Interacting particle systems and random walks on Hecke algebras

Alexey Bufetov

University of Bonn

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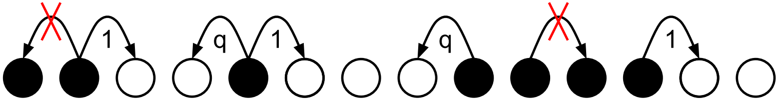
Definition of ASEP



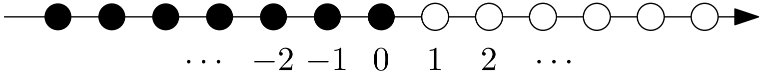
Collection of particles on $\mathbb{Z}$ which evolves in time.

There are two Poisson processes of rates 1 and $q < 1$ associated with each particle.

Each particle jumps one step to the right with rate 1, and jumps one step to the left with rate q, if the neighboring positions are vacant. If the position is occupied by another particle, the jump does not happen.

All Poisson processes are independent.

Step initial condition

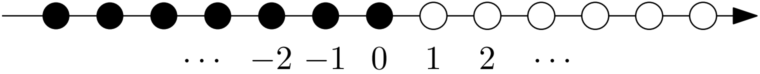


Highly non-stationary initial condition.

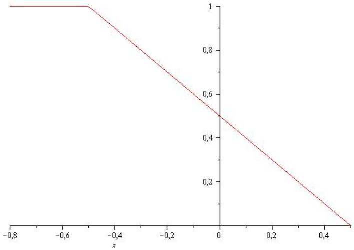
Asymptotic behavior in time ?

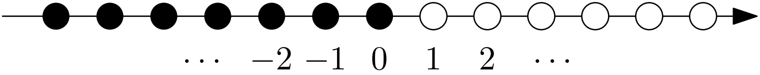
This type of questions: Harris, Liggett, Rost, . . . , . . . , . . .

Step initial condition



Evolution in time ? Density :



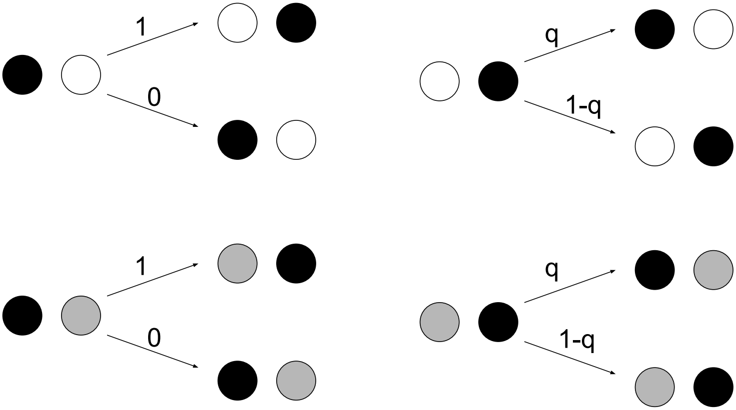


Theorem (Andjel-Vares, Benassi-Fouque, 87)

Let $m = m(t)$, $t \in \mathbb{R}_{\geq 0}$, *be a collection of integers such that* $\lim_{t\to\infty} \frac{m(t)}{t} = y, \qquad y \in \mathbb{R}$. *Then*

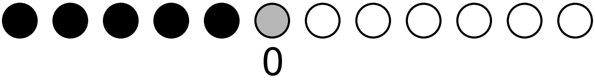
$$\lim_{t\to\infty} P(\eta_t^{asep}(m(t)) = 1) = d(y) := \begin{cases} 0, & y \geq (1-q), \\ \frac{1}{2}\left(1 - \frac{y}{1-q}\right), & -(1-q) < y < (1-q), \\ 1, & y \leq -(1-q). \end{cases}$$

Moreover, for any fixed $L \in \mathbb{Z}_{>0}$ *the random variables* $\{\eta_t^{asep}(m(t)+i)\}_{i=-L,..,L}$ *converge, as* $t \to \infty$, *to i.i.d. Bernoulli distributions with probability of 1 equal to* $d(y)$.



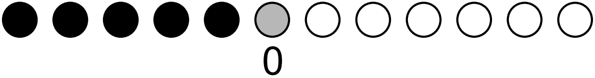
We consider particles of various types (=classes, colors, species).

Set of types is linearly ordered, and a particle of a smaller type interacts with a particle of a larger type as a particle with a hole.



Let us start with this initial condition. Let $S_1(t)$ be the position of the second class particle at time t.

Asymptotics of $S_1(t)$?

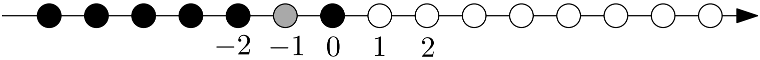


Let us start with this initial condition. Let $S_1(t)$ be the position of the second class particle at time t.

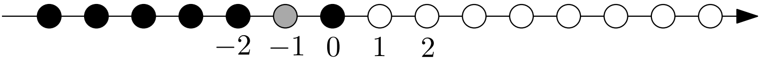
$$\lim_{t\to\infty} \mathrm{Prob}\left(\frac{S_1(t)}{t} < x\right) = d(-x) = \frac{1}{2}\left(1 + \frac{x}{1-q}\right).$$

Uniform distribution on $[-(1-q);(1-q)]$.

P.A. Ferrari-Kipnis'95, P.A. Ferrari-Goncalves-Martin'08.



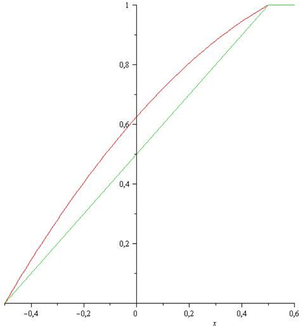
The asymptotic distribution of the second class particle ?

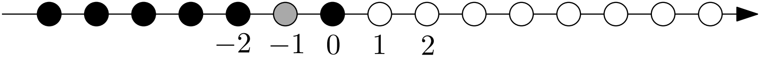


(Borodin-Bufetov'19) The asymptotic distribution of the second class particle

$$\lim_{t\to\infty} \mathrm{Prob}\left(\frac{S_1(t)}{t} < x\right) = d(-x) + (1-q)d(-x)(1-d(-x)).$$

Note the nontrivial dependence on q.





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- $q = 0$: TASEP, Cator-Pimentel'13
- for a class of initial configurations and general q: Borodin-Bufetov'19

Hecke algebra

$W = S_n$, $s_i = (i, i+1)$.

$L(w) :=$ number of inversions in $w \in W$.

Hecke algebra: $\{T_w\}_{w \in W}$ — linear basis

$$\begin{cases} T_s T_w = T_{sw}, & \text{if } L(sw) = L(w) + 1 \\ T_s T_w = (1-q) T_w + q T_{sw}, & \text{if } L(sw) = L(w) - 1. \end{cases}$$

The linear map $I : \mathcal{H} \to \mathcal{H}$

$$I : \sum_w a_w T_w \to \sum_w a_w T_{w^{-1}}$$

satisfies

$$I(h_r h_{r-1} \dots h_2 h_1) = I(h_1) I(h_2) \dots I(h_r), \qquad h_i \in \mathcal{H}.$$

Random walk on Hecke algebra

Generators $\{G_1, \ldots, G_k\}$, each of these generators has an independent exponential clock. When the clock s rings, we multiply G_s to the current position of the random walk $P \in \mathcal{H}$ — our new position is $G_s P$. This is a *random walk on Hecke algebra*.

An element of Hecke algebra

$$h := \sum_w \kappa_w T_w, \qquad \kappa_w \geq 0, \;\; \sum_w \kappa_w = 1,$$

can be interpreted as a random element of W. Random walk on Hecke algebra generates the random walk on W.

Multi-species ASEP / Hecke algebra

$W = S_n$, generators: $\{T_{s_i}\}_{i=1}^{n-1}$. Equivalent language for the description of ASEP: Vocabulary

- Random multi-species configuration — element of Hecke algebra
- Update —- multiplication by T_s
- ASEP evolution — element of S_n generated by random walk on Hecke algebra
- Projection to fewer colors — projection to cosets of parabolic subgroups
- Class-position symmetry — involution I swaps w and w^{-1}.

Other Coxeter groups generate ASEP with a source (hyperoctahedral group), ASEP on a ring (affine Weyl group $\tilde{A}_n$).

How does this help ?

Assume that we want to analyze ASEP which starts from some initial configuration. Lifting it into the multi-species ASEP (in some way), let us say that the initial configuration is given by permutation w. Then we need to study $W(t)T_w$, where $W(t)$ is a random walk which started from identity.

Crucial idea: One can study $I\left(T_{w^{-1}}W(t)\right)$ instead — it has exactly the same distribution ! The benefit is that the continuous time process starts from identity (which leads to step initial condition under projection to fewer types). One needs to analyze the multiplication by $T_{w^{-1}}$ afterwards though....

For general w this is arguably very hard. However, for certain special choices this is quite accessible !

Multi-species ASEP / Hecke algebra

$W = S_n$, generators: $\{T_{s_i}\}_{i=1}^{n-1}$. Equivalent language for the description of ASEP.

- Multi-species ASEP is generated by Hecke algebra: Alcaraz-Rittenberg'93, Alcaraz-Droz-Henkel-Rittenberg'93, ..., Lam'11 , Cantini-de Gier-Wheeler'15, ...
- Class-position symmetry and applications for asymptotic analysis: Angel-Holroyd-Romik'08 (TASEP, $q = 0$), Amir-Angel-Valko'08 (ASEP), Borodin-Bufetov'19 (inhomogeneous stochastic six vertex model).
  Explanation through Hecke algebra: Bufetov'20, Galashin'20; a closely related proof Kuan'20.

What happens if we consider other generators of the random walk on Hecke algebra ?

Mallows measure on S_n

S_n — symmetric group, $L(w)$ — number of inversions in w, and $0 \le q < 1$.

$$\mathrm{Prob}\,(w) = q^{n(n-1)/2 - L(w)} Z.$$

For $q = 0$ this measure is concentrated on one word (longest element), for general q it is "not far" from it for large n.

If we run multi-species ASEP on a finite interval of length n for a long time, it converges to this measure.

Mallows'53

$n \to \infty$: Gnedin-Olshanski'09, Gnedin-Olshanski'11

Other sets of generators also lead to interesting particle systems.

$[a; b] := \{j \in \mathbb{Z} : a \leq j \leq b\}$ the interval between a and b. $S_{a;b} \subset S_n$ permutes the elements from $[a; b]$ only.

Mallows element

$$\mathcal{M}_{a;b} := \sum_{w \in S_{a;b}} Zq^{(b-a+1)(b-a)/2-L(w)} T_w, \qquad \mathcal{M}_{a;b} \in \mathcal{H}(S_n),$$

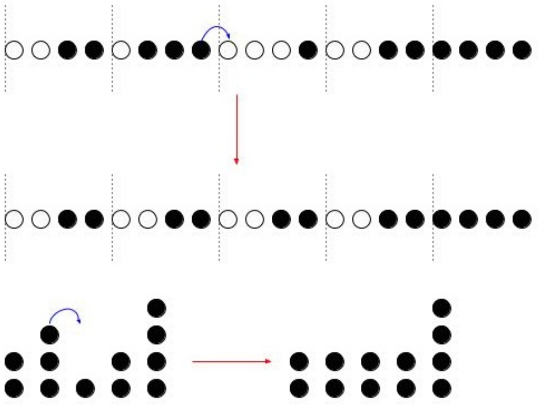
where $L(w)$ is the number of inversions in w. The main property of the element $\mathcal{M}_{a;b}$ is

$$T_w \mathcal{M}_{a;b} = \mathcal{M}_{a;b} T_w = \mathcal{M}_{a;b}, \qquad \text{for any } w \in S_{a;b}.$$

Let $n = NM$, with $M, N \in \mathbb{Z}_{>0}$, and consider the following set of generators of a random walk on the Hecke algebra :

$$\left\{\mathcal{M}_{(x-1)M+1;xM}\mathcal{M}_{xM+1;(x+1)M}\,T_{(xM,xM+1)}\mathcal{M}_{(x-1)M+1;xM}\mathcal{M}_{xM+1;(x+1)M}\right\}_{x=1}^{N-1}.$$

This dynamics generates a multi-species $ASEP(q, M)$.
$q = 0$: M-exclusion TASEP.



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This dynamics generates a multi-species $ASEP(q, M)$.

- Construction is related to the notion of fusion: Kulish-Reshetikhin-Sklyanin'81, Corwin-Petrov'15.
- Single species version of ASEP(q,M) was introduced by Carinci-Giardina-Redig-Sasamoto'15
- Multi-species version of ASEP(q,M) was introduced by Kuan'16
- $M \to \infty$: q-TAZRP (single species version introduced by Sasamoto-Wadati'98).

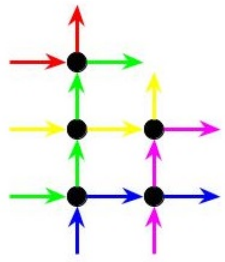
Instead of just $T_{(xM,xM+1)}$ we can have arbitrary interaction between two blocks. This leads to a variety of processes and possible interactions, and one obtains multi-species versions of all these processes. In particular, in $M \to \infty$ limit one recovers the models of Povolotsky'13.

One can consider other random walks on Hecke algebras. In particular, a deterministic random walk on Hecke algebra can also lead to an interesting stochastic process on the Coxeter group.

Stochastic six vertex model

$Y_{s,x} := xT_s + (1-x)T_e$, where e is the identity.

$Y_{(3,4),x}Y_{(4,5),x}Y_{(1,2),x}Y_{(2,3),x}Y_{(3,4),x}$



All the models obtained from random walks on Hecke algebras satisfy class-position symmetry.

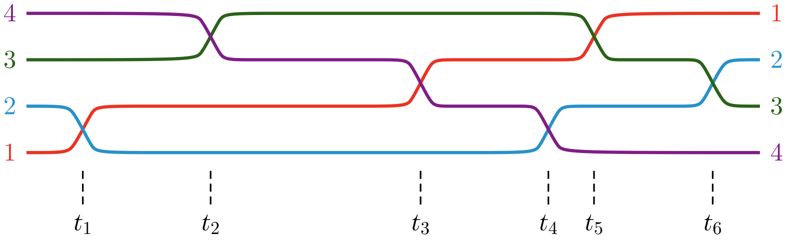
Class-position symmetry was used for asymptotic probabilistic applications in

- Angel-Holroyd-Romik-08: The study of trajectories of particles in multispecies TASEP on an interval.
- Amir-Angel-Valko'08: Joint distribution of various particles started with step initial condition in multispecies TASEP.
- Borodin-Bufetov'19: second class particle in multispecies ASEP with deformed initial condition. Bufetov-P. L. Ferrari'20: second class particle in the TASEP shock under a variety of scalings.
- Bufetov'20: Second class particle in multispecies q-TAZRP with deformed initial conditions. Second-class particle in ASEP with a source and deformed initial condition (comes from BC-Hecke algebra).

The results about limit behavior of second class particles continue the line of research from P. A. Ferrari-Kipnis'95, P. A. Ferrari-Goncalves-Martin'08 (results about limit behavior of second class particle started from a particular initial condition, ASEP), Cator-Pimentel'13 (second class particle started from arbitrary initial condition, TASEP).

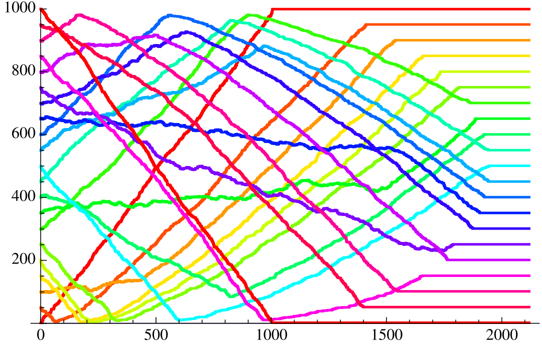
Multispecies TASEP on an interval

- Interval $\{1, 2, \ldots, N\}$. Symmetric group S_N.
- Each transposition $(i, i+1)$ has independent exponential clock.
- When the clock rings, we swap particles at i and $i+1$, but only **if** it will **increase** the number of color-position inversions.



Angel-Holroyd-Romik-08: What's happening as N becomes large?

TASEP on an interval



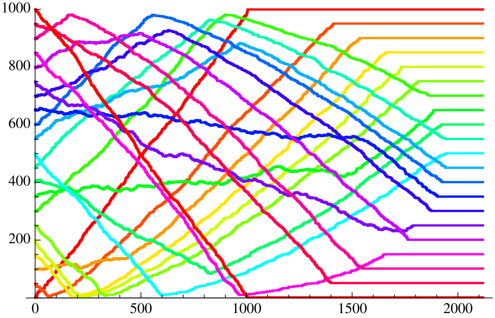
Picture from Angel-Holroyd-Romik-08.
Only 21 out of 1000 trajectories shown.

Theorem. [Angel-Holroyd-Romik] Set $\gamma_y = 1 + 2\sqrt{y(1-y)}$.
If $U_N(k)$ is the **last time the swap** $(k, k+1)$ **happens**, then

$$\frac{U_N(k) - N\gamma_{k/N}}{N^{1/3}(\gamma_{k/N})^{2/3}\left(\frac{k}{N}(1-\frac{k}{N})\right)^{-1/6}} \xrightarrow[N\to\infty]{d} F_2, \qquad \text{(Tracy-Widom distribution)}$$

Proof is based on coupling with TASEP or exponential LPP.

TASEP on an interval



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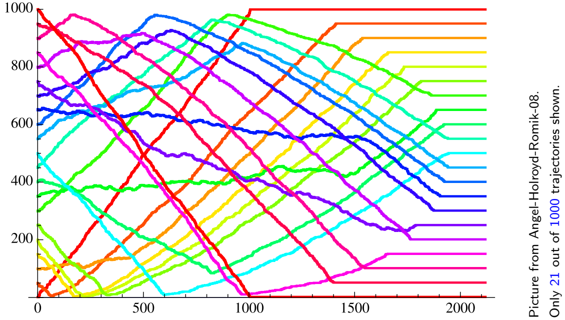
Question. Set T_N^{OSP} — the time when the systems **stops**

[AHR-08]: We have $T_N^{\mathrm{OSP}} \approx 2N$. What are **the fluctuations**?

TASEP on an interval



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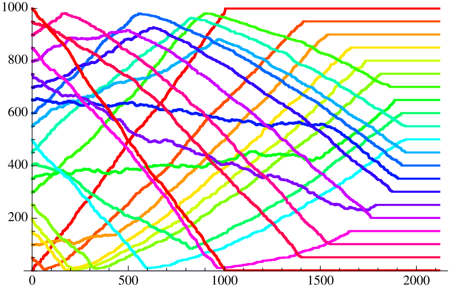
Theorem. Bufetov-Gorin-Romik'20

$$\frac{T_N^{\text{OSP}} - 2N}{2^{1/3}\, N^{1/3}} \xrightarrow[N\to\infty]{d} F_1,$$



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Question. Set T_N^{OSP} — the time when the systems **stops**

Theorem. (Bufetov-Gorin-Romik-20)

$$\frac{T_N^{\mathrm{OSP}} - 2N}{2^{1/3}\, N^{1/3}} \xrightarrow[N\to\infty]{d} F_1,$$

Proof is based on symmetries of interacting particle species Borodin-Gorin-Wheeler'19, Galashin'20, and conjectures of Bisi-Cunden-Gibbons-Romik'20.