Bi-colored Lattice Models, Charge, and the Combinatorics of Weyl Group Multiple Dirichlet Series

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Acknowledgments

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Recap of Definitions

Recall that specifying Weyl group multiple Dirichlet series

$$Z^{(n)}(s, m, \Phi) = \sum_{c=(c_1,\ldots,c_r)\in(\mathcal{O}_S/\mathcal{O}_S^\times)^r,\ c_i\neq 0} \frac{H(c;m)\Psi(c)}{\mathrm{N}c_1^{2s_1}\cdots\mathrm{N}c_r^{2s_r}}$$

for chosen data (Φ, n, F, S, m) is equivalent to specifying the p-parts of the function $H(c;m)$. (See *Weyl Group Multiple Dirichlet Series - Type A Combinatorial Theory*, Brubaker, Bump & Friedberg).

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In *Weyl Group Multiple Dirichlet Series - Type A Combinatorial Theory* by Brubaker, Bump & Friedberg, the p-part of $H(c; m)$ is given two natural combinatorial definitions in terms of Gelfand-Tsetlin patterns, H and H .

Recap of Definitions

First, recall:

Definition

An (uncolored) Gelfand-Tsetlin pattern of rank r is a triangular array of nonnegative integers

$$\begin{array}{ccccccccc} a_{1;1} & & a_{1;2} & & & & a_{1;r} & & a_{1;r+1} \\ & a_{2;1} & & a_{2;2} & & & & a_{2;r} & \\ & & \ddots & & & & \reflectbox{$\ddots$} & & \\ & & & a_{r;1} & & a_{r;2} & & & \\ & & & & a_{r+1;1} & & & & \end{array}$$

such that for $a_{i;j} \quad a_{i+1;j} \quad a_{i;j+1}$ for all natural $i;j < r$.

Recap of Definitions

Example

Here are all 8 of the GT patterns with top row $(2; 1; 0)$:

$$
\begin{array}{ccccc} 2 & & 1 & & 0 \\ & 2 & & 1 & \\ & & 2 & & \end{array};
\quad
\begin{array}{ccccc} 2 & & 1 & & 0 \\ & 2 & & 1 & \\ & & 1 & & \end{array};
\quad
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$$

Recap of Definitions

H and H are defined thus.

For a given Gelfand-Tsetlin pattern , we first define the following functions on the entries of :

$$f\,(\text{entry}) = \begin{cases} 1 & \text{if the entry is equal to its top-right neighbor} \\ v & \text{if the entry is equal to its top-left neighbor} \\ 0 & \text{if the entry is equal to both top neighbors} \\ 1+v & \text{otherwise} \end{cases}$$

where v is a formal variable.

Similarly for the Δ weights,

$$f\,(\text{entry}) = \begin{cases} v & \text{if the entry is equal to its top-right neighbor} \\ 1 & \text{if the entry is equal to its top-left neighbor} \\ 0 & \text{if the entry is equal to both top neighbors} \\ 1+v & \text{otherwise} \end{cases}$$

Recap of Definitions

H and H are defined thus.

Given a GT pattern ,

$$G\,(\ ) = \prod_{n \in} f\,(n)$$

and

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With all of this in mind,

$$H = \sum G\,(\ );$$

and

$$H = \sum G\,(\ );$$

ranging over with a fixed top row.

Recap of Definitions

For Type A,

Theorem (Brubaker-Bump-Friedberg)

For GT patterns with xed top row,

$$H = H ;$$

i.e.

$$\sum G ( ) = \sum G ( ):$$

And in fact, for strict patterns, it holds that

$$G ( ) = G ({}^{0});$$

where ^{0} is the the **Schutzenberger involution** or **evacuation** of , originally defined for standard Young tableau, but defined for Gelfand-Tsetlin patterns by Kirillov and Berenstein.

Recap of Definitions

Going back to the top row $(2, 1, 0)$ example, we have

Example

$$G\begin{pmatrix} 2 & & 1 & & 0 \\ & 2 & & 1 & \\ & & 2 & & \end{pmatrix} = v^3 = G\begin{pmatrix} 2 & & 1 & & 0 \\ & 1 & & 0 & \\ & & 0 & & \end{pmatrix}$$

$$G\begin{pmatrix} 2 & & 1 & & 0 \\ & 2 & & 1 & \\ & & 1 & & \end{pmatrix} = v^2 = G\begin{pmatrix} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 0 & & \end{pmatrix}$$

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$$G\begin{pmatrix} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 1 & & \end{pmatrix} = v(1+v) \neq 0 = G\begin{pmatrix} 2 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{pmatrix}$$

Motivation

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The fix (possibly)?

Refinement, and then seeking a bijection.

Motivation

Recall that in the chapter "Metaplectic Ice" in Multiple Dirichlet Series, L-functions and Automorphic Forms, Brubaker, Bump, Chinta, Friedberg, & Gunnells provide a bijection between strict Gelfand-Tsetlin patterns of fixed top row and admissible states of square ice (or the six-vertex model) with fixed boundary conditions.

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Recall that in the chapter "Metaplectic Ice" in Multiple Dirichlet Series, L-functions and Automorphic Forms, Brubaker, Bump, Chinta, Friedberg, & Gunnells provide a bijection between strict Gelfand-Tsetlin patterns of fixed top row and admissible states of square ice (or the six-vertex model) with fixed boundary conditions.

It thus suffices to work with Tokuyama ice/six-vertex lattice models when one wants to work with strict Gelfand-Tsetlin patterns and Schützenberger involutions.

Motivation

In the literature, there are other types of generalized lattice models that have been described whose partition functions yield more general Whittaker functions compared to the Tokuyama ice model, whose partition function just yields spherical Whittaker functions on $GL_r(F)$.

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In the literature:

Whittaker Function	Additional Statistics
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Metaplectic	Charge
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Our models are refinements of all of the above, and are extremely promising candidates to yield a bijective $H = H$ proof.

Bi-colored Lattice Models

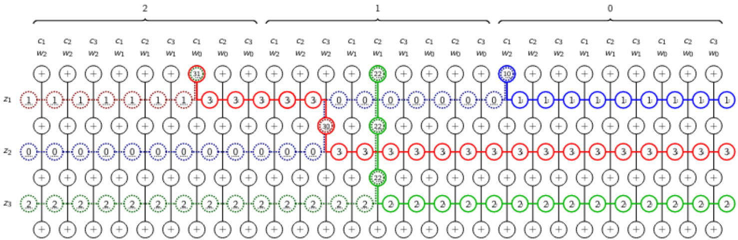
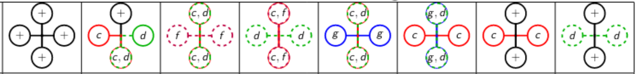


Figure: A bi-colored lattice model state for the non-strict Gelfand-Tsetlin pattern (210, 11, 1).

Γ, Δ vertices and weights



Γ	1	$1+v$	v	-1	$-v$	1	1	1
Δ	1	$1+v$	1	$-v$	-1	v	1	1

Figure: Admissible vertices and their Γ and Δ Boltzmann weights.

Models

As statistics, the models use color and supercolor, just as metaplectic Iwahori ice does, and charge, just as metaplectic ice does.

- Colors move down and to the right.
- Scolors (supercolors) move down and to the left.
- Every horizonal edge has either a color or a scolor.
- Every vertical edge either has both a color and a scolor, or it has neither a color nor a scolor.
- For a fixed $n \geq 1$, we have a function $g(a)$ where a is an integer mod n, where $g(0) = v$, $g(a)g(-a) = -v$ if $a \not\equiv 0$ mod n.

Models - Γ

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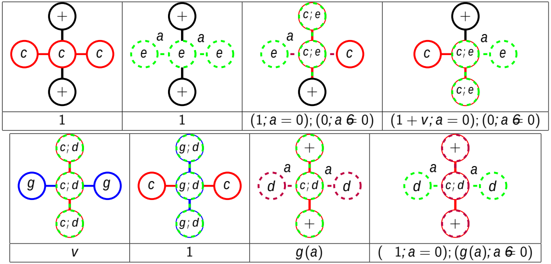

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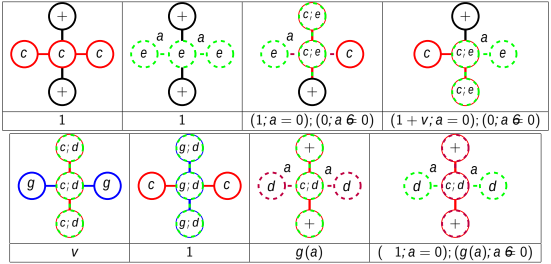

$a \quad = a$, but we set $a \quad = a + 1$ when at the left-most column of a "big column."

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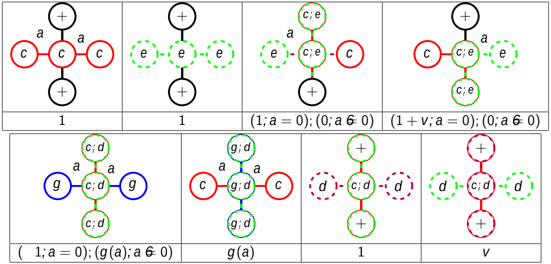

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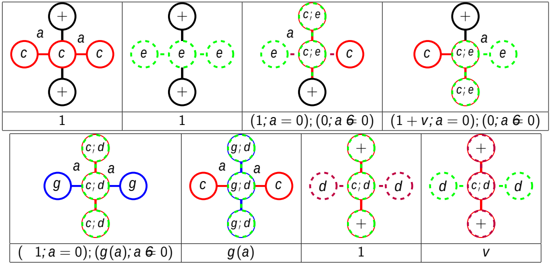

$a \quad = a$, but we set $a \quad = a + 1$ when at the right-most column of a "big column."

A worked-out example (by Andy Hardt)

Choosing $n = 2$, we have two possible values of a. 0 or 1, since a is in $Z=2Z:$
We choose partition $3; 1; 0$.

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Choosing $n = 2$, we have two possible values of a. 0 or 1, since a is in Z=2Z*:*

We choose partition $3; 1; 0$.

The following are bi-colored charged Γ and Δ lattice model states, respectively.

Example: Bi-Colored Charged Lattice Model

Figure: bi-colored charged lattice model.

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Recalling our admissible vertices and their Boltzmann weights, we see that all vertices have weight 1 except for the decorated ones. The triangle has weight $1+v$, and the square and diamond both have weight $g(1)$. Therefore the weight of this bi-colored charged lattice model state is $g(1)^2(1+v)$.

Example: Bi-Colored Charged Lattice Model

Figure: bi-colored charged lattice model.

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Figure: bi-colored charged lattice model.

Recalling our admissible vertices and their Boltzmann weights, we see that all vertices have weight 1 except for maybe the decorated ones. The triangle also has weight 1, the square has weigh v, and diamond has weight $1+v$. Therefore the weight of this bi-colored charged lattice model state is $v(1+v)$:

Example: Equality!

By the rules we stated above for our function $g(a)$:

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For a xed n 1, we have a function $g(a)$ where a is an integer mod n, where $g(0) = v$, $g(a)g(a) = v$ if a 6 0 mod n:

Example: Equality!

By the rules we stated above for our function $g(a)$:
For a fixed $n \geq 1$, we have a function $g(a)$ where a is an integer mod n, where $g(0) = v$, $g(a)g(-a) = -v$ if $a \not\equiv 0 \mod n$.
We have

$$g(1)^2(1+v) = g(1)g(-1)(1+v) = -v(1+v);$$

as desired.
Indeed, $g(1)^2 = g(1)g(-1)$; since $1 \equiv -1 \mod 2$

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Thus, these bi-colored charged lattice models re ne uncolored lattice models.

Conjecture

With all of this in mind, we can now state the main conjecture of the project.

Conjecture

There exists a weight-preserving bijection betweenand bi-colored charged lattice model states with xed boundary conditions.

Future Directions

- Find the bijection!
- Prove solvability by showing this satis es the Yang-Baxter equation.

In Conclusion,

Deye mon, gen mon.
Beyond mountains, there are mountains.
Haitian proverb, as Ben mentioned in a previous seminar talk.

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Thank you!