: $\nu = \{\nu_1, \dots, \nu_n\} \in \mathbb{Z}^n$
Indices of type 2 particles: $p = \{p_1, \dots, p_m\} \in \{1, \dots, n\}^m$.

Definition

The transition probability satisfies the master equation

$$\frac{\mathrm{d}}{\mathrm{d}\,t} G(\nu, p; t) = \sum_{\nu', p'} M_{\nu,p;\nu',p'} G(\nu', p'; t); \qquad t > 0,$$

and initial condition

$$G(\nu, p; 0) = \prod_{i=1}^{n} \delta_{\nu_i, \nu_i^{(0)}} \prod_{j=1}^{m} \delta_{p_j, p_j^{(0)}}$$

2-TASEP master equation

Free equation:

$$\frac{d}{dt}G(\nu, p; t) = \sum_{i=1}^{n} G(\{\nu_1, \ldots, \nu_i - 1, \ldots, \nu_n\}, p; t) - nG(\nu, p; t)$$

Boundary conditions (interaction):

1. ...
2. If $i \notin p$ and $i + 1 \in p$ then

$$G(\{\nu_{i+1} = \nu_i\}, p; t) = G(\{\nu_{i+1} = \nu_i + 1\}, s_i p; t) + G(\{\nu_{i+1} = \nu_i + 1\}, p; t)$$

3. ...

2-TASEP transition probability

Theorem

$$G(\{\mu \to \nu\}, \{p^{(0)} \to p\}; t) =$$

$$\sum_{\pi \in S_n} (-1)^{|\pi|} \oint \prod_{i=1}^{n} \frac{dz_i}{2\pi \mathrm{i}} \prod_{i=1}^{n} \left(\frac{1-z_i}{1-z_{\pi_i}} \right)^i z_{\pi_i}^{\nu_i} z_i^{-\mu_i - 1} e^{(z_i^{-1}-1)t}$$

$$\times \sum_{\rho \in S_m} (-1)^{|\rho|} \oint \prod_{j=1}^{m} \frac{du_j}{2\pi \mathrm{i}} \prod_{i=1}^{m} \left[\left(\frac{1-u_i}{1-u_{\rho_i}} \right)^i \prod_{j=1}^{p_i - 1} (u_{\rho_i} - z_{\pi_j}) \prod_{j=1}^{p_i} \frac{1}{1-z_{\pi_j}} \right]$$

$$\times \prod_{i=1}^{n} (1-z_i)^m \prod_{i=1}^{m} \left[\prod_{j=1}^{p_i^{(0)}} \frac{1}{u_i - z_j} \prod_{j=p_i^{(0)}+1}^{n} \frac{1}{1-z_j} \right]$$

All contours around the origin.

2-TASEP transition probability

Theorem

$$G(\{\mu \to \nu\}, \{p^{(0)} \to p\}; t) =$$

$$\sum_{\pi \in S_n} (-1)^{|\pi|} \oint \prod_{i=1}^{n} \frac{dz_i}{2\pi \mathrm{i}} \prod_{i=1}^{n} \left(\frac{1 - z_i}{1 - z_{\pi_i}} \right)^i z_{\pi_i}^{\nu_i} z_i^{-\mu_i - 1} e^{(z_i^{-1} - 1)t}$$

$$\times \sum_{\rho \in S_m} (-1)^{|\rho|} \oint \prod_{j=1}^{m} \frac{du_j}{2\pi \mathrm{i}} \prod_{i=1}^{m} \left[\left(\frac{1 - u_i}{1 - u_{\rho_i}} \right)^i \prod_{j=1}^{p_i - 1} (u_{\rho_i} - z_{\pi_j}) \prod_{j=1}^{p_i} \frac{1}{1 - z_{\pi_j}} \right]$$

$$\times \prod_{i=1}^{n} (1 - z_i)^m \prod_{i=1}^{m} \left[\prod_{j=1}^{p_i^{(0)}} \frac{1}{u_i - z_j} \prod_{j = p_i^{(0)} + 1}^{n} \frac{1}{1 - z_j} \right]$$

1-TASEP Bethe ansatz
Nested Bethe ansatz
Normalisation

Total crossing: $p_j^{(0)} = j, \quad p_j = n - m + j$

Theorem

The transition probability for total crossing is given by

$$G(\{\mu \to \nu\}, \{p^{(0)} \to p\}; t) = \oint \prod_{i=1}^{m} \frac{dz_i}{2\pi \mathrm{i}} \prod_{j=1}^{n-m} \frac{dw_j}{2\pi \mathrm{i}} \prod_{i=1}^{m} \frac{e^{(z_i^{-1}-1)t}}{(1-z_i)^{n-m}} \prod_{i=1}^{n-m} e^{(w_i^{-1}-1)t} \prod_{i=1}^{m} \prod_{j=1}^{n-m} (w_j - z_i)$$
$$\det\left(z_i^{\nu_{n-m+j}-\mu_i-1}(1-z_i)^{i-j}\right)_{1\le i,j\le m} \det\left(w_i^{\nu_j-\mu_{m+i}-1}(1-w_i)^{i-j}\right)_{1\le i,j\le n-m}$$

Proof by evaluation of simple poles and determinant calculus.

Total crossing (definitions)

Total crossing: $p_j^{(0)} = j, \quad p_j = n - m + j$
Initial coordinates: μ

The probability of total crossing is given by

$$P_{\text{cross}}(\mu; s_1, s_2) = \sum_{s_1 \le \nu_1 < \cdots < \nu_{n-m} < s_2 \le \nu_{n-m+1} < \cdots < \nu_n} G(\{\mu \to \nu\}, \{p^{(0} \to p\}; t),$$

Under random Bernoulli-step initial conditions with parameter ρ

$$P_{\text{B}}(s_1, s_2) = \sum_{\mu_1 < \mu_2 < \cdots < \mu_m < 0} \mathbb{P}(\mu; 0) P_{\text{cross}}(\mu; s_1, s_2)$$

Total crossing (result)

Theorem

The total crossing probability under Bernoulli-step initial conditions is given by

$$P_B(s_1, s_2) = \frac{\rho^m}{m!} \oint \prod_{i=1}^{m} \frac{\mathrm{d}\, z_i}{2\pi \mathrm{i}} \prod_{i=1}^{n-m} \frac{\mathrm{d}\, w_i}{2\pi \mathrm{i}} \prod_{i=1}^{m} \prod_{j=1}^{n-m} (w_j - z_i) \prod_{i \neq j} (z_j - z_i)$$

$$\times \prod_{i=1}^{m} \frac{e^{(z_i^{-1}-1)t} z_i^{s_2}}{(1-z_i)^n (1-(1-\rho)z_i)} \prod_{i=1}^{n-m} \frac{e^{(w_i^{-1}-1)t} w_i^{s_1-i}}{(1-w_i)^{n-m-i+1}} \det\left(w_i^{j-1} - w_i^{s_2-s_1}\right)_{1\leq i,j \leq n-m}$$

Proof by geometric series, symmetrisation identities and determinant calculus.

This expression is amenable for asymptotic analysis which will be a future project.

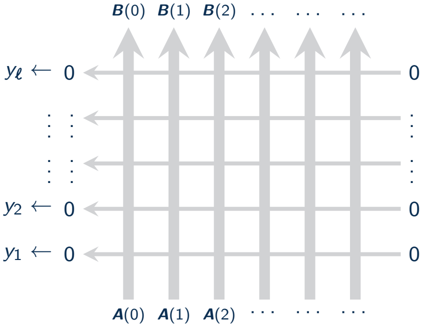
Vertex model approach

With μ and ν compositions, define

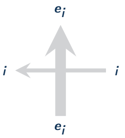
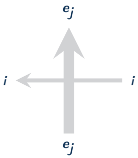
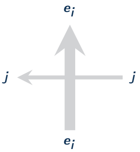
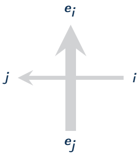
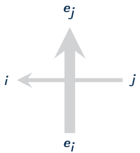
$$\boldsymbol{A}(k) = \sum_{j=1}^{n} \mathbf{1}_{\mu_j = k} \boldsymbol{e}_j, \qquad \boldsymbol{B}(k) = \sum_{j=1}^{n} \mathbf{1}_{\nu_j = k} \boldsymbol{e}_j, \qquad k \in \mathbb{Z}_{\geq 0}\,,$$

Then

$$G(\{\mu \to \nu\}; \{y_1, \ldots, y_\ell\}) =$$



Vertex model approach

 1	 $\dfrac{q^{-1}(1-y)}{1-q^{-1}y}$	 $\dfrac{1-y}{1-q^{-1}y}$
 $\dfrac{1-q^{-1}}{1-q^{-1}y}$	 $\dfrac{y(1-q^{-1})}{1-q^{-1}y}$	

Matrix product approach.

Vertex model approach

Theorem (Borodin-Wheeler)

The limit

$$\lim_{\epsilon\to 0}(-q^{1/2})^{|\mu|-|\nu|+n\ell}G(\{\mu\to\nu\}-\ell;y_1,\dots,y_\ell)\Big|_{y_i\mapsto q^{-1/2}[1+(1-q)\epsilon],\ \ell\mapsto t/\epsilon}$$

exists, and converges to the ASEP probability $G(\{\mu\to\nu\};t)$.

Corollary (Rainbow crossing probability)

The probability that all particles have different colours and exchange their order at time t *is given by*

$$\mathbb{P}_t(\mu\to\nu)=\frac{(1-q)^n}{(2\pi\mathrm{i})^n}\oint dz_1\cdots\oint dz_n\prod_{1\le i<j\le n}\left(\frac{z_j-z_i}{z_j-qz_i}\right)\prod_{i=1}^{n}\exp\left(\frac{(1-q)^2z_it}{(1-z_i)(1-qz_i)}\right)\frac{1}{(1-z_i)(1-qz_i)}\left(\frac{1-qz_i}{1-z_i}\right)^{\mu_i-\nu_i}.$$

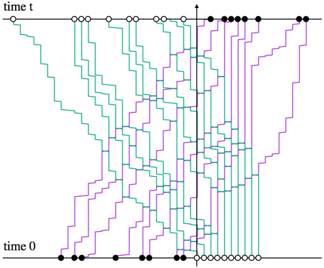
Block symmetrisation leads to m-species ASEP transition probability.

Another generalisation

Two TASEPs can be coupled to form another integrable 2-species model.

Introduced by Arndt-Heinzl-Rittenberg (AHR), the transition rates are

$$
\begin{aligned}
p: &\quad (+,0) \to (0,+) \\
1-p: &\quad (0,-) \to (-,0) \\
1: &\quad (+,-) \to (-,+)
\end{aligned}
$$



Transition probability for total crossing

Initial conditions: assume $x_j^{(0)} < y_k^{(0)}$, i.e. at $t = 0$ all + particles are to the left of all − particles.

Final condition: $x_j > y_k$, i.e. at time t all + particles have passed all − particles

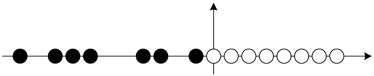
Then

$$G(x_j, y_k, t) = \oint \prod_{j=1}^{n} \mathrm{d}\, z_j \prod_{k=1}^{m} \mathrm{d}\, w_k \; \mathrm{e}^{\Lambda t} \prod_{k=1}^{m} \prod_{j=1}^{n} \frac{1}{q z_j + p w_k}$$
$$\times \det\left(\left(\frac{z_j - 1}{z_i - 1} \right)^{j-1} z_i^{x_j} \right) z_j^{-x_j^{(0)} - 1}$$
$$\times \det\left(\left(\frac{w_k - 1}{w_\ell - 1} \right)^{m-k} w_\ell^{-y_k} \right) w_k^{y_k^{(0)} - 1},$$

with all contours around the origin, and with eigenvalue

$$\Lambda = p \sum_{j=1}^{n} (z_j^{-1} - 1) + q \sum_{k=1}^{m} (w_k^{-1} - 1).$$

Step-Bernoulli condition



Proposition

The total exchange probability $P_{n,m,\rho}(t)$ with Bernoulli initial data is given by

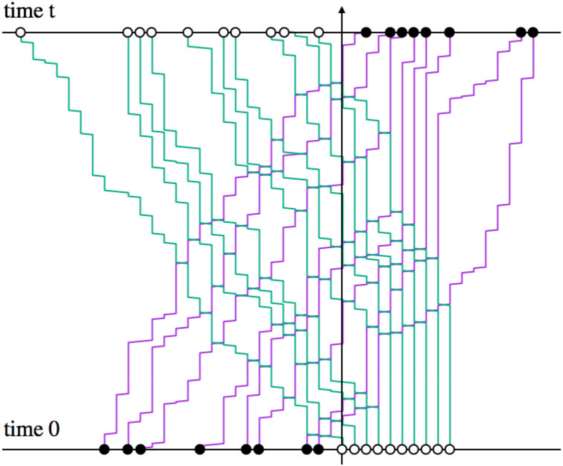
$$P_{n,m,\rho}(t) = \oint \prod_{j=1}^{n} \mathrm{d}\, z_j \prod_{k=1}^{m} \mathrm{d}\, w_k \; \mathrm{e}^{\Lambda_{n,m} t} \times$$

$$\frac{\rho^n \prod\limits_{1\leq i<j\leq n} (z_i - z_j) \prod\limits_{1\leq k<l\leq m} (w_l - w_k) \prod\limits_{j=1}^{n} z_j^{n-j} \prod\limits_{k=1}^{m} w_k^{k-1}}{\prod\limits_{j=1}^{n} (z_j - 1)^{n+1-j} (1-\rho z_j) \prod\limits_{k=1}^{m} (w_k - 1)^k \prod\limits_{j=1}^{n} \prod\limits_{k=1}^{m} \left(q z_j + p w_k \right)},$$

with all contours around the origin.

The w-contours can be readily evaluated if $n > m$ but not when $n < m$

Exchange



Asymptotics

Non-linear fluctuating hydrodynamics (KPZ formalism) suggests a scaling limit of the form

$$n = j_1 t + \alpha t^{1/3} + \beta t^{1/2},$$
$$m = j_2 t + \gamma t^{1/3} + \delta t^{1/2},$$

where $j_{1,2}$, $\alpha, \beta, \gamma, \delta$ are known functions of ρ', and $n < m$.

Need to analyse

$$P_{n,m,\rho}(t) = \underbrace{\oint \dots \oint}_{n \times m} \text{factorised integrand}$$

where n, m, t are large.

Trick: Convert to Fredholm determinant:

$$P_{n,m,\rho}(t) = \det(\mathbb{I} - AB)_{m\times m} = \det(\mathbb{I} - BA)_{L^2(\mathbb{R})},$$

where n, m, t all occur as parameters in BA.

Fredholm determinant

Let K be an integral operator acting on functions $f \in L^2(s, \infty)$ with kernel $K(\zeta, \xi)$ given by

$$(Kf)(\zeta) := \int_s^\infty K(\zeta, \xi) f(\xi)\, \mathrm{d}\,\xi.$$

A Fredholm determinant of the operator $1 + \lambda K$ is formally defined as the series

$$\det(1 + \lambda K)_{L^2(s,\infty)} = 1 + \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} \int_s^\infty \int_s^\infty \cdots \int_s^\infty \det_{1 \le i,j \le k} \left[K(\xi_i, \xi_j) \right] \mathrm{d}\,\xi_1 \ldots \mathrm{d}\,\xi_k.$$

A discrete analogue is defined by

$$\det(1 + \lambda K)_{\ell^2(\mathbb{N})} = 1 + \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} \sum_{x_1=1}^{\infty} \sum_{x_2=1}^{\infty} \cdots \sum_{x_k=1}^{\infty} \det_{1 \le i,j \le k} \left[K(x_i, x_j) \right].$$

Asymptotics

Need to calculate integrals like

$$\mathcal{I}_2 = \oint_1 \mathrm{d}^{n-1} z \, L(\vec{z}) \det(\mathbb{I} - K(\vec{z}))_{\ell^2(\mathbb{N})}$$

with

$$K^c(x, y; \vec{z}) = \oint_1 \frac{\mathrm{d}\,\zeta}{2\pi \mathrm{i}} F^c(\zeta, x) \prod_{j=1}^{n-1} \frac{1 + z_j \zeta}{1 + \zeta} \oint_{0, -\rho', \{-z_j^{-1}\}_{j=1}^{n-1}} \frac{\mathrm{d}\,w}{2\pi \mathrm{i}} G^c(w, y) \prod_{j=1}^{n-1} \frac{1 + w}{1 + z_j w} \frac{1}{w - z},$$

Proposition

For any $(x_1, x_2, \ldots, x_k) \in \mathbb{N}^k$, $\rho \in (0, 1)$, $t > 0$ and $n, m \in \mathbb{N}$, the following equality holds:

$$\oint_1 \mathrm{d}^{n-1} z \, L(\vec{z}) \det \left[K^c(x_i, x_j, \vec{z}) \right]_{1 \le i,j \le k}$$

$$= \oint_1 \mathrm{d}^{n-1} z \, L(\vec{z}) \det \left\{ K_{\mathrm{W}}^c(x_i, x_j) - \left[\sum_{l=1}^{n-1} \prod_{k=1}^{l} (z_k - 1) A_l^c(x_i) \right] B^c(x_j) \right\}_{1 \le i,j \le k}.$$

Asymptotics

$$\mathcal{I}_2 = \mathcal{I}_z \det\left(\mathbb{I} - K(\vec{z} = \vec{1})\right)_{\ell^2(\mathbb{N})} + \text{ lower order}$$

In order to perform asymptotic analysis, we define the rescaled functions

$$\xi = x/\lambda_c t^{1/3}, \qquad \zeta = y/\lambda_c t^{1/3}$$

such that

$$\overline{K}(\xi,\zeta) = (w_c + c)^{\lambda_c t^{1/3}(\xi-\zeta)} \lambda_c t^{1/3} K(\lambda_c t^{1/3}\xi, \lambda_c t^{1/3}\zeta),$$

The rescaled kernel is explicitly described as

$$\overline{K}(\xi,\zeta) = \lambda_c t^{1/3} \oint_1 \frac{\mathrm{d}\,z}{2\pi \mathrm{i}} \left(\frac{z+\rho'}{z+1}\right) \mathrm{e}^{f(z,t,\xi)-f(w_c,t,\xi)} \times$$
$$\oint_{0,-\rho',-1} \frac{\mathrm{d}\,w}{2\pi \mathrm{i}} \left(\frac{w+1}{w+\rho'}\right) \mathrm{e}^{-f(w,t,\zeta)+f(w_c,t,\zeta)+g(w)} \frac{1}{w-z},$$

Sadlle point analysis

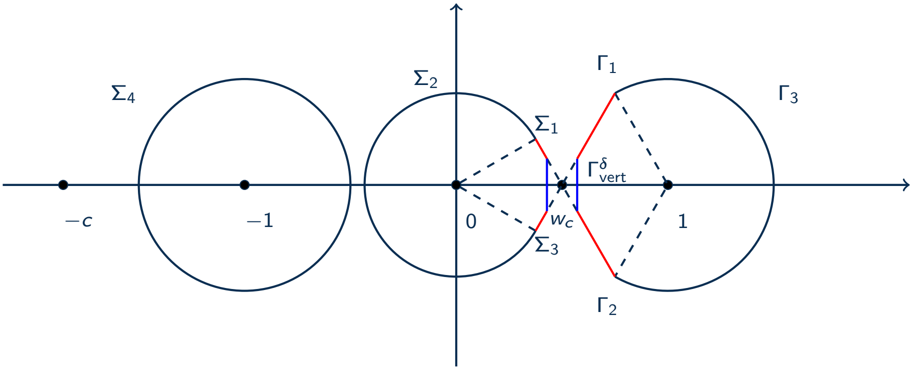


Figure: Saddle point contour ensuring uniform convergence

Theorem

$$\lim_{t\to\infty} \det(1-\overline{K})_{\ell^2(\mathbb{N}/(\lambda_c t^{1/3}))} = \lim_{t\to\infty} \det(1-\overline{K})_{L^2(0,\infty)} = \det(1-A)_{L^2(s,\infty)} = F_2(s)$$

$$A(x,y) = \int_0^\infty \mathrm{Ai}(x+\lambda)\,\mathrm{Ai}(y+\lambda)\,\mathrm{d}\,\lambda$$

and

$$s = \frac{1}{c_2 t^{1/3}}\Big((1+\rho)n - (3-\rho)m + \tfrac{1}{2}(1-\rho)(1-(1-\rho)^2/4)t\Big)$$

Recall

$$\mathcal{I}_2 = \mathcal{I}_z \det\big(\mathbb{I} - K(\vec{z}=\vec{1})\big)_{\ell^2(\mathbb{N})} + \text{ lower order}$$

The integral $\mathcal{I}_z$ converges to a Gaussian

$$\mathcal{I}_2 \to \big(1 - F_G(s')\big)F_2(s) \quad \text{as } t\to\infty$$

Final result

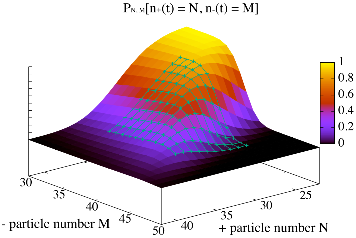
Theorem

In the appropriate scaling limit

$$\lim_{t\to\infty} P_{n,m,\rho}(t) = F_{\mathrm{GUE}}(s) F_{\mathrm{Gauss}}(s'),$$

$$s(n,m;t) =: \frac{1}{c_2 t^{1/3}}\Big((1+\rho)n - (3-\rho)m + \tfrac{1}{2}(1-\rho)(1-(1-\rho)^2/4)t\Big),$$

$$s'(n,m;t) =: \frac{1}{c_{\mathrm{g}} t^{1/2}}\Big(-2(2-\rho)n + 2\rho m + (2-\rho)(1-\rho)\rho t\Big),$$



Conclusion

- Transition probabilities for two-TASEP and two-species AHR model
- Factorised total crossing probabilities
- (First) proof of Nonlinear Fluctuating Hydrodynamics for a two-component mixture
- Interplay between integrability and random matrix theory
- Mix of Gaussian and KPZ modes
- Dynamic poles in integrand