Geometry of polymatroids

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Polymatroids

Definition (polymatroid)

A *polymatroid* P on $[k]$ of type $(a_1, \ldots, a_k)$ is a function $\mathrm{rk}_P \colon 2^{[k]} \to \mathbb{Z}$ satisfying:

- $\mathrm{rk}_P(\emptyset) = 0$,
- If $A_1 \subseteq A_2$, then $\mathrm{rk}_P(A_1) \leq \mathrm{rk}_P(A_2)$,
- $\mathrm{rk}_P(\{i\}) \leq a_i$, and
- for all A_1, A_2, $\mathrm{rk}_P(A_1) + \mathrm{rk}_P(A_2) \geq \mathrm{rk}_P(A_1 \cap A_2) + \mathrm{rk}_P(A_1 \cup A_2)$.

- A polymatroid of type $(1, \ldots, 1)$ is a matroid.

Realizable polymatroids

- A collection of subspaces $S_1, \dots, S_k$ inside a vector space V defines a polymatroid of type $(\mathrm{codim}(S_1), \dots, \mathrm{codim}(S_k))$ by

$$\mathrm{rk}_P(T) = \mathrm{codim}(\cap_{i \in T} S_i).$$

- A *flat* of P is a subset $F \subseteq [k]$ such that $\mathrm{rk}_P(F \cup a) > \mathrm{rk}_P(F)$ for all $a \notin F$.
- Flats of P are in bijection with subspaces of V which arise as intersections of the $\{S_i\}$.

Lifting to hyperplane arrangement

- Every subspace arrangement is some of the flats in a hyperplane arrangement: for each S_i, choose some hyperplanes whose intersection is S_i.
- A matroid M together with a partition of the ground set into blocks $B_1, \dots, B_k$ of size $a_1, \dots, a_k$ defines a polymatroid P of type $(a_1, \dots, a_k)$ via

$$\mathrm{rk}_P(S) = \mathrm{rk}_M\Big(\bigcup_{i \in S} B_i\Big).$$

Multisymmetric lifts

- For a subspace arrangement $S_1, \dots, S_k$, we can choose a_i hyperplanes containing S_i as generically as possible for some $a_i \geq \operatorname{codim}(S_i)$.
- This produces a matroid M on a ground set E of size $a_1 + \cdots + a_k$. There is a natural map $\pi \colon E \to [k]$. We have

$$\operatorname{rk}_M(\pi^{-1}(T)) = \operatorname{codim}(\cap_{i \in T} S_i) = \operatorname{rk}_P(T).$$

- The group $\Gamma = S_{a_1} \times \cdots \times S_{a_k}$ acts on E preserving rk_M.

Multisymmetric lifts

Theorem (Crowley–Huh–L.–Simpson–Wang)

Let P be a polymatroid of type $(a_1, \dots, a_k)$, and let $\pi \colon E \to [k]$ be a map with $|\pi^{-1}(i)| = a_i$. There is a unique matroid M on E such that $\Gamma = S_{a_1} \times \cdots \times S_{a_k}$ acts on E preserving rk_M, and

$$\mathrm{rk}_M(\pi^{-1}(T)) = \mathrm{rk}_P(T).$$

- Construction has appeared many times in the literature, beginning with Helgason in 1972.
- The Γ-fixed flats of M are exactly the inverse images of flats of P.
- Compatible with direct sum, deletion, contraction, ...

Definition (Independence polytope)

The independence polytope $I(P)$ of a polymatroid of type $(a_1, \ldots, a_k)$ is

$$I(P) := \{x \in \mathbb{R}^k_{\geq 0} \colon \sum_{i \in S} x_i \leq \mathrm{rk}_P(S) \text{ for all } S\}.$$

- $I(P)$ is contained in $[0, a_1] \times \cdots \times [0, a_k]$.
- Let

$$p_\pi \colon \mathbb{R}^E \to \mathbb{R}^k, \quad p_\pi(x) = \left(\sum_{j \in \pi^{-1}(i)} x_j \right).$$

  Then

$$I(M) = p_\pi^{-1}(I(P)) \cap [0, 1]^{a_1 + \cdots + a_k}.$$

Wonderful compactifications of subspace arrangements

- Given a subspace arrangement in V with $\cap S_i = 0$ and all subspaces proper, the De Concini–Procesi *wonderful compactification* is the variety obtained from $\mathbb{P}V$ by blowing up the strict transforms of the proper non-empty flats, in increasing order of dimension.
- This is a smooth projective variety which is a simple normal crossings compactification of $\mathbb{P}V \setminus \cup \mathbb{P}S_i$.
- Studying wonderful compactifications of hyperplane arrangements has been very fruitful.

Example

If $k = 1$, then the wonderful compactification is $\mathbb{P}V$.

Cohomology ring of the wonderful variety

Theorem (De Concini–Procesi)

The cohomology ring of the wonderful compactification of a subspace arrangement realizing a matroid P is

$$\frac{\mathbb{Z}[x_F, y_i]_{F \text{ proper non-empty flat, } i \in [k]}}{\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3}, \text{ where}$$

$$\mathcal{I}_1 = \langle x_{F_1} x_{F_2} : F_1, F_2 \text{ incomparable} \rangle,$$

$$\mathcal{I}_2 = \langle x_F \prod_{i \in T} y_i^{u_i} : \mathrm{rk}_P(F \cup T) \leq \mathrm{rk}_P(F) + \sum u_i, F \text{ flat}, \ F \cap T = \emptyset \rangle,$$

$$\mathcal{I}_3 = \langle y_i + \sum_{F \ni i} x_F - y_j + \sum_{G \ni j} x_G \rangle,$$

- This only depends on P. We define $A^\bullet(P)$ to be the graded ring given by this presentation for any (loopless) polymatroid.

From hyperplane arrangements to subspace arrangements

Example (Hall–Rado polymatroid)

Given a hyperplane arrangement, we can form the subspace arrangement that consists of all flats of the hyperplane arrangement. This produces a polymatroid whose ground set is the set of proper non-empty flats of the hyperplane arrangement.

The wonderful variety of this subspace arrangement is the wonderful variety of the hyperplane arrangement, and so the Chow ring of the polymatroid coincides with the Chow ring of the matroid.

- When P is realizable, this ring satisfies the Kähler package: Poincaré duality, Hard Lefschetz, and the Hodge–Riemann relations.

Theorem (Pagaria–Pezzoli)

For any (loopless) P, $A^\bullet(P)$ satisfies the Kähler package.

- Their proof is difficult.

The Bergman fan of a polymatroid

- Let P be a (loopless) polymatroid of type $(a_1, \ldots, a_k)$, and let M be the multisymmetric lift on ground set E.
- The Bergman fan of P is a $(\mathrm{rk}(P) - 1)$-dimensional (quasiprojective) unimodular fan in $\mathbb{R}^E/\mathbb{R} \cdot (1, \ldots, 1)$.
- The Bergman fan of P is a coarsening of the Bergman fan of M.
- The Chow ring of the toric variety associated to the Bergman fan of P is $A^\bullet(P)$.
- The Bergman fan of P is *not* the tropicalization of the complement of the subspace arrangement complement when P is realizable, which typically does not embed into a torus.

Kähler package for $A^\bullet(P)$

Theorem (Ardila–Denham–Huh)

Whether the Chow ring of a unimodular (quasiprojective) fan satisfies the Kähler package depends only on the support of the fan.

Theorem (Adiprasito–Huh–Katz)

The Chow ring of the Bergman fan of a matroid satisfies the Kähler package.

Corollary (Crowley–Huh–L.–Simpson–Wang)

For any (loopless) P, $A^\bullet(P)$ satisfies the Kähler package.

The polypermutohedral variety

- The Bergman fan of every polymatroid of type $(a_1, \ldots, a_k)$ lives inside the *polypermutohedral fan*, which is the Bergman fan of the boolean polymatroid of type $(a_1, \ldots, a_k)$ (which has $\mathrm{rk}(T) = \sum_{i \in T} a_i$).
- The *polypermutohedral variety* $X_{a_1,\ldots,a_k}$ is the smooth projective projective toric variety associated to the polypermutohedral fan. It is the wonderful compactification of the boolean arrangement.
- The wonderful compactification of any subspace arrangement whose polymatroid has type $(a_1, \ldots, a_k)$ embeds into $X_{a_1,\ldots,a_k}$.
- The group $GL_{a_1} \times \cdots \times GL_{a_k}$ acts on $X_{a_1,\ldots,a_k}$, and in particular the subgroup $S_{a_1} \times \cdots \times S_{a_k}$.
- The polypermutohedral fan is the normal fan of $p_\pi^{-1}(Q) \cap \mathbb{R}^E_{\geq 0}$, where $Q = \mathrm{Conv}\{(\sigma(1), \ldots, \sigma(k)) \colon \sigma \in S_k\}$ is the usual permutohedron.

Bergman class of a polymatroid

- Each realization of a polymatroid P of type $(a_1, \ldots, a_k)$ defines a homology class on $X_{a_1,\ldots,a_k}$. This class depends only on P.
- We can extend this to define a homology class $[\Sigma_P] \in H_{\mathrm{rk}(P)-1}(X_{a_1,\ldots,a_k})$ for any (loopless) P.
- The Bergman fan of P is balanced when every maximal cone is given the constant weight 1, and so defined a Minkowski weight.
- We have $A^\bullet(P) = A^\bullet(X_{a_1,\ldots,a_k})/\operatorname{ann}([\Sigma_P])$.

Valuative group of polymatroids

Definition (Valuative group of polymatroids of type $(a_1, \ldots, a_k)$)

$\mathrm{Val}^{\bullet}_{a_1,\ldots,a_k}$ is the subgroup of functions on $\mathbb{R}^k$ generated by indicator functions of independence polytopes of polymatroids of type $(a_1, \ldots, a_k)$.

- $\mathrm{Val}^{\bullet}_{a_1,\ldots,a_k}$ is graded by rank.
- Many functions of polymatroids factor through $\mathrm{Val}^{\bullet}_{a_1,\ldots,a_k}$.

Valuative group of polymatroids

Theorem (L.–Eur)

The map $P \mapsto [\Sigma_P]$ induces an isomorphism from the subgroup of $\mathrm{Val}^r_{a_1,\dots,a_k}$ generated by loopless polymatroids to $H_{r-1}(X_{a_1,\dots,a_k})$.

- In the matroid case, this was proven by Hampe.
- Feichtner–Yuzvinsky give a basis for $A^\bullet(X_{a_1,\dots,a_k})$, which can be used to give a basis for $\mathrm{Val}^r_{a_1,\dots,a_k}$.

Corollary

The Bergman class of a matroid M on E lies in the image of the pullback map from $A^\bullet(X_{a_1,\dots,a_k})$ if and only if M is multisymmetric.

The polystellahedral variety

- In (L.–Eur), we introduce a $(a_1 + \ldots + a_k)$-dimensional smooth projective toric variety $\tilde{X}_{a_1,\ldots,a_k}$, called the polystellahedral variety, which contains $X_{a_1,\ldots,a_k}$ as a divisor.
- For any polymatroid P on $[k]$, the polytope $p_\pi^{-1}(I(P)) \cap \mathbb{R}_{\geq 0}^{a_1+\ldots+a_k}$ defines a line bundle on $\tilde{X}_{a_1,\ldots,a_k}$.
- There is a class $[\tilde{\Sigma}_P] \in H_{\mathrm{rk}(P)}(\tilde{X}_{a_1,\ldots,a_k})$, the augmented Bergman class, whose restriction to the divisor $X_{a_1,\ldots,a_k}$ is $[\Sigma_P]$.

Theorem (L.–Eur)

The map $P \mapsto [\tilde{\Sigma}_P]$ induces an isomorphism from $\mathrm{Val}^r_{a_1,\ldots,a_k}$ to $H_r(\tilde{X}_{a_1,\ldots,a_k})$.

- In the matroid case, this was proven in Eur–Huh–L.

Proof idea

- $K(\tilde{X}_{a_1,\ldots,a_k})$ has a polyhedral description in terms of translates of $p_\pi^{-1}(I(P)) \cap \mathbb{R}_{\geq 0}^{a_1+\cdots+a_k}$ for *any* polymatroid on $[k]$.
- Construct an "exceptional isomorphism" $\phi\colon K(\tilde{X}_{a_1,\ldots,a_k}) \to H^\bullet(\tilde{X}_{a_1,\ldots,a_k})$ which has $\phi([p_\pi^{-1}(I(P^\perp)) \cap \mathbb{R}_{\geq 0}^{a_1+\cdots+a_k}]) = [\tilde{\Sigma}_P] +$ lower order terms, which gives well-definedness.
- For surjectivity, use that $H^\bullet(\tilde{X}_{a_1,\ldots,a_k})$ is generated in degree 2 and that products of augmented Bergman classes of polymatroids are augmented Bergman classes.
- For injectivity, need show that there are no relations among the $p_\pi^{-1}(I(P)) \cap \mathbb{R}_{\geq 0}^{a_1+\cdots+a_k}$ for polymatroids of type $(a_1, \ldots, a_k)$ beyond those coming from valuativity.
- There is birational map $\tilde{X}_{1,\ldots,1} \to \tilde{X}_{a_1,\ldots,a_k}$. Pulling back reduces to showing that there are no relations among the augmented Bergman classes of the multisymmetric lifts, which is the matroid case of the theorem.

Missing ingredient: tautological bundles of polymatroids

- In Berget–Eur–Spink–Tseng and Eur–Huh–L., vector bundles associated to matroids are constructed, and computations involving these vector bundles recover many other algebro-geometric computations related to matroids.
- Results in Eur–L. indicate a candidate for the Chern classes of tautological bundles of polymatroids, but it's unclear how to compute anything.
- Ehrhart-theoretic results in Cameron–Fink indicate that intersection numbers involving tautological classes of polymatroids should be related to bases activities of polymatroids, in the sense of Kálmán.

Thank you!

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