Quasi-polynomial representations of double affine Hecke algebras and a generalization of Macdonald polynomials

Joint work with Jasper Stokman and Vidya Venkateswaran

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- The representation π, which we recall below, is called the polynomial representation of $\mathcal{H}$.

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- We refer to elements of $\mathcal{Q}$ as *quasi-polynomials*, and to π_g as the quasi-polynomial representation.
- We first recall the basics of Macdonald polynomials, and then explain our generalization.

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- Here W_0 is the Weyl group of the underlying finite root system Φ_0 and L_0 is its coroot lattice.

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- In particular $\mathbb{Z}\delta$ is central and $\mathcal{W}$ contains two copies of the lattice L_0 with a commutator pairing into $\mathbb{Z}\delta$.

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$$T_i(x^{\lambda}) = k_i x^{s_i(\lambda)} + (k_i - k_i^{-1}) \frac{1 - x^{-(\lambda,\alpha_i)\alpha_i^{\vee}}}{1 - x^{\alpha_i^{\vee}}} x^{\lambda}. \tag{1}$$

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- Then (1) is the polynomial representation π of $\mathcal{H}$, which turns out to be faithful.

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- In view of the isomorphism $L \approx L_0 + \mathbb{Z}\delta$ we can regard an element p of $\mathcal{X}$ as Laurent polynomials in $x_1 = x^{\alpha_1^\vee}, \cdots, x_r = x^{\alpha_r^\vee}$ and $q = x^\delta$.

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- Then H is an affine Hecke algebra, which is a flat deformation of the group algebra of $W = W_0 \ltimes L_0$.

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- The simultaneous eigenfunctions of the Y_κ are the Macdonald polynomials.

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- $$g_i(e) = \begin{cases} k_i & \text{if } (\alpha_i, e) \in \mathbb{Z}, \\ h_i(e) & \text{else.} \end{cases} \tag{3}$$

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- $\mathcal{X}_\mathcal{O}$ stable under π_g, and this defines a subrepesentation $\pi^\mathcal{O} = \pi_g^\mathcal{O}$.

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The operators $Y_\kappa^{\mathcal{O}} = \pi^{\mathcal{O}}(y^\kappa)$ can be simultaneously diagonalized on $\mathcal{X}_{\mathcal{O}}$ and the common eigenspaces have dimension 1 over $K_{\mathcal{O}}$.

Generalizations of Madonald polynomials

- Thus we obtain a decomposition of π_g as a direct sum of $\pi^{\mathcal{O}}$.
- The definition of $\pi^{\mathcal{O}}$ involves a *fixed* central element $q_s = x^{s\delta}$ and only a *finite* number of values γ_j of the g-functions.
- Thus $\pi^{\mathcal{O}}$ can be defined over the generic field $K_{\mathcal{O}} = \mathbb{Q}(q, k_i, q_s, \gamma_j)$.

Theorem (S.-Stokman-Venkateswaran)

The operators $Y_\kappa^{\mathcal{O}} = \pi^{\mathcal{O}}(y^\kappa)$ can be simultaneously diagonalized on $\mathcal{X}_{\mathcal{O}}$ and the common eigenspaces have dimension 1 over $K_{\mathcal{O}}$.

- These common eigenfunctions are our generalizations of Macdonald polynomials.

Remarks

- If $\mathcal{O}$ is contained in the rational points $L \otimes_{\mathbb{Z}} \mathbb{Q}$ then our generalizations of Macdonald polynomials can be viewed as elements of the group algebra of a lattice L' with finite index in L.

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- If $\mathcal{O}$ is contained in the rational points $L \otimes_{\mathbb{Z}} \mathbb{Q}$ then our generalizations of Macdonald polynomials can be viewed as elements of the group algebra of a lattice L' with finite index in L.
- Upon suitable specialization of the q-parameters, this yields the polynomials studied by Chinta-Gunnells.
- A further specialization of the g-parameters to suitable Gauss sums yields the metaplectic Iwahori-Whittaker functions.

Some Metaplectic polynomials for GL(3)

Formulas for $E_\lambda^{(n)}(x)$, $1 \leq n \leq 5$ and $\lambda \in \mathbb{Z}^3$ of weight at most 2.

$E_{(0,0,0)}^{(1)}(x) = 1$
$E_{(0,0,0)}^{(2)}(x) = 1$
$E_{(0,0,0)}^{(3)}(x) = 1$
$E_{(0,0,0)}^{(4)}(x) = 1$
$E_{(0,0,0)}^{(5)}(x) = 1$

$E_{(1,0,0)}^{(1)}(x) = x_1$
$E_{(1,0,0)}^{(2)}(x) = x_1$
$E_{(1,0,0)}^{(3)}(x) = x_1$
$E_{(1,0,0)}^{(4)}(x) = x_1$
$E_{(1,0,0)}^{(5)}(x) = x_1$

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$$E^{(1)}_{(0,1,0)}(x) = \frac{(k-1)(k+1)}{k^4 q-1} x_1 + x_2$$
$$E^{(2)}_{(0,1,0)}(x) = \frac{(k-1)(k+1)}{k(kq^2+\epsilon)} x_1 + x_2$$
$$E^{(3)}_{(0,1,0)}(x) = \frac{(k-1)(k+1)g_1}{k^4 g_1^3 q^3+1} x_1 + x_2$$
$$E^{(4)}_{(0,1,0)}(x) = \frac{(k-1)(k+1)g_1}{k^4 g_1^3 q^4+1} x_1 + x_2$$
$$E^{(5)}_{(0,1,0)}(x) = \frac{(k-1)(k+1)g_1}{k^4 g_1^3 q^5+1} x_1 + x_2$$

$$E^{(1)}_{(0,0,1)}(x) = \frac{(k-1)(k+1)}{qk^2-1} x_1 + \frac{(k-1)(k+1)}{qk^2-1} x_2 + x_3$$
$$E^{(2)}_{(0,0,1)}(x) = -\frac{(k-1)(k+1)}{k(k+\epsilon q^2)} x_1 + \frac{(k-1)(k+1)}{q^2+\epsilon k} x_2 + x_3$$
$$E^{(3)}_{(0,0,1)}(x) = -\frac{(k-1)(k+1)g_1^2}{k^2 g_1^3 q^3+1} x_1 + \frac{(k-1)(k+1)g_1}{k^2 g_1^3 q^3+1} x_2 + x_3$$
$$E^{(4)}_{(0,0,1)}(x) = -\frac{(k-1)(k+1)g_1^2}{k^2 g_1^3 q^4+1} x_1 + \frac{(k-1)(k+1)g_1}{k^2 g_1^3 q^4+1} x_2 + x_3$$
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$$E^{(1)}_{(1,0,1)}(x) = \frac{(k-1)(k+1)}{k^4q-1}x_1x_2 + x_3x_1$$
$$E^{(2)}_{(1,0,1)}(x) = \frac{(k-1)(k+1)}{k(kq^2+\epsilon)}x_1x_2 + x_3x_1$$
$$E^{(3)}_{(1,0,1)}(x) = \frac{(k-1)(k+1)g_1}{k^4g_1^3q^3+1}x_1x_2 + x_3x_1$$
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$$E^{(4)}_{(1,1,0)}(x) = x_1x_2$$
$$E^{(5)}_{(1,1,0)}(x) = x_1x_2$$

$$E^{(1)}_{(2,0,0)}(x) = x_1^2 + \frac{q(k-1)(k+1)}{qk^2-1}x_1x_2 + \frac{q(k-1)(k+1)}{qk^2-1}x_3x_1$$
$$E^{(2)}_{(2,0,0)}(x) = x_1^2$$
$$E^{(3)}_{(2,0,0)}(x) = x_1^2$$
$$E^{(4)}_{(2,0,0)}(x) = x_1^2$$
$$E^{(5)}_{(2,0,0)}(x) = x_1^2$$

Some Metaplectic polynomials for GL(3)

$$E^{(1)}_{(0,2,0)}(x) = \frac{(k-1)(k+1)}{(qk^2-1)(qk^2+1)}x_1^2 + x_2^2 + \frac{(k-1)(k+1)\left(k^4q^2+qk^2-q-1\right)}{(qk^2+1)(qk^2-1)^2}x_1x_2 + \frac{(k-1)^2(k+1)^2q}{(qk^2+1)(qk^2-1)^2}x_3x_1 + \frac{q(k-1)(k+1)}{qk^2-1}x_3x_2$$

$$E^{(2)}_{(0,2,0)}(x) = \frac{(k-1)(k+1)}{(q^2k^2-1)(q^2k^2+1)}x_1^2 + x_2^2$$

$$E^{(3)}_{(0,2,0)}(x) = \frac{(k-1)(k+1)g_1^2}{k^2g_1^3+q^6}x_1^2 + x_2^2$$

$$E^{(4)}_{(0,2,0)}(x) = \frac{(k-1)(k+1)}{k(q^8k+\epsilon)}x_1^2 + x_2^2$$

$$E^{(5)}_{(0,2,0)}(x) = \frac{(k-1)(k+1)g_2}{k^4g_2^3q^{10}+1}x_1^2 + x_2^2$$

Some Metaplectic polynomials for GL(3)

$$E^{(1)}_{(0,0,2)}(x) = \frac{(k-1)(k+1)}{(kq-1)(kq+1)}x_1^2 + \frac{(k-1)(k+1)}{(kq-1)(kq+1)}x_2^2 + x_3^2 + \frac{(q+1)(k-1)^2(k+1)^2}{(kq-1)(kq+1)(qk^2-1)}x_1x_2 + \frac{(q+1)(k-1)(k+1)}{(kq-1)(kq+1)}x_3x_1 + \frac{(q+1)(k-1)(k+1)}{(kq-1)(kq+1)}x_3x_2$$

$$E^{(2)}_{(0,0,2)}(x) = \frac{(k-1)(k+1)}{(kq^2-1)(kq^2+1)}x_1^2 + \frac{(k-1)(k+1)}{(kq^2-1)(kq^2+1)}x_2^2 + x_3^2$$

$$E^{(3)}_{(0,0,2)}(x) = -\frac{(k-1)(k+1)g_1}{k^4g_1^3+q^6}x_1^2 + \frac{(k-1)(k+1)k^2g_1^2}{k^4g_1^3+q^6}x_2^2 + x_3^2$$

$$E^{(4)}_{(0,0,2)}(x) = -\frac{(k-1)(k+1)}{k(\epsilon q^8+k)}x_1^2 + \frac{(k-1)(k+1)}{q^8+\epsilon k}x_2^2 + x_3^2$$

$$E^{(5)}_{(0,0,2)}(x) = -\frac{(k-1)(k+1)g_2^2}{k^2g_2^3q^{10}+1}x_1^2 + \frac{(k-1)(k+1)g_2}{k^2g_2^3q^{10}+1}x_2^2 + x_3^2$$