Schubert polynomials and the inhomogeneous TASEP on a ring

Donghyun Kim

UC Berkeley/ Harvard

(joint work with Lauren Williams)

Asymmetric Exclusion Process

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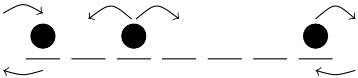

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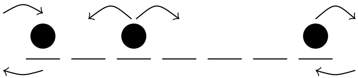

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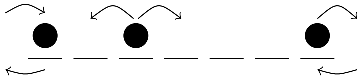
- There are many variants of the ASEP and these were studied with various mathematical approaches, for example, Bethe Ansatz, quadratic algebras, combinatorics, orthogonal polynomials, random matrices, stochastic differential equations and hydrodynamic limits.
- The ASEP has many applications in a broad range including protein synthesis, traffic flow, formation shocks, surface growth, and sequence alignments.



The inhomogenous TASEP definition

- Consider a lattice with n sites arranged in a ring. Let $St(n)$ denote the $n!$ labelings of the lattice by distinct numbers $1, 2, \ldots, n$, where each number i is called a *particle of weight* i.

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- The *inhomogeneous TASEP on a ring of size n* is a Markov chain with state space $St(n)$ where at each time t a swap of two adjacent particles may occur: a particle of weight i on the left swaps its position with a particle of weight j on the right with transition rate $r_{i,j}$ given by:

$$r_{i,j} = \begin{cases} x_i - y_{n+1-j} & \text{if } i < j \\ 0 & \text{otherwise.} \end{cases}$$

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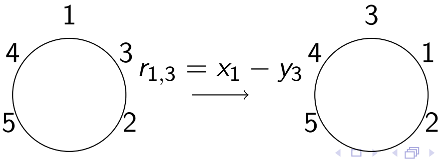

Cantini proved that the inhomogenous TASEP is a solvable lattice model. ("Inhomogenous Multispecies TASEP on a ring with spectral parameters", 2016)

The inhomogeneous TASEP $n = 3$

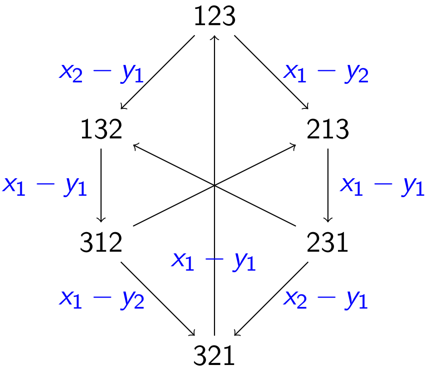

Figure: The transition diagram for the inhomogeneous TASEP for $n = 3$

Renormalized steady state probabilities

- The steady state probabilities for $n = 3$ inhomogeneous TASEP
  States 123, 231, 312: $\frac{x_1 - y_1}{6x_1 + 3x_2 - 6y_1 - 3y_2}$

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$$\psi_{123\ldots n} = \prod_{i<j} (x_i - y_{n+1-j})^{j-i-1}.$$

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- Schubert polynomials? $x_1 - y_1 = \mathfrak{S}_{(2,1)}$, $x_1 + x_2 - y_1 - y_2 = \mathfrak{S}_{(1,3,2)}$

Double Schubert polynomials definition

Definition: double Schubert polynomials

For the longest permutation $\sigma_0 \in S_n$

$$\mathfrak{S}_{\sigma_0}(x;y) = \prod_{i+j\leq n}(x_i - y_j)$$

for generic $\sigma \in S_n$

$$\mathfrak{S}_{\sigma}(x;y) = \partial_{\sigma^{-1}\sigma_0}\mathfrak{S}_{\sigma_0}(x;y)$$

where $\partial_\sigma = \partial_{i_1}\partial_{i_2}\cdots\partial_{i_l}$ ($s_{i_1}s_{i_2}\cdots s_{i_l}$ is a reduced decomposition of σ)

$$(\partial_i P)(x_1,\dots,x_n) = \frac{P(\dots,x_i,x_{i+1},\dots) - P(\dots,x_{i+1},x_i,\dots)}{x_i - x_{i+1}}$$

Table $n = 4$

State w	Probability ψ_w
1234	$(x_1 - y_1)^2(x_1 - y_2)(x_2 - y_1)$
1324	$(x_1 - y_1)\mathfrak{S}_{1432}$
1342	$(x_1 - y_1)(x_2 - y_1)\mathfrak{S}_{1423}$
1423	$(x_1 - y_1)(x_1 - y_2)(x_2 - y_1)\mathfrak{S}_{1243}$
1243	$(x_1 - y_2)(x_1 - y_1)\mathfrak{S}_{1342}$
1432	$\mathfrak{S}_{1423}\mathfrak{S}_{1342}$

Table $n = 5,\ y_i = 0$

State w	Probability ψ_w
12345	$x^{(6,3,1)}$
12354	$x^{(5,2,0)}\mathfrak{S}_{13452}$
12435	$x^{(4,1,0)}\mathfrak{S}_{14532}$
12453	$x^{(4,1,1)}\mathfrak{S}_{14523}$
12534	$x^{(5,2,1)}\mathfrak{S}_{12453}$
12543	$x^{(3,0,0)}\mathfrak{S}_{14523}\mathfrak{S}_{13452}$
13245	$x^{(3,1,1)}\mathfrak{S}_{15423}$
13254	$x^{(2,0,0)}\mathfrak{S}_{15423}\mathfrak{S}_{13452}$
13425	$x^{(3,2,1)}\mathfrak{S}_{15243}$
13452	$x^{(3,3,1)}\mathfrak{S}_{15234}$
13524	$x^{(2,1,0)}(\mathfrak{S}_{164325} + \mathfrak{S}_{25431})$
13542	$x^{(2,2,0)}\mathfrak{S}_{15234}\mathfrak{S}_{13452}$
14235	$x^{(4,2,0)}\mathfrak{S}_{13542}$
14253	$x^{(4,2,1)}\mathfrak{S}_{12543}$
14325	$x^{(1,0,0)}(\mathfrak{S}_{1753246} + \mathfrak{S}_{265314} + \mathfrak{S}_{2743156} + \mathfrak{S}_{356214} + \mathfrak{S}_{364215} + \mathfrak{S}_{365124})$
14352	$x^{(1,1,0)}\mathfrak{S}_{15234}\mathfrak{S}_{14532}$
14523	$x^{(4,3,1)}\mathfrak{S}_{12534}$
14532	$x^{(1,1,1)}\mathfrak{S}_{15234}\mathfrak{S}_{14523}$
15234	$x^{(5,3,1)}\mathfrak{S}_{12354}$
15243	$x^{(3,1,0)}(\mathfrak{S}_{146325} + \mathfrak{S}_{24531})$
15324	$x^{(2,1,1)}(\mathfrak{S}_{15432} + \mathfrak{S}_{164235})$
15342	$x^{(2,2,1)}\mathfrak{S}_{15234}\mathfrak{S}_{12453}$
15423	$x^{(3,2,0)}\mathfrak{S}_{12534}\mathfrak{S}_{13452}$
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- For $y_i = 0$, Conjecture 1 has been proved by Arita and Mallick in 2012 by giving a monomial expansion formula in terms of multiline queues as conjectured by Ayyer and Linusson.
- Later, Cantini in 2016 generalized the model by putting y-parameters and established a solvability. Then he made the above conjectures.

Multiline queues

- A *multiline queue* Q is an $L \times n$ array in which each of the Ln positions is either vacant or occupied by a ball. The number of balls in each row is weakly increasing from top to bottom.

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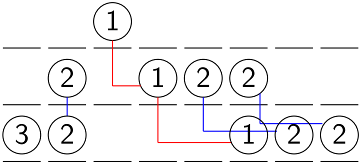
- (Eg)



type: $(2, 2, 1, 4, 4, 4, 2, 3)$

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- Let $z_{r,i}$ be the number of $i - covered$ vacancies on row r. We define a weight of a multiline queue Q as follows

$$weight(Q) = \prod_{1 \leq i < r \leq L} (\frac{x_r}{x_i})^{z_{r,i}}$$

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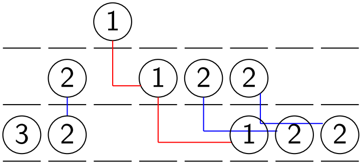

$$weight(Q) = (\frac{x_2}{x_1})(\frac{x_3}{x_1})^2.$$

Multiline queues

Theorem(Arita, Mallick)

The steady state probability ψ_w is proportional to a weighted sum over multiline queues of type w

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Limits

- Cannot deal with the model with general y_i's. It is an open problem to define a weight putting y_i's.
- Hard to explain the appearance of Schubert polynomials except for a special case.

Special case (Inverse of Grassmannian permutations)

Theorem(K, Williams)

If w is an inverse of Grassmannian permutation that starts with 1, then there exists a (weight-preserving) bijection between multiline queues of type w and certain flagged semistandard young tableaux.

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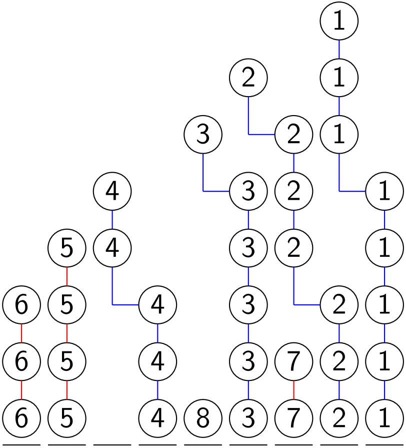
Definition: pattern avoidance

We say that a permutation π avoids a pattern σ, or σ-avoiding, if π does not contain a subsequence in a same relative order with σ. Eg. $(1,4,2,6,3,5,7)$ is not 2143-avoiding $\rightarrow (1,\underline{4},\underline{2},\underline{6},3,\underline{5},7)$

Bijection



1	1	4	≤ 6
2	3	6	≤ 6
3	4	≤ 7	
6	≤ 8		

Integrability

Cantini introduced a familiy $\psi_w(z_1, \cdots, z_n)$ that recovers the steady-state probability ψ_w by taking the leading coefficient in z-variables (specializing to $z = \infty$). We call $\psi_w(z_1, \cdots, z_n)$ a *z-deformed steady-state probability*.

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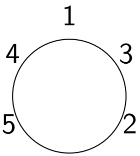
- Let V be a vector space with formal basis $e_1, \cdots, e_n$ where e_i represents the particle type i.
- Consider $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ $(V_i = V)$, and represent each state of the inhomogeneous TASEP as a basis of this space
  Eg)



$$\longrightarrow \quad e_1 \otimes e_3 \otimes e_2 \otimes e_5 \otimes e_4$$

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- Let $\mathcal{M}$ be the Markov matrix of the inhomogenous TASEP, then $\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2 + \cdots + \mathcal{M}_n$ is the sum of local terms $\mathcal{M}_i$. The matrix $\mathcal{M}_i$ acts on $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ by acting locally on $V_i \otimes V_{i+1}$ parts.

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Suppose we have an operator $R_i(a, b)$ for formal variables a and b acting locally on $V_i \otimes V_{i+1}$ such that

$$R_i(a, a) = 1, \frac{dR_i(a, b)}{da}|_{a=b=\infty} \propto \mathcal{M}_i$$

and a vector $\psi(z_1, \cdots, z_n)$ that satisfies

$$R_i(z_i, z_{i+1})\psi(z_1, \cdots, z_n) = s_i\psi(z_1, \cdots, z_n)$$

where s_i acts by exchanging z_i and z_{i+1}. We call the above equation, *exchange equation*.

Integrability

Claim

$\psi(z_1, \cdots, z_n)|_{z=\infty}$ is proportional to the steady state probability of the inhomogeneous TASEP.

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Proof.

Differentiating the exchange equation with z_i and plugging in $z = \infty$ gives

$$\mathcal{M}_i \psi(z_1, \cdots, z_n)|_{z=\infty} = \partial_{i+1} \psi(z_1, \cdots, z_n)|_{z=\infty} - \partial_i \psi(z_1, \cdots, z_n)|_{z=\infty}.$$

Summing over $i = 1$ to n completes the proof. □

Integrability

- Cantini found such operator $R_i(a, b)$ satisfying the additional two equations (unitary relation, braid Yang-Baxter equation)

$$R_i(a,b)R_i(b,a) = 1$$
$$R_i(b,c)R_{i+1}(a,c)R_i(a,b) = R_{i+1}(a,b)R_i(a,c)R_{i+1}(b,c).$$

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- The vector $\psi(z_1, \cdots, z_n)$ is the common eigenvector of scattering matrices

$$S_i = \mathcal{R}R_{i-2}(z_i, z_{i-1}) \cdots R_{i+1}(z_i, z_{i+2})R_i(z_i, z_{i+1})$$

where $\mathcal{R}$ acts by rotation

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where $\mathcal{R}$ acts by rotation

$$\mathcal{R}(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = (v_n \otimes v_1 \otimes \cdots \otimes v_{n-1})$$

- S_i and S_j commute by braid Yang-Baxter equation

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The initial condition is given by

$$\psi_{(1,2,\cdots,n)}(z_1,\cdots,z_n)$$
$$=\prod_{1\leq i<j\leq n}(x_i-y_{n+1-j})^{j-i-1}\prod_{i=1}^{n}\Big(\prod_{j=1}^{i-1}(z_i-x_j)\prod_{j=i+1}^{n}(z_i-y_{n+1-j})\Big)$$

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Expanding the exchange equation component by component gives

$$\psi_{s_i w}(z_1,\cdots,z_n) = \pi_i(w_i, w_{i+1}; n)\psi_w(z_1,\cdots,z_n) \quad \text{if } w_i > w_{i+1},$$

where $\pi_i(\beta,\alpha;n)$ is the *isobaric divided difference operator* defined by

$$\pi_i(\beta,\alpha;n)G(\mathbf{z}) = \frac{(z_i - y_{n+1-\beta})(z_{i+1} - x_\alpha)}{x_\alpha - y_{n+1-\beta}}\,\frac{G(\mathbf{z}) - s_i G(\mathbf{z})}{z_i - z_{i+1}}.$$

$n = 3$ example

We can compute every component $\psi_w(z_1, \cdots, z_n)$ of $\psi(z_1, \cdots, z_n)$ from the initial condition by applying sequences of isobarbic divided difference operators.

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$$((x_1 + x_2 - y_1 - y_2)z_3 z_1 + (x_1 x_2 - y_1 y_2)(z_3 + z_1) - x_1 x_2 y_1 - x_1 x_2 y_2 + x_1 y_1 y_2$$

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Taking the leading coefficients in z-variables (specializing to $z = \infty$) gives $\psi_{(1,2,3)} = x_1 - y_1$ and $\psi_{(3,2,1)} = (x_1 + x_2 - y_1 - y_2)$.

Main results

Definition(K, Williams)

We say that $w \in S_n$ is a *evil-avoiding*, if: $w_1 = 1$; w avoids the patterns 2413, 3214, 4132, and 4213. We say $w \in St(n, k)$ if w is evil-avoiding and w^{-1} has exactly k *descents*.

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For $w \in St(n,k)$, the steady-state probability ψ_w is given as a trivial factor times product of k double Schubert polynomials.

Eg) $w = (1,2,5,4,3)$, $w^{-1} = (1,2,5,4,3)$.
$w \in St(5,2)$.

$$\psi_w = (x_1 - y_1)(x_1 - y_2)(x_1 - y_3)\mathfrak{S}_{(1,4,5,2,3)}(x;y)\mathfrak{S}_{(1,3,4,5,2)}(x;y)$$

Main results

$\to$ The number of evil-avoiding permutations in S_n is $\frac{(2+\sqrt{2})^{n-1}+(2-\sqrt{2})^{n-1}}{2}$. Previously the formula (explaining the appearance of (double) Schubert polynomials) for n out of $n!$ states were known by Cantini.

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- Cantini gave an explicit formula for $\psi_w(z_1, \cdots, z_n)$ for the permutation w of the form $w(n,h) := (1, h+1, h+2, \ldots, n, h, h-1, \ldots, 2)$
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- We will present an explicit formula for $\psi_w(z_1, \cdots, z_n)$ for evil-avoiding permutations.
- We introduce z-Schubert polynomials $\mathfrak{S}^n_\lambda(z;x;y)$ to do that.

Lemma, Double Schubert polynomials

Code $c(w) = (a_1, \cdots, a_n)$ of a permutation $w \in S_n$ is an integer vector such that a_i is the number of $w_i > w_j$ for $j > i$.
Example $c((5,1,3,4,2)) = (4,0,1,1,0)$ and $c((1,3,4,2)) = (0,1,1,0)$

Proposition (K, Williams)

Let w and w' be permutations such that $c(w) = (c_1, \ldots, c_n)$ and $c(w') = (M, c_1, \ldots, c_n)$. If M is "sufficiently big" then we have the equation

$$\mathfrak{S}_{w'}(x; y) = \mathfrak{S}_w(x_{\hat{1}}; y) \prod_{k=1}^{M} (x_1 - y_k).$$

$x_{\hat{1}}$ means we shift indices of x-variables by 1 $(x_1 \to x_2, x_2 \to x_3, \cdots)$
Example)

$$\mathfrak{S}_{(5,1,3,4,2)} = \mathfrak{S}_{(1,3,4,2)}(x_{\hat{1}}) \prod_{i=1}^{4} (x_1 - y_i)$$

- We say that a partition λ is *valid* for n if λ is properly contained in a $length(\lambda) \times (n - length(\lambda))$ rectangle.

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- $\mathfrak{S}^n_\lambda(z;x;y)$ is defined recursively from $\mathfrak{S}^{n-1}_{\lambda'}(z;x;y)$ where λ' is the partition obtained by deleting the first part of λ

$$\mathfrak{S}^n_\lambda(z;x;y) = \partial_{n-\lambda_1-mul(\lambda)} \cdots \partial_1 \Big(\mathfrak{S}^{n-1}_{\lambda'}(\sigma^{\lambda_1-\lambda_2+1} z; x_{\hat{1}}; y) \\ \times \prod_{l=1}^{n-mul(\lambda)} (x_1 - y_l) \prod_{i=1}^{(\lambda_1-\lambda_2+1)} \prod_{m=2}^{n-\lambda_1-mul(\lambda)+1} (z_i - x_m) \Big).$$

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z-Schubert polynomials $\mathfrak{S}^n_\lambda(z;x;y)$

- Leading coefficient of $\mathfrak{S}^n_\lambda(z;x;y)$ (specializing to $z=\infty$) is a double Schubert polynomial. And there is a simple rule to get a code of the permutation indexing double Schubert polynomial.
  (Eg.)

$$\mathfrak{S}^4_{(1,1)}(\mathrm{z};\mathrm{x};\mathrm{y}) \to \mathfrak{S}_{(1,3,4,2)}$$
$$\mathfrak{S}^5_{(2,1,1)}(\mathrm{z};\mathrm{x};\mathrm{y}) \to \mathfrak{S}_{(1,3,5,4,2)}$$

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- Recursive construction of z-Schubert polynomials is consistent with the equations of Schubert polynomials.

$$\mathfrak{S}_{(1,3,5,4,2)} = \partial_2\partial_1(\mathfrak{S}_{(5,1,3,4,2)}) = \partial_2\partial_1(\mathfrak{S}_{(1,3,4,2)}(x_{\hat{1}})\prod_{i=1}^{4}(x_1-y_i))$$

$$\mathfrak{S}^5_{(2,1,1)}(\mathrm{z};\mathrm{x};\mathrm{y}) = \partial_2\partial_1(\mathfrak{S}^4_{(1,1)}(\mathrm{z};\mathrm{x}_{\hat{1}};\mathrm{y})\prod_{i=1}^{4}(x_1-y_i)\prod_{i=1}^{2}\prod_{m=2}^{3}(z_i-x_m))$$

Main results

For $w \in St(n,k)$, let $c(w^{-1}) = (c_1, \ldots, c_n)$; and denote descents positions by $a_1, \ldots, a_k$. For $1 \le i \le k$, define
$\lambda^i = (n - a_i)^{a_i} - (\underbrace{0, \cdots, 0}_{a_{i-1}}, c_{a_{i-1}+1}, c_{a_{i-1}+2}, \ldots, c_{a_i})$.

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For $w \in St(n,k)$ we have

$$\psi_w(z_1, \cdots, z_n) = \text{"Trivial Factor"} \times \prod_{i=1}^{k} \mathfrak{S}^n_{\lambda_i}(\sigma^{t_i}(z); x; y)$$

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$\rightarrow$ As a corollary, we have that ψ_w for $w \in St(n,k)$ is given as a trvial factor times product of k Schubert polynomials.

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Proof idea

- We first prove the theorem for $St(n, 1)$. (Inverse of Grassmannian permutations that start with 1)

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- Use factorization theorem by Cantini
  $\rightarrow$ Suppose that w splits as $w^{(1)}w^{(2)}\cdots w^{(k)}$ such that every component in $w^{(i)}$ is bigger than $w^{(j)}$ if $i > j$.
  Then $\psi_w(z) = \bar{\psi}_{w^{(1)}}(z)\bar{\psi}_{w^{(2)}}(z)\cdots\bar{\psi}_{w^{(k)}}(z)$

Future directions

- Give a combinatorial formula for $\mathfrak{S}^n_\lambda(z; x; y)$. Leading coefficient is a double Schubert polynomial for a vexillary permutation (2143-avoiding) $\to$ flagged semistandard young tableau

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- Define *z*-Schubert polynomials in a more general setting. And understand $\psi_w(\mathrm{z})$ for *w* other than evil-avoiding permutations (as a sum of *z*-Schubert polynomials). I have checked that this is possible for some states whose probability is a sum of two Schubert polynomials.

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- Geometric interpretation?

Thanks for your attention!

Extended abstract is available (https://arxiv.org/abs/2102.00560)

- C. Arita and K. Mallick, Matrix product solution of an inhomogeneous multi-species TASEP, Journal of Physics A: Mathematical and Theoretical 46 (2013), no. 8, 085002.
- L. Cantini, Inhomogenous multispecies TASEP on a ring with spectral parameters, arXiv:1602.07921.
- T. Lam and L. Williams, A Markov chain on the symmetric group that is Schubert positive?, Experimental Mathematics 21 (2012), no. 2, 189-192.