THE LOCAL MOTIVIC MONODROMY CONJECTURE FOR SIMPLICIAL NONDEGENERATE SINGULARITIES

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ABSTRACT. We prove the local motivic monodromy conjecture for singularities that are nondegenerate with respect to a simplicial Newton polyhedron. It follows that all poles of the local topological zeta functions of such singularities correspond to eigenvalues of monodromy acting on the cohomology of the Milnor fiber of some nearby point, as do the poles of Igusa's local p-adic zeta functions for large primes p.

CONTENTS

1. INTRODUCTION

Throughout, let $\Bbbk$ be a field of characteristic 0, and let $f \in \Bbbk[x_1, \ldots, x_n]$ be a regular function whose vanishing locus X_f contains $0 \in \mathbb{A}^n$. The coefficients of f are contained in a finitely generated subfield $\Bbbk' \subset \Bbbk$, so we may choose an embedding $\Bbbk' \subset \mathbb{C}$, view f as a holomorphic function on $\mathbb{C}^n$, and consider the Milnor fiber $\mathcal{F}_x$, with its monodromy action, for any geometric point $x \in X_f$. The characteristic polynomial of the induced action on $H^*(\mathcal{F}_x, \mathbb{C})$ is independent of all choices and its zeros are the *eigenvalues of monodromy* of f at x. The monodromy is quasi-unipotent, so all such eigenvalues of monodromy are roots of unity. We say that $\exp(2\pi i \alpha)$ is a *nearby eigenvalue of monodromy* of f if 0 lies in the Zariski closure of the locus of points $x \in X_f$ such that $\exp(2\pi i\alpha)$ is an eigenvalue of monodromy of f at x.

The local motivic zeta function is a subtle invariant of the singularity of f at 0, introduced by Denef and Loeser [DL98]. Let $K^{\hat{\mu}}$ be the Grothendieck ring of $\Bbbk$-varieties with good $\hat{\mu}$-action, where $\hat{\mu} = \varprojlim \mu_m$ is the inverse limit of the groups of mth roots of unity, and let $\mathcal{M}^{\hat{\mu}} := K^{\hat{\mu}}[\mathbb{L}^{-1}]$ be the associated motivic ring obtained by inverting $\mathbb{L} := [\mathbb{A}^1]$. Then the local motivic zeta function $Z_{\mathrm{mot}}(T) \in \mathcal{M}^{\hat{\mu}}[\![T]\!]$ is expressible non-uniquely as the formal power series expansion of a rational function in $\mathcal{M}^{\hat{\mu}}\big[T, \frac{1}{1-\mathbb{L}^a T^b}\big]_{(a,b)\in\mathbb{Z}\times\mathbb{Z}_{>0}, a/b\in\mathcal{P}}$, for some finite $\mathcal{P} \subset \mathbb{Q}$. Any such $\mathcal{P}$ is a *set of candidate poles* for $Z_{\mathrm{mot}}(T)$, as defined in [BV16, BN20].

Local Motivic Monodromy Conjecture. *There is a set of candidate poles $\mathcal{P} \subset \mathbb{Q}$ for $Z_{\mathrm{mot}}(T)$ such that, for every $\alpha \in \mathcal{P}$, $\exp(2\pi i\alpha)$ is a nearby eigenvalue of monodromy.*

Note that the notion of poles is subtle in this context because $K^{\hat{\mu}}$ is not an integral domain; in particular, it is unclear whether the intersection of two sets of candidate poles for $Z_{\mathrm{mot}}(T)$ is necessarily a set of candidate

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poles. Our main result (Theorem 1.1.1) confirms the local motivic monodromy conjecture for singularities that are nondegenerate with respect to a simplicial Newton polyhedron.

1.1. **Statement of main results.** For $u = (u_1, \dots, u_n)$ in $\mathbb{Z}^n_{\geq 0}$, let $x^u := x_1^{u_1} \cdots x_n^{u_n}$, and write $f = \sum_u a_u x^u$. The Newton polyhedron of f is the Minkowski sum $\mathrm{Newt}(f) := \mathrm{conv}\{u : a_u \neq 0\} + \mathbb{R}^n_{\geq 0}$. For each face F of $\mathrm{Newt}(f)$, we consider $f|_F := \sum_{u \in F} a_u x^u$. Then f is *nondegenerate* if, for all compact faces F, the vanishing locus of $f|_F$ has no singularities in the complement of the coordinate hyperplanes in $\mathbb{A}^n$.

For any face F of $\partial\,\mathrm{Newt}(f)$ that meets the interior of the orthant $\mathbb{R}^n_{>0}$, let $C_F := \overline{\mathbb{R}_{\geq 0} F}$ be the closure of the cone spanned by F. The set of all faces of such cones forms a fan Δ whose support is the positive orthant $\mathbb{R}^n_{\geq 0}$. We say that $\mathrm{Newt}(f)$ is *simplicial* if Δ is a simplicial fan.

Theorem 1.1.1. *Suppose that* $\mathrm{Newt}(f)$ *is simplicial and* f *is nondegenerate. Then there is a set of candidate poles* $\mathcal{P} \subset \mathbb{Q}$ *for* $Z_{\mathrm{mot}}(T)$ *such that, for every* $\alpha \in \mathcal{P}$, $\exp(2\pi i \alpha)$ *is a nearby eigenvalue of monodromy.*

In other words, the local motivic monodromy conjecture is true for any nondegenerate singularity with a simplicial Newton polyhedron. This was known previously for $n = 2$ [BN20]. Our definition of simplicial Newton polyhedron agrees with that in [JKYS19]. A *convenient* Newton polyhedron, i.e., one that intersects each of the coordinate axes [Kou76], is simplicial if and only if each of its compact faces is a simplex.

Remark 1.1.2. The methods used in the proof of Theorem 1.1.1 are discussed in Section 1.4. Roughly speaking, we have one collection of arguments, presented in Sections 3-4 that prove existence of eigenvalues corresponding to candidate poles associated to many facets of $\mathrm{Newt}(f)$. Another collection of arguments, presented in Section 5, shows that certain such candidate poles are *fake* and can be removed to give a smaller set of candidate poles. The assumption that the Newton polyhedron is simplicial is used in our arguments that produce eigenvalues of monodromy. Our arguments for reducing the size of the set of candidate poles is presented in somewhat greater generality. Variations on our eigenvalue arguments also yield a proof of the local motivic monodromy conjecture for many nondegenerate singularities with non-simplicial Newton polyhedra, including all cases where $n = 3$. The details are combinatorially involved and will appear in a follow-up paper. All of the critical new ideas appear already in the proof of the simplicial case that we present here, on its own, in an effort to balance generality with clarity.

The local motivic monodromy conjecture is a motivic analogue of the local p-adic and topological monodromy conjectures, and the following cases of the latter conjectures are consequences of Theorem 1.1.1.

The local motivic zeta function specializes to the local topological zeta function $Z_{\mathrm{top}}(s) \in \mathbb{Q}(s)$ by expanding $Z_{\mathrm{mot}}(T)$ as a power series in $\mathbb{L} - 1$ and then setting $T \mapsto \mathbb{L}^{-s}$ and $[Y] \mapsto \chi(Y/\hat{\mu})$ [DL98, §2.3]. It follows that the poles of $Z_{\mathrm{top}}(s)$ are contained in every set of candidate poles for $Z_{\mathrm{mot}}(T)$.

Theorem 1.1.3. *Suppose* $\mathrm{Newt}(f)$ *is simplicial and* f *is nondegenerate. If* α *is a pole of* $Z_{\mathrm{top}}(s)$ *then* $\exp(2\pi i \alpha)$ *is a nearby eigenvalue of monodromy*

This confirms the local topological monodromy conjecture [DL92, Conjecture 3.3.2] for singularities that are nondegenerate with respect to a simplicial Newton polyhedron.

If $f \in \mathbb{Z}_p[x_1, \dots, x_n]$ has good reduction mod p, meaning that $\overline{f} \in \mathbb{F}_p[x_1, \dots, x_n]$ is nondegenerate with $\mathrm{Newt}(\overline{f}) = \mathrm{Newt}(f)$, then $Z_{\mathrm{mot}}(T)$ also specializes to the Igusa local p-adic zeta function $Z_{(p)}(s) \in \mathbb{Q}(p^s)$, which is viewed as a global meromorphic function in the complex variable s. In this case, the real part of any pole of $Z_{(p)}(s)$ is contained in every set of candidate poles for $Z_{\mathrm{mot}}(T)$.

Theorem 1.1.4. *Suppose* $f \in \mathbb{Z}_p[x_1, \dots, x_n]$, $\mathrm{Newt}(f)$ *is simplicial, and* f *is nondegenerate with good reduction mod* p. *If* α *is a pole of* $Z_{(p)}(s)$, *then* $\exp(2\pi i \Re(\alpha))$ *is a nearby eigenvalue of monodromy.*

If $f \in \mathbb{Z}[x_1, \dots, x_n]$ is nondegenerate, then f has good reduction mod p for all but finitely many primes p. In this sense, Theorem 1.1.4 implies that the local p-adic monodromy conjecture holds for nondegenerate singularities with simplicial Newton polyhedra.

1.2. **Background and motivation.** We now discuss the background and motivation for the local monodromy conjectures. In particular, we recall the definitions of the local motivic, p-adic, and topological zeta functions and how they relate to the geometry of embedded log resolutions. We also recall A'Campo's formula for the zeta function of monodromy at 0 for a nondegenerate singularity.

1.2.1. *Archimedean zeta functions.* The motivation for the local monodromy conjectures comes from a theorem of Malgrange concerning the following archimedean analogues of local zeta functions. Suppose $\Bbbk = \mathbb{R}$ or $\mathbb{C}$, and let Φ be a smooth function supported on a compact set that does not contain any critical points of f other than 0. Consider the function

$$Z_\Phi(s) := \int_{\Bbbk^n} \Phi(x)|f(x)|^{\delta s} dx,$$

where $\delta = 1$ if $\Bbbk = \mathbb{R}$ and $\delta = 2$ if $\Bbbk = \mathbb{C}$. This integral converges for $s \in \mathbb{C}$ with $\Re(s) > 0$.

Theorem 1.2.1 ([Mal74])**.** *The function $Z_\Phi(s)$ extends to a meromorphic function on $\mathbb{C}$ whose poles are rational numbers. Moreover, if α is a pole of $Z_\Phi(s)$ then $\exp(2\pi i\alpha)$ is an eigenvalue of monodromy.*

Furthermore, every such eigenvalue of monodromy appears as a pole of $Z_\Phi(s)$ for some Φ.

1.2.2. *Log resolutions and zeta functions of monodromy.* Let $h\colon \mathcal{F}_x \to \mathcal{F}_x$ denote the monodromy action on the Milnor fiber of f at $x \in X_f$. The zeta function of monodromy of f at x is then

$$\zeta_x(t) := \frac{\det\left(1 - th^* \mid H^{\text{even}}(\mathcal{F}_x, \mathbb{C})\right)}{\det\left(1 - th^* \mid H^{\text{odd}}(\mathcal{F}_x, \mathbb{C})\right)}. \tag{1}$$

When f is nondegenerate, the zeta function of monodromy at 0 may be expressed in terms of the numerical data of a log resolution and the topological Euler characteristics of the strata in the fiber, as follows.

Let $\pi\colon Y \to \mathbb{A}^n$ be a proper morphism that is an isomorphism away from X_f, such that the support of $D := \pi^{-1}(X_f)$ is a divisor with simple normal crossings. Let $D_0, \dots, D_r$ be the irreducible components of D, numbered so that D_0 is the strict transform of X_f. The associated *numerical data* of this log resolution are the pairs of integers (N_i, ν_i), where N_i and $\nu_i - 1$ are the orders of vanishing of $\pi^*(f)$ and $\pi^*(dx_1 \wedge \cdots \wedge dx_n)$, respectively, along D_i.

For $I \subset \{0, \dots, r\}$, let $D_I := \bigcap_{i \in I} D_i$ and $D_I^\circ := D_I \smallsetminus \bigcup_{j \notin I} D_{I \cup \{j\}}$. We then define

$$E_I := D_I \cap \pi^{-1}(0) \text{ and } E_I^\circ = D_I^\circ \cap \pi^{-1}(0)$$

for the corresponding closed and locally closed strata in the fiber over 0.

Theorem 1.2.2 ([A'C75])**.** *Suppose f is nondegenerate. Then the zeta function of monodromy acting on the cohomology of $\mathcal{F}_0$ is*

$$\zeta_0(t) = \prod_{i=1}^{r} (1 - t^{N_i})^{\chi(E_i^\circ)}. \tag{2}$$

Note that the exponents $\chi(E_i^\circ)$ can be positive or negative, and there can be a great deal of cancellation in simplifying this rational function expression for $\zeta_0(t)$ down to a quotient of two relatively prime polynomials. In particular, it is difficult to determine from the numerical data of the log resolution whether any given root of $1 - t^{N_i}$ is an eigenvalue of monodromy.

1.2.3. *Igusa's local $\mathfrak{p}$-adic zeta functions.* Let $\mathbb{k}$ be a finite extension of $\mathbb{Q}_p$, equipped with the unique extension of the p-adic valuation and its associated norm. Let $R \subset \mathbb{k}$ be the valuation ring, with $\mathfrak{p} \subset R$ the maximal ideal. For instance, if $f \in \mathbb{Q}_p[x_1, \dots, x_n]$, then $R = \mathbb{Z}_p$ and $\mathfrak{p} = p\mathbb{Z}_p$. Igusa introduced and studied the local zeta function

$$Z_{\mathfrak{p}}(s) := \int_{\mathfrak{p}^n} |f(x)|^s dx,$$

in his proof of a conjecture of Borewicz and Shafarevich [BS66, p. 63] on rationality of generating functions for the number of solutions mod p^m to a polynomial equation with integer coefficients. Here, dx denotes the normalized Haar measure on the compact additive group $\mathfrak{p}^n$. Note that $Z_{\mathfrak{p}}(s)$ is a nonarchimedean analogue of the asymptotic integrals $Z_\Phi(s)$; the role of Φ is played by the indicator function of the compact subset $\mathfrak{p}^n$. Igusa proved that $Z_{\mathfrak{p}}(s)$ is a rational function in q^{-s} [Igu75], where $q = |R/\mathfrak{p}|$, and that its real poles other than -1 are all of the form $\alpha_i := -\nu_i/N_i$, where (N_i, ν_i) is the numerical data associated to some exceptional divisor in a given log resolution [Igu78]. Denef gave a second proof of the rationality of $Z_{\mathfrak{p}}(s)$, using p-adic cell decompositions [Den84].

Remark 1.2.3. Note that some sources in the literature define a local $\mathfrak{p}$-adic zeta function by integrating with respect to the restriction of the normalized Haar measure on $\mathbb{Z}_{\mathfrak{p}}^n$; the result differs from our $Z_{\mathfrak{p}}(s)$ by a factor of q^{-n}. Such renormalizations do not affect the poles of the local zeta functions.

Typically, very few of the rational numbers α_i associated to the numerical data in a log resolution are actually poles of $Z_{\mathfrak{p}}(s)$. In the archimedean setting, this is explained by Malgrange's theorem (Theorem 1.2.1), since many rational numbers that appear in this way do not correspond to eigenvalues of monodromy.

Both Denef [Den85] and Igusa [Igu88] observed that the analogue of Malgrange's theorem seems to hold for $Z_{\mathfrak{p}}(s)$; in all examples that had been computed, whenever α_j is a pole of $Z_{\mathfrak{p}}(s)$, the corresponding root of unity $\exp(2\pi i \alpha_j)$ is an eigenvalue of monodromy. Loeser proved that this is true for $n = 2$ [Loe88], and for certain nondegenerate singularities in higher dimensions [Loe90]. By the early 1990s, the expectation that this nonarchimedean analogue of Malgrange's theorem should hold was known as the monodromy conjecture. See, e.g., [Den91a, Conjecture 4.3], [Den91b, Conjecture 2.3.2], and [Vey93, p. 546–547]. We will follow the usual convention and call this the *local $\mathfrak{p}$-adic monodromy conjecture*, to distinguish it from the topological and motivic variants that followed.

Local $\mathfrak{p}$-adic Monodromy Conjecture. *Suppose $\mathbb{k}$ is a number field. For all but finitely many primes $\mathfrak{p} \subset \mathcal{O}_{\mathbb{k}}$, if α is a pole of $Z_{\mathfrak{p}}(s)$ then $\exp(2\pi i \Re(\alpha))$ is a nearby eigenvalue of monodromy.*

Interest in this conjecture has persisted through the decades [Nic10, VS22]. Bories and Veys proved it for $n = 3$ when f is nondegenerate and of good reduction [BV16, Theorem 0.12]. There has been little progress in higher dimensions.

1.2.4. *Good reduction.* The local $\mathfrak{p}$-adic zeta function has a particularly simple expression when $X_f \subset \mathbb{A}^n$ has an embedded log resolution with good reduction mod $\mathfrak{p}$. Most results on the local $\mathfrak{p}$-adic monodromy conjecture for $n \geq 3$, including those of [BV16] and our Theorem 1.1.4, have a good reduction hypothesis.

Suppose the log resolution $\pi \colon Y \to \mathbb{A}^n$ factors through a closed embedding $Y \hookrightarrow \mathbb{P}^m \times \mathbb{A}^n$ over $\mathbb{k}$. Let $\mathbb{P}^m_R$ and $\mathbb{A}^n_R$ denote the projective and affine spaces of dimension m and n, respectively, over R. Let X_R and Y_R be the closures of X and Y in $\mathbb{A}^n_R$ and $\mathbb{P}^m_R \times \mathbb{A}^n_R$. Then π extends naturally to a projective morphism $\pi_R \colon Y_R \to X_R$. Let $\overline{X}$ and $\overline{Y}$ be the respective special fibers of X_R and Y_R. Base change to $\mathbb{F}_q = R/\mathfrak{p}$ gives a projective morphism $\overline{\pi} \colon \overline{Y} \to \overline{X}$. Let $\overline{D}_i$ be the special fiber of the closure of D_i in Y_R.

Definition 1.2.4. *The resolution $\pi \colon Y \to \mathbb{A}^n$ has* good reduction *mod $\mathfrak{p}$ if*

- *$\overline{Y}$ is smooth in a neighborhood of $\overline{\pi}^{-1}(0)$;*

- $\overline{D}_0, \dots, \overline{D}_r$ *are smooth and distinct over* $\mathbb{F}_q$*, and they meet each other transversely.*

Note that, if f and π are defined over a number field K, then π has good reduction mod $\mathfrak{p}$ for all but finitely many primes $\mathfrak{p}$ in the ring of integers $\mathcal{O}_K$ [Den87, Theorem 2.4].

Any resolution with good reduction mod $\mathfrak{p}$ gives rise to a pleasant formula for $Z_{\mathfrak{p}}(s)$ in terms of the numerical data of the resolution and the number of $\mathbb{F}_q$-points in the strata of the fiber over 0. Let

$$\overline{E}_I^\circ = \{x \in \overline{\pi}^{-1}(0) : x \in \overline{D}_i \text{ if and only if } i \in I\}.$$

Theorem 1.2.5 ([Den87, Theorem 3.1])**.** *Suppose* π *has good reduction mod* $\mathfrak{p}$*. Then*

$$Z_{\mathfrak{p}}(s) = \sum_{I \subset \{0,\dots,r\}} (q-1)^{|I|} |\overline{E}_I^\circ(\mathbb{F}_q)| \prod_{i \in I} \frac{q^{-N_i s - \nu_i}}{1 - q^{-N_i s - \nu_i}}. \tag{3}$$

Note that point counts over finite fields are analogous to topological Euler characteristics over $\mathbb{C}$; both are additive with respect to disjoint unions and multiplicative with respect to products. The role of $|\overline{E}_i^\circ(\mathbb{F}_q)|$ in (3) is analogous to that of $\chi(E_i^\circ)$ in (2). When $|\overline{E}_i^\circ(\mathbb{F}_q)|$ vanishes, then a term in (3) involving a pole at α_i vanishes, and when the Euler characteristic $\chi(E_i^\circ)$ vanishes, a term in (2) involving the multiplicity of the corresponding eigenvalue of monodromy vanishes. For more explicit connections, see [Den91a].

1.2.5. *Local topological zeta functions.* The analogy between Euler characteristics and the point counts over finite fields that appear in formulas for the local zeta functions in cases of good reduction lead to the topological zeta functions of Denef and Loeser. Heuristically, these are limits of local $\mathfrak{p}$-adic zeta functions. The local topological zeta function is defined as follows:

$$Z_{\mathrm{top}}(s) := \sum_{I \subset \{0,\dots,r\}} \chi(E_I^\circ) \prod_{i \in I} \frac{1}{N_i s + \nu_i}.$$

It is independent of the choice of resolution [DL92, Theorem 2.1.2].

Suppose f has coefficients in a number field K. Then poles of $Z_{\mathrm{top}}(s)$ give rise to poles of most local $\mathfrak{p}$-adic zeta functions. More precisely, after clearing denominators, we may assume that f has coefficients in the ring of integers. In this case, if α is a pole of $Z_{\mathrm{top}}(s)$ then, for all but finitely many primes $\mathfrak{p}$ in the ring of integers, there are infinitely many unramified extensions $\Bbbk \,|\, K_{\mathfrak{p}}$ such that α is a root of the local $\mathfrak{p}$-adic zeta function of f over $\Bbbk$. See [DL92, Theorem 2.2].

Local Topological Monodromy Conjecture ([DL92, Conjecture 3.3.2])**.** *If* α *is a pole of* $Z_{\mathrm{top}}(s)$ *then* $\exp(2\pi i \alpha)$ *is a nearby eigenvalue of monodromy.*

We note that local topological zeta functions have a pleasantly simple expression for singularities that are nondegenerate [DL92, §5]. For the corresponding formula for the local $\mathfrak{p}$-adic zeta function of a nondegenerate singularity with good reduction mod $\mathfrak{p}$, see [DH01, Theorem 4.2].

One naturally expects that the local topological monodromy conjecture should be easier to prove than the local $\mathfrak{p}$-adic monodromy conjecture, even in the good reduction case, and experience does bear this out. For instance, both conjectures are known in the special case when f is nondegenerate and $n = 3$. However, the proof of the topological case [LVP11] preceded the proof of the $\mathfrak{p}$-adic case [BV16] by a few years and is considerably shorter. See [VS22, Exercise 3.65] for an example of a nondegenerate hypersurface (for $n = 5$) with a real pole of its local $\mathfrak{p}$-adic zeta functions that is not a pole of its local topological zeta function.

1.2.6. *The local motivic zeta function.* The local motivic zeta function of f at 0 is a formal power series with coefficients in a localization of the $\hat{\mu}$-equivariant Grothendieck ring of varieties and can be characterized in terms of an embedded log resolution, as follows.

Let $\mu_m := \operatorname{Spec} \Bbbk[t]/(t^m - 1)$ denote the group of mth roots of unity over $\Bbbk$, and let

$$\hat{\mu} := \varprojlim \mu_m.$$

An action of $\hat{\mu}$ on a $\Bbbk$-variety Y is *good* if the action factors through μ_m for some m, and Y is covered by invariant affine opens. The *Grothendieck ring* $K^{\hat{\mu}}$ is additively generated by classes $[Y]$, where Y is a $\Bbbk$-variety with good $\hat{\mu}$-action, subject to the relations:

- if Z is a closed $\hat{\mu}$-invariant subvariety, then $[Y] = [Y \smallsetminus Z] + [Z]$;
- if $W \to Y$ is a $\hat{\mu}$-equivariant $\mathbb{A}^m$-bundle, then $[W] = [\mathbb{A}^m \times Y]$.

In the second relation, $\hat{\mu}$ acts trivially on $\mathbb{A}^m$. Multiplication in the Grothendieck ring $K^{\hat{\mu}}$ is given by $[Y] \cdot [Z] = [Y \times Z]$, with the diagonal $\hat{\mu}$-action on $Y \times Z$.

For each nonempty subset $I \subset \{0, \ldots, r\}$, let $m_I := \gcd\{N_i : i \in I\}$. Then D_I° is covered by Zariski open subsets $U \subset Y$ on which $\pi^* f$ is of the form ug^{m_I}, where u is a unit on U, and g is a regular function. Consider the Galois cover $\widetilde{D}_I^\circ \to D_I^\circ$, with Galois group μ_{m_I} whose restriction to such an open set $D_I^\circ \cap U$ is

$$\{(z, y) \in \mathbb{A}^1 \times (D_I^\circ \cap U) : z^{m_I} = u^{-1}\}.$$

Then $\widetilde{D}_I^\circ$ comes with the evident good $\hat{\mu}$-action that factors through μ_{m_I} and commutes with the projection to $\mathbb{A}^n$. Let

$$\widetilde{E}_I^\circ := \widetilde{D}_I^\circ \times_{\mathbb{A}^n} \{0\}$$

be the induced Galois cover of the fiber of D_I° over 0, with the good $\hat{\mu}$-action that it inherits from $\widetilde{D}_I^\circ$.

Let $\mathbb{L} := [\mathbb{A}^1]$, and set $\mathcal{M}^{\hat{\mu}} = K^{\hat{\mu}}[\mathbb{L}^{-1}]$. The local motivic zeta function of f at 0 is the formal power series expansion in $\mathcal{M}^{\hat{\mu}}[\![T]\!]$ of the following rational function in $\mathcal{M}^{\hat{\mu}}(T)$:

$$Z_{\mathrm{mot}}(T) = \sum_{I \subset \{0, \ldots, r\}} (\mathbb{L} - 1)^{|I|} [\widetilde{E}_I^\circ] \prod_{i \in I} \frac{\mathbb{L}^{-\nu_i} T^{N_i}}{1 - \mathbb{L}^{-\nu_i} T^{N_i}}.$$

Note, in particular, that $Z_{\mathrm{mot}}(T)$ is contained in the subring of $\mathcal{M}^{\hat{\mu}}[\![T]\!]$ generated over $\mathcal{M}^{\hat{\mu}}$ by T and $\left\{\frac{1}{1 - \mathbb{L}^{-\nu_i} T^{N_i}} : 0 \leq i \leq r\right\}$. This subring depends on the choice of a log resolution, but the power series $Z_{\mathrm{mot}}(T)$ is independent of all choices.

Remark 1.2.6. In the literature, an additional multiplicative factor of $\mathbb{L}^{-n}$ sometimes appears in the definition of the local motivic zeta function. See, e.g. [RV03, (0.1.2)] and [VS22, Theorem 3.18]. Other versions differ from ours by a factor of $\mathbb{L} - 1$ [BN20, Corollary 5.3.2] . These renormalizations do not affect the sets of candidate poles and are not relevant to the local motivic monodromy conjecture.

Grothendieck rings of varieties are not integral domains [Poo02], so some care is required in defining poles of $Z_{\mathrm{mot}}(T)$. Various notions are possible. See, for instance, [RV03, §4]. We follow the now standard convention and state the local motivic monodromy conjecture in terms of sets of candidate poles, as in [BV16, BN20].

Definition 1.2.7. *Let $\mathcal{P}$ be a finite set of rational numbers. Then $\mathcal{P}$ is a* set of candidate poles *for $Z_{\mathrm{mot}}(T)$ if $Z_{\mathrm{mot}}(T)$ is contained in*

$$\mathcal{M}^{\hat{\mu}}\left[T, \frac{1}{1 - \mathbb{L}^a T^b}\right]_{(a,b) \in \mathbb{Z} \times \mathbb{Z}_{>0}, a/b \in \mathcal{P}}.$$

Roughly speaking, if α satisfies any reasonable notion of being a pole of $Z_{\mathrm{mot}}(T)$, then it is contained in every set of candidate poles.

Remark 1.2.8. In practice, passing to an embedded log resolution π is not a useful way of computing local zeta functions; this typically introduces many exceptional divisors whose numerical data correspond neither to poles of the zeta function nor to eigenvalues of monodromy. One obtains more efficient expressions for the local zeta functions of nondegenerate singularities by first proving that they can be computed from a log smooth partial resolution [BN20] or a stacky resolution [Que22].

1.3. **Relations to prior results.** The local monodromy conjectures remain wide open in general, despite the persistent efforts of many mathematicians over a period of decades. Perhaps most surprising is that the local topological monodromy conjecture remains open for isolated nondegenerate singularities, even though there are well-known and relatively simple combinatorial formulas for both the characteristic polynomial of monodromy [A'C75, Theorem 4] and the local topological zeta function [DL92, Theorem 5.3]. Nevertheless, there is a vast literature on the local monodromy conjectures, far more than can reasonably be reviewed here. We give only a brief and largely ahistorical review of prior work closely related to our main theorems, and recommend the excellent survey articles [Nic10, VS22] for more detailed discussions and further references.

1.3.1. *Local monodromy conjectures in low dimensions.* For $n = 2$, Bultot and Nicaise proved the local motivic monodromy conjecture in full generality [BN20, Theorem 8.2.1].

For nondegenerate singularities when $n = 3$, Lemahieu and Van Proeyen proved the local topological monodromy conjecture [LVP11]. Bories and Veys used the same arguments for existence of eigenvalues and developed new arguments to reduce the size of sets of candidate poles, proving the local $\mathfrak{p}$-adic monodromy conjecture [BV16]. They also proved a *naive* variant of the local motivic monodromy conjecture for nondegenerate singularities with $n = 3$. In the naive variant, the ring $K^{\hat{\mu}}$ is replaced with the ordinary Grothendieck ring of varieties (without $\hat{\mu}$-action); the local motivic zeta function specializes to the local naive motivic zeta function by setting $[Y] \mapsto [Y/\hat{\mu}]$.

Esterov, Lemahieu, and Takeuchi introduced new arguments for both existence of eigenvalues and cancellation of poles for local topological zeta functions of nondegenerate singularities, especially for $n = 4$, and stated a conjecture for how these should generalize to higher dimensions [ELT22, Conjecture 1.3]. Recently, while this paper was in the final stages of preparation, Quek produced a naive motivic upgrade for some of the pole cancellation arguments from [ELT22], giving a new proof of the main result of Bories and Veys for $n = 3$. Quek also suggested a different way in which the pole cancellation statements for small n might generalize to higher dimensions [Que22, Question 5.1.8]. Neither of these predictions are correct. See Examples 2.2.1 and 2.2.2.

We also note that one of the claimed results for cancellation of poles in [ELT22], namely their Proposition 3.7, which is a special case of their Theorem 4.3, is also incorrect as stated. A single facet that is a B_1*-pyramid not of compact type* in the sense of their Definition 3.1 can contribute a nontrivial pole to the local topological zeta function, and this happens already for $n = 4$. See Example 2.2.3. We use a different notion of non-compact B_1-facet that agrees with that of Quek [Que22, Definition 1.1.7], specializes to that of Lemahieu and Van Proeyen when $n = 3$ [LVP11, Definition 3], and also has the expected pole cancellation property for candidate poles contributed only by a single B_1-facet in higher dimensions. See Theorem 5.1.2. For compact faces, [ELT22, Proposition 3.7] is correct. From the authors, we understand that the mistake in the corresponding statement for non-compact facets does not seriously affect the other arguments in their paper.

1.3.2. *Global zeta functions and strong monodromy conjectures.* There are *global* versions of the local motivic, $\mathfrak{p}$-adic, and topological zeta functions and their associated monodromy conjectures. See, e.g., [DL92] for a discussion of the local and global topological zeta functions. The difference between the local and global motivic zeta functions is illustrated by [BN20, Theorems 8.3.2 and 8.3.5]. The global zeta functions are invariants of $X_f \subset \mathbb{A}^n$, while the local zeta functions are invariants of its germ at 0.

Replacing "local" by "global" in each of the local monodromy conjectures gives rise to its global counterpart. There are also *strong* versions of the local and global monodromy conjectures proposing that the (real parts of the) poles of the corresponding zeta functions are zeros of the Bernstein-Sato polynomial b_f. If α is a zero of b_f then $\exp(2\pi i\alpha)$ is an eigenvalue of monodromy, and all eigenvalues of monodromy occur in this way [Mal74]. It is also conjectured that the orders of poles of local zeta functions are bounded by the multiplicities of zeros of b_f [DL92, Conjecture 3.3.1′].

The strong local and global motivic monodromy conjectures are known for $n = 2$ [BN20]. Loeser has given a combinatorial condition on Newton polyhedra that guarantees that each candidate pole associated to a facet is a zero of the Bernstein-Sato polynomial [Loe90]. Nondegenerate polynomials with such Newton polyhedra therefore satisfy the strong local and global motivic monodromy conjectures.

Aside from this, we note that if X_f is smooth aside from an isolated singularity at 0, then each local monodromy conjecture at 0 implies the corresponding global monodromy conjecture. The Newton polyhedra whose nondegenerate singularities are isolated were classified by Kouchnirenko [Kou76]. Furthermore, if f has such a Newton polyhedron and $f|_F$ has no singularities outside the coordinate hyperplanes for *all* faces F of $\mathrm{Newt}(f)$, not just the compact faces, then X_f is smooth away from 0. Thus the global motivic monodromy conjecture for isolated singularities with simplicial Newton polyhedra that satisfy this stronger nondegeneracy condition follows from Theorem 1.1.1.

1.3.3. *Further variants of the local zeta functions and monodromy conjectures.* There are also monodromy and holomorphy conjectures for $\mathfrak{p}$-adic zeta functions *twisted* by a character, and topological analogues of twisted $\mathfrak{p}$-adic zeta functions. For discussions of these variants, see, e.g., [Den91b]. Another variant is the topological zeta function for a variety equipped with a holomorphic form that plays the role of $\Phi(x)dx$ in Malgrange's archimedean zeta functions [Vey07].

Our results on eigenvalues of monodromy in Sections 3-4 are applicable to all such variants.

1.4. **Methods and structure of the paper.** We conclude the introduction with a brief overview of our approach to the local motivic monodromy conjecture and outline the content of each section of the paper.

1.4.1. *Key definitions.* We first recall the notion of candidate poles and candidate eigenvalues. In the literature, a candidate pole and candidate eigenvalue is associated to each facet of $\mathrm{Newt}(f)$. For our purposes, it is important to extend these notions to a wider class of faces of $\mathrm{Newt}(f)$. To be precise, let G be a proper face of $\mathrm{Newt}(f)$ that contains the vector $\mathbf{1} = (1, ..., 1)$ in its linear span, denoted $\mathrm{span}(G)$. Let ψ_G be the unique linear function on $\mathrm{span}(G)$ with value 1 on G. Then

$$\alpha_G := -\psi_G(\mathbf{1})$$

is the *candidate pole* associated to G, and $\exp(2\pi i\alpha_G)$ is the corresponding *candidate eigenvalue* of monodromy. If G' contains G as a face, then G' also contains $\mathbf{1}$ in its linear span and the candidate poles associated to G and G' are the same.

Definition 1.4.1. *Let* $\mathrm{Contrib}(\alpha)$ *be the set of faces of* $\mathrm{Newt}(f)$ *that contribute the candidate pole* α.

Then $\{\alpha \in \mathbb{R} : \mathrm{Contrib}(\alpha) \neq \emptyset\} \cup \{-1\}$ is a set of candidate poles for $Z_{\mathrm{mot}}(T)$ [BN20, Corollary 8.3.4]. This set of candidate poles is standard in the literature. The key difference here is that we consider faces in $\mathrm{Contrib}(\alpha)$ of arbitrary codimension, not just facets. This change in perspective is crucial in what follows.

Let C be a cone in Δ. The rays of C are the union of rays through vertices in $\mathrm{Newt}(f)$ and rays disjoint from $\mathrm{Newt}(f)$ that contain a coordinate vector e_ℓ for some $1 \leq \ell \leq n$. In particular, for each ray of C, there is a corresponding distinguished generator: either the corresponding vertex of $\mathrm{Newt}(f)$, or the corresponding coordinate vector e_ℓ. We let $\mathrm{Gen}(C)$ be the set of distinguished generators of the rays of C.

We say that a vertex A in G is an *apex* with *base direction* e_ℓ^* if $\langle e_\ell^*, A\rangle > 0$, and $\langle e_\ell^*, V\rangle = 0$ for all $V \in \mathrm{Gen}(C_G)$ with $V \neq A$.

Definition 1.4.2. *A face G of* $\mathrm{Newt}(f)$ *is B_1 if it has an apex A with base direction e_ℓ^*, and $\langle e_\ell^*, A\rangle = 1$.*

The notion of B_1 was introduced for simplicial facets in [LVP11, Definition 3]. For arbitrary facets, our definition agrees with [Que22, Definition 1.1.7] but is stronger than [ELT22, Definition 3.1]. All of these definitions of B_1-facets agree when $\mathrm{Newt}(f)$ is simplicial. Note that the base direction e_ℓ^* determines the apex A. The converse is not true. A B_1-face may have several apices, and when the face is not a facet, each of those apices can have multiple base directions. We introduce the following definition.

Definition 1.4.3. *A face G of* $\mathrm{Newt}(f)$ *is UB_1 if there exists an apex A in G with a unique choice of base direction e_ℓ^*, and $\langle e_\ell^*, A\rangle = 1$.*

Theorems 1.4.6 and 1.4.7, show the importance of the notion of UB_1-faces. Note that every B_1-facet is UB_1; the distinction between B_1 and UB_1 is only relevant when considering higher codimension faces.

1.4.2. *Eigenvalue multiplicities and local h-polynomials.* The starting point for our work is the the third author's nonnegative formula for the multiplicities of eigenvalues of monodromy at 0 when f is nondegenerate and $\mathrm{Newt}(f)$ is convenient [Sta17, Section 6.3]. Specializing [Sta17, Theorem 6.20] from equivariant mixed Hodge numbers to equivariant multiplicities, one obtains a combinatorial formula with nonnegative integer coefficients for the multiplicities of the eigenvalues of monodromy on the reduced cohomology of $\mathcal{F}_0$.

Assume that $\mathrm{Newt}(f)$ is simplicial. If we forget the lattice structure of the fan Δ, we may view Δ as encoding a triangulation of a simplex, e.g., by slicing with a transverse hyperplane. We note that in the non-simplicial case, one may choose an appropriate simplicial refinement of Δ and apply a similar argument to the one below. Then the combinatorial formula for eigenvalues is a sum over cones C in Δ of a contribution that is a product of two nonnegative factors, one coming from Ehrhart theory (the number of lattice points in a polyhedral set). The other factor is the evaluation of the local h-polynomial $\ell(\Delta, C; t)$ at $t = 1$. These local h-polynomials were first introduced and studied by Stanley in the special case where $C = 0$ and later generalized by Athanasiadis, Nill, and Schepers [Ath12a, Nil12]. They have nonnegative, symmetric integer coefficients and naturally appear when applying the decomposition theorem to toric morphisms. See [Sta92, Theorem 5.2], [KS16, Theorem 6.1] and [dCMM18].

This formula for eigenvalue multiplicities in the convenient nondegenerate case offers fundamental advantages over earlier approaches to existence of eigenvalues. Whereas the formulas of A'Campo and Varchenko for zeta functions of monodromy typically involve a great deal of cancellation, the third author's formula is a sum of nonnegative terms. Moreover, for each face G in $\mathrm{Contrib}(\alpha)$, there is a canonically associated *essential face* $E \subset G$. See Definition 3.3.1. Then the Ehrhart factor in the summand associated to C_E for the multiplicity of $\exp(2\pi i\alpha)$ is strictly positive. Thus, either $\exp(2\pi i\alpha)$ is an eigenvalue of monodromy or $\ell(\Delta, C_E; t)$ is zero. There are a number of simple sufficient conditions for the nonvanishing of $\ell(\Delta, C_E; t)$; for instance, if E meets the interior of the positive orthant then $\ell(\Delta, C_E; 0) = 1$. The general problem of classifying when local h-polynomials vanish was posed by Stanley [Sta92, Problem 4.13]. See [dMGP$^+$20] for a classification with $n \leq 4$ for $E = \emptyset$ and partial results in higher dimensions.

1.4.3. *A nonnegative formula for nearby eigenvalues.* In Section 3, we extend the results of [Sta17] to the case where $\mathrm{Newt}(f)$ is simplicial but not necessarily convenient. In this setting, the singularity of X_f at 0 may not be isolated, and the Milnor fibers at 0 and at nearby points may have cohomology in multiple positive degrees.

In this setting, we consider $\widetilde{\chi}(\mathcal{F}_x) := \sum_i (-1)^i \widetilde{H}^i(\mathcal{F}_x, \mathbb{C})$ as a virtual representation, where $\widetilde{H}$ denotes reduced cohomology. Now $\exp(2\pi i\alpha)$ has a multiplicity $\widetilde{m}_x(\alpha)$, which may be positive or negative, as an eigenvalue in this virtual representation. We consider these multiplicities not only at 0 but also at a general point x_I in each coordinate subspace $\mathbb{A}^I$ contained in X_f. The idea of studying the eigenvalues at these points was first introduced when $n = 3$ in [LVP11], and further developed in [ELT22]. We give a nonnegative formula for the alternating sum

$$\sum_{\mathbb{A}^I \subset X_f} (-1)^{n-1-|I|} \widetilde{m}_{x_I}(\alpha).$$

See Theorem 3.2.1 for a precise statement.

Theorem 3.2.1 implies, in particular, the remarkable fact that the corresponding alternating product of monodromy zeta functions is a polynomial, i.e.,

$$\prod_{\mathbb{A}^I \subset X_f} \left(\frac{\zeta_{x_I}(t)}{1-t} \right)^{(-1)^{n-1-|I|}} \in \mathbb{Z}[t].$$

From this perspective, the theorem provides a nonnegative formula for the vanishing order of this polynomial at $\exp(2\pi i \alpha)$. See Remark 3.2.2 for the precise formula.

Just as in the convenient case, this nonnegative formula is a sum over cones C in Δ, and each of the terms is once again an Ehrhart factor times $\ell(\Delta, C; 1)$. Moreover, for each $G \in \mathrm{Contrib}(\alpha)$, we have essential face $E \subset G$, and the Ehrhart factor in the C_E-summand for the multiplicity of $\exp(2\pi i\alpha)$ is strictly positive. We deduce the following corollary. See Corollary 3.3.2 for an equivalent statement.

Corollary 1.4.4. *Suppose* $\mathrm{Newt}(f)$ *is simplicial and* f *is nondegenerate. Let* $G \in \mathrm{Contrib}(\alpha)$. *Let* E *be the essential face of* G. *If* $\ell(\Delta, C_E; t)$ *is nonzero, then* $\sum_{\mathbb{A}^I \subset X_f} (-1)^{n-1-|I|} \widetilde{m}_{x_I}(\alpha) > 0$. *In particular,* $\exp(2\pi i\alpha)$ *is a nearby eigenvalue of monodromy (for reduced cohomology).*

This motivates a detailed study of necessary conditions for vanishing of $\ell(\Delta, C_E; t)$ when E is the essential face associated to some face $G \in \mathrm{Contrib}(\alpha)$. In this situation, we also have some additional structure which is crucial for our arguments. The face $C_G \smallsetminus C_E \in \mathrm{lk}_\Delta(C_E)$ admits what we call a *full partition*. See Lemma 3.3.4 and Definition 4.1.2.

1.4.4. *A necessary condition for vanishing of the local h-polynomial.* Motivated by the results of Section 3, in Section 4 we undertake a detailed (and necessarily technical) investigation of the conditions under which $\ell(\Delta, C'; t)$ vanishes, where C' is a cone in Δ that is contained in a cone that admits a full partition. This section is self-contained and applies to any local h-polynomial of a geometric triangulation. See [LPS22] for further work on necessary conditions for the vanishing of the local h-polynomial in a more general setting, for quasi-geometric homology triangulations.

Recall that $\ell(\Delta, C'; t)$ is naturally identified with the Hilbert function of a module $L(\Delta, C')$ [Ath12b, Ath12a], as follows. Consider the ideal in the face ring $\mathbb{Q}[\mathrm{lk}_\Delta(C')]$ generated by monomials x^C such that $C + C'$ meets the interior of the orthant $\mathbb{R}^n_{>0}$. Then $L(\Delta, C')$ is the image of this ideal in the quotient of $\mathbb{Q}[\mathrm{lk}_\Delta, C']$ by a special linear system of parameters. Thus $\ell(\Delta, C'; t) = 0$ if and only if every such monomial is contained in the ideal generated by a special linear system of parameters. When C admits a full partition, we can associate a distinguished monomial with image in $L(\Delta, C')$. We show that the image of this distinguished monomial specializes to a top degree cohomology class, given by the refined self-intersection of a compact T-invariant subvariety of half-dimension in a toric variety. An explicit calculation shows that this self-intersection is not zero. This calculation is purely combinatorial, and its proof constitutes the majority of the section.

Using this calculation, we prove the following theorem, which is an immediate consequence of Theorem 4.1.3. A cone C in $\mathrm{lk}_\Delta(C')$ is a *U-pyramid* if it meets the interior of the positive orthant and there exists a ray $A \in C$ such that $(C \sqcup C') \smallsetminus A$ is contained in a unique coordinate hyperplane in $\mathbb{R}^n$, i.e., C is a pyramid with a *unique* base direction with respect to the apex A. See Definition 4.1.1.

Theorem 1.4.5. *Let Δ be a simplicial fan supported on $\mathbb{R}^n_{\geq 0}$. Let C' be a cone in Δ, and let $C \in \mathrm{lk}_\Delta(C')$. If $\ell(\Delta, C'; t) = 0$ and C admits a full partition, then C is a U-pyramid.*

When G is compact and $C = C_G \smallsetminus C_E \in \mathrm{lk}_\Delta(C_E)$, the condition that G is UB_1 is equivalent to the condition that C is a U-pyramid. See Lemma 3.3.3. This leads to the following theorem, which is our main result on existence of eigenvalues.

Theorem 1.4.6. *Suppose* $\mathrm{Newt}(f)$ *is simplicial and f is nondegenerate. Let $\alpha \in \mathbb{R}$. Then either every face in* $\mathrm{Contrib}(\alpha)$ *is UB_1, or $\exp(2\pi i\alpha)$ is a nearby eigenvalue of monodromy (for reduced cohomology).*

Section 3 ends with a proof that this theorem follows from Theorem 1.4.5; the proof is completed in Section 4.

1.4.5. *The local formal zeta function and its candidate poles.* In Section 5, we prove the following theorem, which is complementary to Theorem 1.4.6.

Theorem 1.4.7. *Suppose* $\mathrm{Newt}(f)$ *is simplicial and f is nondegenerate. Let*

$$\mathcal{P} = \{\alpha \in \mathbb{R} : \mathrm{Contrib}(\alpha) \neq \emptyset\} \cup \{-1\}, \text{ and } \mathcal{P}' = \{\alpha \in \mathcal{P} : \alpha \notin \mathbb{Z}_{<0}, \text{every face in } \mathrm{Contrib}(\alpha) \text{ is } UB_1\}.$$

Then $\mathcal{P} \smallsetminus \mathcal{P}'$ is a set of candidate poles for $Z_{\mathrm{mot}}(T)$.

Note that Theorem 1.1.1 follows directly from Theorems 1.4.6 and 1.4.7, using the fact that 1 is an eigenvalue of monodromy on $H^0(\mathcal{F}_0, \mathbb{C})$.

Our starting point for the proof of Theorem 1.4.7 is the formula for $Z_{\mathrm{mot}}(T)$ in [BN20, Theorem 8.3.5], which expresses $Z_{\mathrm{mot}}(T)$ as a sum over lattice points of the dual fan of $\mathrm{Newt}(f)$. We introduce the *local formal zeta function* $Z_{\mathrm{for}}(T)$, which is a power series over a polynomial ring that specializes to $Z_{\mathrm{mot}}(T)$. The advantage of working with $Z_{\mathrm{for}}(T)$ is that an intersection of two sets of candidate poles of $Z_{\mathrm{for}}(T)$ is a set of candidate poles (Lemma 5.3.5), so it suffices to show that for each $\alpha \notin \mathbb{Z}_{<0}$ such that $\mathrm{Contrib}(\alpha)$ consists entirely of UB_1-faces, there is a set of candidate poles for $Z_{\mathrm{for}}(T)$ not containing α.

Because $\mathrm{Newt}(f)$ is simplicial, every face in $\mathrm{Contrib}(\alpha)$ contains a unique minimal face of $\mathrm{Contrib}(\alpha)$. In Section 5.6, for each minimal face M of $\mathrm{Contrib}(\alpha)$, we build a small polyhedral neighborhood $N_{M,\leq\delta}$ around the dual cone σ_M. One may naturally assign a contribution to $Z_{\mathrm{for}}(T)$ to any polyhedral cone supported in $\mathbb{R}^n_{\geq 0}$, in particular, we may consider the contribution of $N_{M,\leq\delta}$. This allows us to restrict our attention to faces of $\mathrm{Contrib}(\alpha)$ that contain M.

It is known that if all facets contributing α are B_1 and admit a set of *consistent base directions* [ELT22, Definition 4.2], then α is not a pole of the local topological zeta function [ELT22, LVP11] or local naive motivic zeta function [Que22, Theorem A]. In our setup, the consistent base directions assumption is equivalent to the condition that all faces of $\mathrm{Contrib}(\alpha)$ that contain M admit a common base direction. In this case, analogous techniques to [ELT22] show that the contribution of $N_{M,\leq\delta}$ admits a set of candidate poles not containing α.

Unfortunately, one can not expect a set of consistent base directions to exist in general. Instead, the UB_1 assumption will allow us to apply these techniques locally and then compatibly pieces things together to get the required global statement.

More precisely, we construct a complete fan in which every face is labelled by a base direction. The UB_1 assumption ensures that our labeling is done with some compatibility properties. For each cone τ in the fan, we reduce to considering the contribution of the intersection $\tau \cap N_{M,\leq\delta}$. We then use the choice of base direction assigned to τ to locally apply analogous techniques to [ELT22] and show that the contribution of $\tau \cap N_{M,\leq\delta}$ admits a set of candidate poles not containing α, completing the proof. For example, if the consistent base directions assumption holds, then one can choose the fan to be $\mathbb{R}^n$, considered as a fan with a single cone.

There are two main technical aspects to the proof. First, we must find a collection of properties for the complete fan to satisfy, and prove that they imply that the rough argument above works. Second, we must prove the existence of such a fan. Roughly speaking, we will construct the fan as the dual fan to a polytope that is a deformation of an explicitly constructed polytope with vertices indexed by saturated chains of faces of $\mathrm{Contrib}(\alpha)$ which contain M.

Finally, observe that $\exp(2\pi i\alpha)$ appearing as a zero or pole of the monodromy zeta function implies that $\exp(2\pi i\alpha)$ is an eigenvalue of monodromy, but the converse is not true. When $\mathrm{Newt}(f)$ is simplicial and X_f is nondegenerate, we prove that there is a set of candidate poles $\mathcal{P}$ such that $\exp(2\pi i\alpha)$ is a zero or pole of the monodromy zeta function at the generic point of some coordinate subspace $\mathbb{A}^I \subset X_f$, for all $\alpha \in \mathcal{P} \smallsetminus \mathbb{Z}_{<0}$. This stronger statement about when certain monodromy zeta functions are sufficient to detect eigenvalues is not true when $\mathrm{Newt}(f)$ is not simplicial and $n \geq 4$. In such cases, there may be poles of local topological zeta functions such that $\exp(2\pi i\alpha)$ appears as a zero or pole of the monodromy zeta function only at points along strata that are properly contained in coordinate subspaces. See [ELT22, Example 7.5]. For one combinatorial approach to detecting such eigenvalues, see [Est21].

1.5. **Notation.** We now set up some additional notation which we will use for the remainder.

A face of $\mathrm{Newt}(f)$ is *interior* if it meets $\mathbb{R}^n_{>0}$. If G is a face of a proper interior face of $\mathrm{Newt}(f)$, let ψ_G be the unique linear function on $\mathrm{span}(G)$ with value 1 on G. These assemble into a function ψ on $\mathbb{R}^n_{\geq 0}$ that is piecewise linear with respect to Δ. The *lattice distance* of G to the origin is the smallest positive integer ρ_G such that $\rho_G\psi_G$ is $\mathbb{Z}$-linear, i.e., lies in $\mathrm{Hom}(\mathrm{span}(G) \cap \mathbb{Z}^n, \mathbb{Z})$. Note that $\rho_G = 1$ when $G = \emptyset$.

For a polyhedral cone C, let ∂C denote its boundary, defined to be the union of all faces of C of dimension strictly less than $\dim C$. Let $C^\circ = C \smallsetminus \partial C$ denote the relative interior of a polyhedral cone. A nonzero vector v in $\mathbb{Z}^n$ is *primitive* if it generates the group $\mathbb{R}v \cap \mathbb{Z}^n$. Recall that C is *simplicial* if it is a pointed cone generated by $\dim C$ rays.

A *geometric triangulation* of a simplex is a subdivision of a geometric simplex into a union of geometric simplices that meet along shared faces.

For a positive integer ℓ, we write $[\ell] = \{1, \dots, \ell\}$.

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2. Examples

2.1. **Basic examples.** Our first two examples are intended to serve as a guide to the main constructions in the paper. In these examples, $\mathrm{Newt}(f)$ is simplicial and f is supported at the vertices of $\mathrm{Newt}(f)$, and hence f is nondegenerate [BO16].

Below, $\widetilde{E}(\mathcal{F}_x) \in \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ is an alternative encoding of $\frac{\zeta_x(t)}{1-t}$ (see (4) in Section 3). The local formal zeta function $Z_{\mathrm{for}}(T)$ lies in a quotient ring of $\mathbb{Z}[L, L^{-1}][Y_K : K$ non-empty compact face of $\mathrm{Newt}(f)][\![T]\!]$, where L, T, Y_K are formal variables. See Definition 5.3.1.

Example 2.1.1. Let $f(x, y) = y^2 - x^3$. Then f has an isolated cusp at 0, and $\mathrm{Newt}(f)$ is convenient and has a unique compact facet F with vertices $v = (3, 0)$ and $w = (0, 2)$. Note that $\alpha_F = -5/6$, and F is not B_1. Theorem 1.4.6 says that $\exp(2\pi i\alpha_F)$ is an eigenvalue of monodromy for f at 0.

Then Δ is the trivial fan, and $\ell(\Delta, C; t)$ equals 1 if $C = C_F$, and equals 0 otherwise. We have monodromy zeta function $\zeta_0(t) = \frac{1-t}{1-t+t^2}$, and $\widetilde{E}(\mathcal{F}_0) = [1/6] + [5/6]$. The local formal zeta function is

$$Z_{\mathrm{for}}(T) = \frac{(L-1)\left(Y_F L^{-5}T^6 + Y_v L^{-2}T^3(1 + L^{-3}T^3) + Y_w L^{-1}T^2(1 + L^{-2}T^2 + L^{-4}T^{-4})\right)}{1 - L^{-5}T^6}.$$

The local motivic zeta function $Z_{\mathrm{mot}}(T)$ is

$$\frac{(\mathbb{L}-1)\big(\big(\frac{(\mathbb{L}-1)\mathbb{L}^{-1}T}{1-\mathbb{L}^{-1}T}+[Y_F(1)]\big)\mathbb{L}^{-5}T^6+[\mu_3]\mathbb{L}^{-2}T^3(1+\mathbb{L}^{-3}T^3)+[\mu_2]\mathbb{L}^{-1}T^2(1+\mathbb{L}^{-2}T^2+\mathbb{L}^{-4}T^{-4})\big)}{1-\mathbb{L}^{-5}T^6},$$

where $Y_F(1)$ is an elliptic curve minus 6 points, with a free μ_6-action, and $Y_F(1)/\mu_6$ is isomorphic to $\mathbb{P}^1$ minus 3 points. After simplification, the local naive motivic zeta function is

$$\frac{(\mathbb{L}-1)(\mathbb{L}^{-1}T^2-\mathbb{L}^{-4}T^5+\mathbb{L}^{-4}T^6-\mathbb{L}^{-6}T^7)}{(1-\mathbb{L}^{-1}T)(1-\mathbb{L}^{-5}T^6)}.$$

For $p \notin \{2,3\}$, f has good reduction mod p, and the local p-adic zeta function is

$$Z_{(p)}(s)=\frac{(p-1)(p^{5s+5}-p^{2s+2}+p^{s+2}-1)}{(p^{s+1}-1)(p^{6s+5}-1)}.$$

The local topological zeta function is $Z_{\mathrm{top}}(s)=\frac{(4s+5)}{(s+1)(6s+5)}$.

Example 2.1.2. Let $f=\{x^2-y^2z\}\in\mathbb{C}[x,y,z]$. Then X_f is the Whitney umbrella. There are three coordinate subspaces contained in X_f: $0=\mathbb{A}^{\emptyset}$, $\mathbb{A}^{\{2\}}=\{x=z=0\}$ and $\mathbb{A}^{\{3\}}=\{x=y=0\}$. The singular locus of X_f is $\mathbb{A}^{\{3\}}$. In particular, f does not have an isolated singularity at the origin and $\mathrm{Newt}(f)$ is not convenient. The nonempty compact faces of $\mathrm{Newt}(f)$ are a 1-dimensional face F with vertices $v=(2,0,0)$ and $w=(0,2,1)$. Note that F is UB_1 and $(1,1,1)\notin\mathrm{span}(F)$. There are two unbounded interior facets: $F_1=\{x+y=2\}=F+\mathbb{R}_{\geq 0}e_3$ and $F_2=\{x+2z=2\}=F+\mathbb{R}_{\geq 0}e_2$, with $F_1\cap F_2=F$.

Then $\alpha_{F_1}=-1$, and F_1 is not B_1. Theorem 1.4.6 predicts that $\exp(2\pi i\alpha_{F_1})=1$ is a nearby eigenvalue of monodromy for reduced cohomology. Also, $\alpha_{F_2}=-3/2$, and F_2 is UB_1. Then Theorem 1.4.7 predicts that there is a set of candidate poles for $Z_{\mathrm{mot}}(T)$ not containing $-3/2$. In this particular case, the corresponding candidate eigenvalue $\exp(2\pi i\alpha_{F_2})=-1$ is a nearby eigenvalue of monodromy.

The fan Δ has two maximal cones C_{F_1} and C_{F_2} intersecting in a unique interior 2-dimensional face C_F. We have

$$\ell(\Delta,C;t)=\begin{cases}1 & \text{if } C=C_{F_1} \text{ or } C=C_{F_2},\\ 1+t & \text{if } C=C_F,\\ 0 & \text{otherwise.}\end{cases}$$

The monodromy zeta function at a general point x_I of each coordinate subspace $\mathbb{A}^I$ is given by: $\zeta_0(t)=(1-t)(1+t)$, $\zeta_{x_{\{2\}}}=1-t$, $\zeta_{x_{\{3\}}}=1$. Then $\prod_{\mathbb{A}^I\subset X_f}\left(\frac{\zeta_{x_I}(t)}{1-t}\right)^{(-1)^{n-1-|I|}}=1-t^2$. Equivalently, $\widetilde{E}(\mathcal{F}_0)=[1/2]$, $\widetilde{E}(\mathcal{F}_{x_{\{2\}}})=0$, $\widetilde{E}(\mathcal{F}_{x_{\{3\}}})=-1$, and $\sum_{\mathbb{A}^I\subset X_f}(-1)^{n-1-|I|}\widetilde{E}(\mathcal{F}_{x_I})=1+[1/2]$. The local formal zeta function is

$$Z_{\mathrm{for}}(T)=\frac{L^{-3}T^2(L-1)^3(Y_v(1-L^{-1}T)+L^{-1}T(1-L^{-1}))}{(1-L^{-1})^2(1-L^{-1}T)(1-L^{-2}T^2)}.$$

The local motivic zeta function is

$$Z_{\mathrm{mot}}(T)=\frac{\mathbb{L}^{-3}T^2(\mathbb{L}-1)^3([\mu_2](1-\mathbb{L}^{-1}T)+\mathbb{L}^{-1}T(1-\mathbb{L}^{-1}))}{(1-\mathbb{L}^{-1})^2(1-\mathbb{L}^{-1}T)(1-\mathbb{L}^{-2}T^2)}.$$

After simplifying, the local naive motivic zeta function is

$$\frac{\mathbb{L}^{-1}T^2(\mathbb{L}-1)(1-\mathbb{L}^{-2}T)}{(1-\mathbb{L}^{-1}T)(1-\mathbb{L}^{-2}T^2)}.$$

For $p \neq 2$, f has good reduction mod p, and the local p-adic zeta function is

$$Z_{(p)}(s) = \frac{(p-1)(p^{s+2}-1)}{(p^{s+1}-1)^2(p^{s+1}+1)}.$$

The local topological zeta function is $Z_{\mathrm{top}}(s) = \frac{(s+2)}{2(s+1)^2}$.

2.2. **Counterexamples.** We now present counterexamples to [ELT22, Conjecture 1.8], [Que22, Conjecture 5.1.8], and [ELT22, Proposition 3.7]. The polyhedral computations in these examples were done using polymake [GJ00], and the computation of the zeta functions can be verified using Sage code of [VS12].

Example 2.2.1. Let

$$f(x_1, x_2, x_3, x_4, x_5, x_6) = x_1^8 + x_2^5 + x_3^{24} + x_4^{13} + x_5^{17} + x_6^{14} + x_3x_5x_6 + x_2^3x_4 + x_4x_5x_6 + x_1x_3^2x_4x_6.$$

Then f is a nondegenerate polynomial with an isolated singularity at 0 whose Newton polyhedron is simplicial and convenient, with 16 compact facets and 10 vertices. There are five compact facets containing the face with vertices $\{(8,0,0,0,0,0), (0,5,0,0,0,0), (0,0,1,0,1,1), (0,3,0,1,0,0)\}$, each of which contributes the candidate pole $-69/40$. All of these facets are B_1, and no other facets contribute $-69/40$. Two of these facets are obtained by adding either $\{(0,0,0,0,0,14), (0,0,0,1,1,1)\}$ or $\{(0,0,0,0,0,14), (1,0,2,1,0,1)\}$ to the above face. The existence of these two facets implies that the condition in [ELT22, Conjecture 1.3] is not satisfied, so the conjecture predicts that $\exp(2\pi i(-69/40))$ is a eigenvalue of monodromy. But this is not one of the 1912 eigenvalues of monodromy at the origin.

The local topological zeta function of f is $Z_{\mathrm{top}}(s) = -\frac{6142656s^3-2948088s^2-93769198s-115234075}{17(s+1)(104s+157)^2(168s+275)}$, which does not have $-69/40$ as a pole. One can deduce from Theorem 5.1.2 that there is a set of candidate poles for $Z_{\mathrm{mot}}(T)$ which does not contain $-69/40$.

Example 2.2.2. Let

$$f(x_1, x_2, x_3, x_4, x_5) = x_1^{21} + x_2^{22} + x_3^{24} + x_4^6 + x_5^{12} + x_1x_2 + x_2x_3^6x_5^5 + x_3x_4x_5 + x_3^2x_5^9.$$

Then f is a nondegenerate polynomial with an isolated singularity at the origin whose Newton polyhedron is simplicial and convenient, with 10 compact facets and 9 vertices. Every compact facet contains the face with vertices $\{(1,1,0,0,0), (0,0,1,1,1)\}$, and the candidate pole of every facet is -2. These ten facets have a choice of compatible apices in the sense of [Que22, Definition 5.1.5], so [Que22, Conjecture 5.1.8] predicts that $\{-1\}$ is a set of candidate poles for the local naive motivic zeta function.

The local topological zeta function of f is $Z_{\mathrm{top}}(s) = \frac{7s^2+45s+96}{24(s+1)(s+2)^2}$, so any set of candidate poles for the local naive motivic zeta function contains -2. When $n = 6$, there are counterexamples to [Que22, Conjecture 5.1.8] whose candidate pole is not an integer.

Example 2.2.3. In [ELT22, Definition 3.1.3], the authors give a different definition of a B_1-facet when the facet is non-compact. They say that a facet F with unbounded directions S is a *B_1-facet of non-compact type* if the image of F under the projection $\mathbb{R}^n \to \mathbb{R}^{n-|S|}$ forgetting the variable indexed by S is a B_1-facet. Then [ELT22, Proposition 3.7] claims that if a pole $\alpha \neq -1$ is contributed only by a single B_1-facet, then α is not a pole of $Z_{\mathrm{top}}(s)$. Consider the nondegenerate polynomial

$$f(x_1, x_2, x_3, x_4) = x_1x_3 + x_2x_3 + x_2x_4^5 + x_4^6.$$

Then $\mathrm{Newt}(f)$ has 4 vertices and 8 facets, one of which is compact. There is B_1-facet of non-compact type with vertices $\{(1,0,1,0), (0,1,1,0), (0,1,0,5)\}$ and with unbounded directions e_1 and e_2 whose candidate pole is $-6/5$. It is the only facet whose candidate is $-6/5$, so [ELT22, Proposition 3.7] claims that $-6/5$ is not a pole of $Z_{\mathrm{top}}(s)$. In fact, $Z_{\mathrm{top}}(s) = \frac{6}{(s+1)(5s+6)}$.

At the origin, the monodromy zeta function is 1. The singular locus of X_f is the set of points of the form $\{(c,-c,0,0)\}$. At any $c \neq 0$, the monodromy zeta function is equal to $-(t-1)(t^4+t^3+t^2+t+1)$, so $\exp(2\pi i(-6/5))$ is an eigenvalue of monodromy.

3. A nonnegative formula for nearby eigenvalues

Here and throughout, $f \in \Bbbk[x_1, \dots, x_n]$ is a nondegenerate polynomial that vanishes at 0. In this section and also in Section 4, we assume that $\mathrm{Newt}(f)$ is simplicial. Eigenvalues of monodromy and applications to the local monodromy conjectures for nondegenerate singularities with non-simplicial Newton polytopes will be addressed elsewhere.

For a geometric point x in the hypersurface X_f, we write $\widetilde{m}_x(\alpha)$ to denote the multiplicity of $\exp(2\pi i\alpha)$ in the virtual representation $\widetilde{\chi}(\mathcal{F}_x) := \sum_i (-1)^i \widetilde{H}^i(\mathcal{F}_x, \mathbb{C})$, where $\widetilde{H}$ denotes reduced cohomology. We define

$$\widetilde{E}(\mathcal{F}_x) := \sum_{[\alpha]\in\mathbb{Q}/\mathbb{Z}} \widetilde{m}_x(\alpha)[\alpha] \in \mathbb{Z}[\mathbb{Q}/\mathbb{Z}], \tag{4}$$

where $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ is the group algebra of $\mathbb{Q}/\mathbb{Z}$. Note that $\widetilde{E}(\mathcal{F}_x)$ encodes information equivalent to that in the monodromy zeta function $\zeta_x(t)$. This alternative notation is useful when studying how the monodromy action interacts with finer invariants of the cohomology of the Milnor fiber, such as its mixed Hodge structure [Sta17, Section 6.3]. The additive structure of $\widetilde{E}(\mathcal{F}_x)$ and the restriction to reduced cohomology is also more natural from a combinatorial perspective: it aligns with standard formulas for local h-polynomials and thereby leads to the nonnegative formula for nearby eigenvalues that we prove here.

Let x_I denote a general point in the coordinate subspace $\mathbb{A}^I \subset \mathbb{A}^n$, for $I \subset [n]$. The main result of this section is a formula with nonnegative integer coefficients for

$$\sum_{\mathbb{A}^I \subset X_f} (-1)^{n-1-|I|} \widetilde{E}(\mathcal{F}_{x_I}).$$

The set of coordinate subspaces contained in X_f depends only on $\mathrm{Newt}(f)$. Explicitly, $\mathbb{A}^I \subset X_f$ if and only if $\mathbb{R}^I_{\geq 0} \cap \mathrm{Newt}(f) = \emptyset$.

3.1. The local h-polynomial.

Recall from the introduction that if we forget the lattice structure of the fan Δ, we may view Δ as encoding a triangulation of a simplex, e.g., by slicing with a transverse hyperplane. For C a cone in Δ, let $\sigma(C)$ be the smallest face of $\mathbb{R}^n_{\geq 0}$ containing C. We use the following definition of the local h-polynomial (see, for example, [KS16, Lemma 4.12]).

Definition 3.1.1. *Let C be a cone in Δ. Then*

$$\ell(\Delta, C; t) := \sum_{C \subset C'} (-1)^{\mathrm{codim}(C')} t^{\mathrm{codim}(C) - e(C')} (t-1)^{e(C')},$$

where $e(C') := \dim \sigma(C') - \dim C'$ is called the excess *of C'.*

The local h-polynomial has several important properties. Most important for our purposes is that its coefficients are nonnegative integers [Ath12a]. We refer the reader to [KS16] for more details and a more general setting. Observe that

$$\ell(\Delta, C; 1) := \sum_{\substack{C \subset C' \\ e(C')=0}} (-1)^{\mathrm{codim}(C')}. \tag{5}$$

Recall that every cone C in Δ has a set $\mathrm{Gen}(C)$ of distinguished generators of its rays.

Definition 3.1.2. *Let C be a cone in Δ with* $\mathrm{Gen}(C) = \{w_1, \ldots, w_r\}$*. Let*

$$\mathrm{Box}^{\circ}_C := \Big\{ w \in \mathbb{Z}^n : w = \sum_{i=1}^{r} \lambda_i w_i, 0 < \lambda_i < 1 \Big\} \quad \textit{and} \quad \mathrm{Box}_C = \Big\{ w \in \mathbb{Z}^n : w = \sum_{i=1}^{r} \lambda_i w_i, 0 \leq \lambda_i < 1 \Big\}.$$

Note that $\mathrm{Box}^{\circ}_C = \{0\}$ if $C \cap \mathrm{Newt}(f) = \emptyset$, and $\mathrm{Box}_C := \cup_{C' \subset C} \mathrm{Box}^{\circ}_{C'}$.

3.2. **Nearby eigenvalues along coordinate subspaces.** We now state our nonnegative formula. Recall that ψ is the unique piecewise linear function on $\mathbb{R}^n_{\geq 0}$ with value 1 on all interior faces of $\partial \,\mathrm{Newt}(f)$.

Theorem 3.2.1. *Suppose* $\mathrm{Newt}(f)$ *is simplicial and f is nondegenerate. Then*

$$\sum_{\mathbb{A}^I \subset X_f} (-1)^{n-1-|I|} \widetilde{E}(\mathcal{F}_{x_I}) = \sum_{C \in \Delta} \ell(\Delta, C; 1) \sum_{w \in \mathrm{Box}^{\circ}_C} [-\psi(w)]. \tag{6}$$

Remark 3.2.2. Theorem 3.2.1 can be restated in terms of monodromy zeta functions at x_I, as follows:

$$\prod_{\mathbb{A}^I \subset X_f} \left(\frac{\zeta_{x_I}(t)}{1-t} \right)^{(-1)^{n-1-|I|}} = \prod_{C \in \Delta} \prod_{w \in \mathrm{Box}^{\circ}_C} (1 - \exp(-2\pi i \psi(w)) t)^{\ell(\Delta, C; 1)}.$$

For the remainder, we work in terms of $\widetilde{E}(\mathcal{F}_{x_I})$, for the reasons discussed above.

We deduce Theorem 3.2.1 from Varchenko's formula for the monodromy zeta function of a nondegenerate singularity [Var76, Theorem 4.1]. We will use the following notation. If C is a cone in Δ, then let C_{unb} be the largest face of C such that $C_{\mathrm{unb}} \cap \mathrm{Newt}(f) = \emptyset$.

Proposition 3.2.3. *The multiplicities of eigenvalues of monodromy of f at 0 are given by*

$$\widetilde{E}(\mathcal{F}_0) = \sum_{\substack{C \in \Delta \\ C_{\mathrm{unb}} = \{0\} \\ e(C) = 0}} (-1)^{\dim C + 1} \sum_{w \in \mathrm{Box}_C} [-\psi(w)].$$

Proof. If $C_{\mathrm{unb}} = \{0\}$, then $C = C_G$ for some simplex G. Let ρ_G be the lattice distance from G to the origin. Then

$$\sum_{w \in \mathrm{Box}_G} [-\psi(w)] = \mathrm{Vol}(G) \sum_{i=0}^{\rho_G - 1} [i/\rho_G],$$

where $\mathrm{Vol}(G)$ is the normalized volume of G (see, for example, [Sta17, Examples 4.12-4.13]). The proposition then follows from Varchenko's formula for the monodromy zeta function [Var76, Theorem 4.1]. □

Given $c = (c_1, \ldots, c_n) \in \mathbb{k}^n$, let $f_c(x_1, \ldots, x_n) := f(x_1 + c_1, \ldots, x_n + c_n)$.

Proof of Theorem 3.2.1. Consider a coordinate subspace $\mathbb{A}^I$ in X_f. Let $J = [n] \smallsetminus I$, and consider the projection map $\mathrm{pr}_J \colon \mathbb{R}^n \to \mathbb{R}^J$ and the polyhedron $P_J := \mathrm{pr}_J(\mathrm{Newt}(f)) \subset \mathbb{R}^J$. Observe that P_J is simplicial. Let Δ_J denote the corresponding fan. Then $\Delta_J = \{\mathrm{pr}_J(C) : C \in \Delta, \mathbb{R}^I_{\geq 0} \subset C_{\mathrm{unb}}\}$.

Suppose $C \in \Delta$ and $\mathbb{R}^I_{\geq 0} \subset C_{\mathrm{unb}}$. Let $\sigma_J(\mathrm{pr}_J(C))$ be the smallest face of $\mathbb{R}^J_{\geq 0}$ containing $\mathrm{pr}_J(C)$, and let $e_J(\mathrm{pr}_J(C))$ be the corresponding excess. Then

$$e_J(\mathrm{pr}_J(C)) = \dim \sigma_J(\mathrm{pr}_J(C)) - \dim \mathrm{pr}_J(C) = (\dim \sigma(C) - |I|) - (\dim(C) - |I|) = e(C).$$

We define a bijection $\phi \colon \mathrm{Box}_C \to \mathrm{Box}_{\mathrm{pr}_J(C)}$ as follows. Write $\mathrm{Gen}(C) = \{w_1, \ldots, w_r\} \cup \{e_i : i \in I\}$. If $w = \sum_{i=1}^r \lambda_i w_i + \sum_{i \in I} \mu_i e_i \in \mathrm{Box}_C$, then $\phi(w) = \sum_{i=1}^r \lambda_i \,\mathrm{pr}_J(w_i)$. Observe that $\psi(w) = \psi_J(\phi(w))$, where ψ_J is the unique piecewise linear function on $\mathbb{R}^J$ with value 1 on all interior faces of ∂P_J.

Let x_I be a general point in $\mathbb{A}^I$. Then $\mathcal{F}_{x_I}$ is the Milnor fiber of f_{x_I} at the origin. It follows from [ELT22, Proposition 7.2] and its proof that f_{x_I} is nondegenerate with Newton polyhedron equal to $P_J \times \mathbb{R}^I_{\geq 0}$. In particular, the bounded faces of $\mathrm{Newt}(f_{x_I})$ are the bounded faces of P_J.

Then Proposition 3.2.3 implies that

$$\widetilde{E}(\mathcal{F}_{x_I}) = \sum_{\substack{C \in \Delta \\ C_{\mathrm{unb}} = \mathbb{R}^I_{\geq 0} \\ e(C)=0}} (-1)^{\dim C - |I| + 1} \sum_{w \in \mathrm{Box}_C} [-\psi(w)].$$

We compute

$$\begin{aligned}
\sum_{\mathbb{A}^I \subset X_f} (-1)^{n-1-|I|} \widetilde{E}(\mathcal{F}_{x_I}) &= \sum_{\mathbb{A}^I \subset X_f} (-1)^{n-1-|I|} \sum_{\substack{C \in \Delta \\ C_{\mathrm{unb}} = \mathbb{R}^I_{\geq 0} \\ e(C)=0}} (-1)^{\dim C - |I| + 1} \sum_{w \in \mathrm{Box}_C} [-\psi(w)] \\
&= \sum_{\substack{C \in \Delta \\ e(C)=0}} (-1)^{\operatorname{codim} C} \sum_{w \in \mathrm{Box}_C} [-\psi(w)] \\
&= \sum_{C' \in \Delta} \sum_{w \in \mathrm{Box}^\circ_{C'}} [-\psi(w)] \sum_{\substack{C' \subset C \in \Delta \\ e(C)=0}} (-1)^{\operatorname{codim} C} \\
&= \sum_{C' \in \Delta} \sum_{w \in \mathrm{Box}^\circ_{C'}} [-\psi(w)] \ell(\Delta, C'; 1).
\end{aligned}$$

Here the final equality follows from (5). □

We will use the following remark in the next section in the proof of Theorem 1.4.6.

Remark 3.2.4. Let g be a nondegenerate polynomial with Newton polyhedron P and Milnor fiber $\widehat{\mathcal{F}}_0$ at the origin. By [Var76, Theorem 4.1], $\widetilde{E}(P) := \widetilde{E}(\widehat{\mathcal{F}}_0)$ depends only on P, and not on the choice of g.

Consider a coordinate subspace $\mathbb{A}^I \subset X_f$, and a general point x_I in $\mathbb{A}^I$. Let $J = [n] \smallsetminus I$, and consider the projection map $\mathrm{pr}_J \colon \mathbb{R}^n \to \mathbb{R}^J$ and the polyhedron $P_J := \mathrm{pr}_J(\mathrm{Newt}(f)) \subset \mathbb{R}^J$. Then a corollary of the proof above is that $\widetilde{E}(\mathcal{F}_{x_I}) = \widetilde{E}(P_J \times \mathbb{R}^I_{\geq 0})$. The latter is equal to $\widetilde{E}(P_J)$. We deduce that if the coefficient of $[\alpha]$ in $\widetilde{E}(P_J)$ is nonzero for some such choice of I, then $\exp(2\pi i \alpha)$ is a nearby eigenvalue of monodromy for reduced cohomology.

3.3. **A vanishing result for nearby eigenvalues of monodromy.** We now proceed to prove Theorem 1.4.6.

Let C be a cone in Δ with $\mathrm{Gen}(C) = \{w_1, \dots, w_r\}$. We consider the function

$$\mathrm{box}_C \colon \mathrm{span}(C) \cap \mathbb{Z}^n \to \mathrm{Box}_C \text{ defined by } \mathrm{box}_C\left(\sum_{i=1}^r \lambda_i w_i\right) = \sum_{i=1}^r \{\lambda_i\} w_i,$$

where $\lambda_i \in \mathbb{Q}$ and $\{\lambda_i\}$ denotes the fractional part of λ_i.

Throughout this section, fix some α and let $G \in \mathrm{Contrib}(\alpha)$. Recall that ψ_G is the unique linear function on $\mathrm{span}(G)$ with value 1 on G and $\alpha_G = -\psi_G(\mathbf{1})$ is the candidate pole associated to G.

Definition 3.3.1. *The* essential face $E = E(G)$ *is the unique face of* G *such that* $\mathrm{box}_{C_G}(\mathbf{1})$ *is in* $\mathrm{Box}^\circ_{C_E}$.

Equivalently, one may verify that E is the unique face of G with $\mathrm{Gen}(C_E) = \{w_i : a_i \notin \mathbb{Z}\}$. Note that, for any lattice point $w \in \mathrm{span}(C_G) \cap \mathbb{Z}^n_{\geq 0}$, $[\psi_G(w)] = [\psi(\mathrm{box}_{C_G}(w))]$ in $\mathbb{Q}/\mathbb{Z}$. We deduce that

$$[\alpha_G] = [-\psi(\mathrm{box}_G(\mathbf{1}))] \text{ in } \mathbb{Q}/\mathbb{Z}. \tag{7}$$

We deduce the following corollary of Theorem 3.2.1, which is equivalent to Corollary 1.4.4.

Corollary 3.3.2. *Suppose* $\mathrm{Newt}(f)$ *is simplicial and* f *is nondegenerate. Let* $G \in \mathrm{Contrib}(\alpha)$. *Let* E *be the essential face of* G. *If* $\ell(\Delta, C_E; t)$ *is nonzero, then the coefficient of* $[\alpha]$ *in* $\widetilde{E}(\mathcal{F}_{x_I})$ *is nonzero at a general point* x_I *of some coordinate subspace* $\mathbb{A}^I \subset X_f$.

Proof. By (7), $\ell(\Delta, C_E; 1)[\alpha_G]$ is a term in the right-hand side of (6). The result now follows from the nonnegativity of the local h-polynomial. □

Recall that a vertex A in G is an *apex* with *base direction* e^*_ℓ if $\langle e^*_\ell, A\rangle > 0$, and $\langle e^*_\ell, V\rangle = 0$ for all $V \in \mathrm{Gen}(C_G)$ with $V \neq A$. Recall that G is UB_1 if there exists an apex A in G with a unique choice of base direction e^*_ℓ, and $\langle e^*_\ell, A\rangle = 1$. The following definition is a special case of Definition 4.1.1.

Assume that G is compact. Then $C_G \smallsetminus C_E$ in $\mathrm{lk}_\Delta(C_E)$ is a *U-pyramid* if there exists an apex A in G with a unique choice of base direction e^*_ℓ, and $A \notin \mathrm{Gen}(C_E)$.

Lemma 3.3.3. *Suppose* $\mathrm{Newt}(f)$ *is simplicial and* f *is nondegenerate. Let* $G \in \mathrm{Contrib}(\alpha)$ *be a compact face. Let* E *be the essential face of* G. *Then* G *is* UB_1 *if and only if* $C_G \smallsetminus C_E$ *in* $\mathrm{lk}_\Delta(C_E)$ *is a* U*-pyramid.*

Proof. Let $\mathrm{Gen}(C_G) = \{w_1, \dots, w_r\}$, and uniquely write $\mathbf{1} = \sum_{i=1}^r \lambda_i w_i$, for some $\lambda_i \in \mathbb{Q}$. Let w_i be an apex with a base direction e^*_ℓ. Let $h = \langle e^*_\ell, w_i\rangle \in \mathbb{Z}_{>0}$. Then $\lambda_i = 1/h \in \mathbb{Z}_{>0}$. The result then follows since

$$w_i \notin C_E \iff \lambda_i \notin \mathbb{Q} \iff h = 1.$$

□

Assume that G is compact. Let $\mathcal{A}_G$ be the subcone of C_G generated by all apices of G. Recall that for a cone C in Δ, $\sigma(C)$ denotes the smallest face of $\mathbb{R}^n_{\geq 0}$ containing C. We identify faces of $\mathbb{R}^n_{\geq 0}$ with their corresponding subsets of $[n]$. Let $J = \{\ell \in [n] :$ there exists an apex with base direction $e^*_\ell\}$. The following definition is a special case of Definition 4.1.2.

A *full partition* of $C_G \smallsetminus C_E \in \mathrm{lk}_\Delta(C_E)$ is a decomposition

$$C_G \smallsetminus C_E = \mathcal{A}_G \sqcup C_1 \sqcup C_2,$$

for some cones C_1,C_2, such that

(1) $\sigma(\mathcal{A}_G \sqcup C_1 \sqcup C_E) = [n]$,
(2) $\sigma(C_2 \sqcup C_E) = [n] \smallsetminus J$.

Lemma 3.3.4. *Suppose* $\mathrm{Newt}(f)$ *is simplicial and* f *is nondegenerate. Let* $G \in \mathrm{Contrib}(\alpha)$ *be a compact face. Let* E *be the essential face of* G. *Then* $C_G \smallsetminus C_E \in \mathrm{lk}_\Delta(C_E)$ *admits a full partition.*

Proof. Let $\mathrm{Gen}(C_G) = \{w_1, \dots, w_r\}$, and uniquely write $\mathbf{1} = \sum_{i=1}^r \lambda_i w_i$, for some $\lambda_i \in \mathbb{Q}$. We have a decomposition

$$C_G \smallsetminus C_E = \mathcal{A}_G \sqcup C_1 \sqcup C_2, \tag{8}$$

where $\mathcal{A}_G = \{w_i : w_i \text{ is an apex}\}$, $C_1 = \{w_i : w_i \notin \mathcal{A}_G, \lambda_i \in \mathbb{Z}_{>0}\}$, and $C_2 = \{w_i : \lambda_i \in \mathbb{Z}_{\leq 0}\}$. Note that $w_i \in \mathcal{A}_G$ implies that $\lambda_i = 1$. We claim that (8) is a full partition.

For each w_i in $\mathrm{Gen}(C_G)$, write $(w_i)_\ell \in \mathbb{Z}_{\geq 0}$ for the ℓth coordinate of w_i. For each coordinate $\ell \in [n]$, we have

$$1 = \sum_{i=1}^n \lambda_i (w_i)_\ell. \tag{9}$$

Consider $\ell \in [n]$. If $\ell \notin \sigma(\mathcal{A}_G \sqcup C_1 \sqcup C_E)$, then the right-hand side of (9) is a sum of nonpositive terms, a contradiction. We conclude that $\sigma(\mathcal{A}_G \sqcup C_1 \sqcup C_E) = [n]$.

It remains to show that $\sigma(C_2 \sqcup C_E) = [n] \smallsetminus J$. It follows from the definitions that $\sigma(C_2 \sqcup C_E) \subset [n] \smallsetminus J$. It remains to prove that $[n] \smallsetminus \sigma(C_2 \sqcup C_E) \subset J$. Suppose that $\ell \in [n] \smallsetminus \sigma(C_2 \sqcup C_E)$. Then all terms on the right-hand side of (9) are nonnegative integers, and we deduce that there is a unique index k such that $\lambda_k = (w_k)_\ell = 1$ and $\lambda_i(w_i)_\ell = 0$ for $i \neq k$. If $(w_i)_\ell \neq 0$ for some $i \neq k$, then $\lambda_i = 0$ and hence $\ell \in \sigma(C_2)$, a contradiction. We deduce that w_i is an apex with base direction e_ℓ^*. That is, $\ell \in J$. □

The following corollary is immediate from Theorem 1.4.5, together with Lemma 3.3.3 and Lemma 3.3.4. Here Theorem 1.4.5 is an immediate consequence of Theorem 4.1.3, whose proof is the subject of Section 4.

Corollary 3.3.5. *Suppose* $\mathrm{Newt}(f)$ *is simplicial and* f *is nondegenerate. Let* $G \in \mathrm{Contrib}(\alpha)$ *be a compact face. Let* E *be the essential face of* G*. If* $\ell(\Delta, C_E; t)$ *is zero, then* G *is* UB_1*.*

Proof of Theorem 1.4.6. Suppose that $G \in \mathrm{Contrib}(\alpha)$ and G is not UB_1. We want to show that $\exp(2\pi i\alpha)$ is a nearby eigenvalue of monodromy for reduced cohomology.

Our strategy is to reduce to the case when G is compact and apply our results above. Let $I \subset [n]$ be determined by $(C_G)_{\mathrm{unb}} = \mathbb{R}^I_{\geq 0}$, and let $J = [n] \smallsetminus I$. Consider the projection map $\mathrm{pr}_J \colon \mathbb{R}^n \to \mathbb{R}^J$ and the polyhedron $P_J := \mathrm{pr}_J(\mathrm{Newt}(f)) \subset \mathbb{R}^J$. The faces of $\mathbb{R}^J_{\geq 0}$ not intersecting P_J are in bijection with the faces of $\mathbb{R}^n_{\geq 0}$ not intersecting $\mathrm{Newt}(f)$ that contain I. Observe that $\mathrm{pr}_J(G)$ is compact. Since C_G is simplicial and G is not UB_1, $\mathrm{pr}_J(G)$ is not UB_1. One deduces from the definitions that $\mathrm{pr}_J(G)$ has a candidate pole α.

Let g be a nondegenerate polynomial with Newton polyhedron P_J and Milnor fiber denoted $\widehat{\mathcal{F}}$. Corollary 3.3.2 and Corollary 3.3.5 imply that the coefficient of $[\alpha]$ in $\widetilde{E}(\widehat{\mathcal{F}}_{y_{\hat{I}}})$ is nonzero at a general point $y_{\hat{I}}$ of some coordinate subspace $\mathbb{A}^{\hat{I}} \subset Y_f$, where $\hat{I} \subset J$.

Let $I' = I \cap \hat{I}$ and $J' = [n] \smallsetminus I'$. Let $\mathcal{F}'$ be the Milnor fiber of a nondegenerate polynomial with Newton polyhedron $P_{J'}$. By Remark 3.2.4, the coefficient of $[\alpha]$ in $\widetilde{E}(\mathcal{F}')$ is nonzero, and $\exp(2\pi i\alpha)$ is a nearby eigenvalue of monodromy for f, as desired. □

4. A necessary condition for vanishing of the local h-polynomial

4.1. **Overview.** In this section, we prove a necessary condition for vanishing of the the local h-polynomial of a geometric triangulation of a simplex. The section is self-contained and combinatorial in nature. As such, the notation used is independent from the rest of the paper. In Section 4.2, we recall the combinatorial commutative algebra interpretation of the local h-polynomial. In Section 4.3, we reduce our result to proving a positivity result, Proposition 4.3.5, which we then prove in Section 4.4.

Let $\sigma \colon \mathcal{S} \to 2^{[n]}$ be a geometric triangulation of a simplex. An element $G \in \mathcal{S}$ is *interior* if $\sigma(G) = [n]$. Let E be a face of $\mathcal{S}$ and let $F \in \mathrm{lk}_{\mathcal{S}}(E)$ be a face. Then F is a *pyramid* with apex $A \in F$ if $F \sqcup E$ is interior and $(F \sqcup E) \smallsetminus A$ is not interior. Let

$$\mathcal{A}_F := \{A \in F : F \text{ is a pyramid with apex } A\}, \quad \text{and} \quad V_A = V_A(F) := [n] \smallsetminus \sigma((F \sqcup E) \smallsetminus A)$$

for $A \in \mathcal{A}_F$. In what follows, we identify simplices with their sets of vertices.

Definition 4.1.1. *We say that* F *is a* U-pyramid *if* $|V_A| = 1$ *for some* $A \in \mathcal{A}_F$*.*

Definition 4.1.2. *Consider a decomposition of* F *of the form:*

$$F = F_1 \sqcup F_2 \sqcup \mathcal{A}_F.$$

We say F *is a* full partition *if* $F_1 \sqcup \mathcal{A}_F \sqcup E$ *is interior and* $\sigma(F_2 \sqcup E) = [n] \smallsetminus \bigcup_{A \in \mathcal{A}_F} V_A$*.*

Recall that $\ell(\mathcal{S}, E; t)$ denotes the corresponding local h-polynomial. See Definition 3.1.1. Our goal is to prove the following theorem.

Theorem 4.1.3. *Let $\sigma\colon \mathcal{S} \to 2^{[n]}$ be a geometric triangulation of a simplex, and fix a face $E \in \mathcal{S}$. Let $F \in \mathrm{lk}_{\mathcal{S}}(E)$ be a face that admits a full partition $F = F_1 \sqcup F_2 \sqcup \mathcal{A}_F$. If the coefficient of $t^{|F_1|+|\mathcal{A}_F|}$ in $\ell(\mathcal{S}, E; t)$ is zero, then F is a U-pyramid.*

Our strategy is as follows: assume that F admits a full partition and is not a U-pyramid. We argue that the nonvanishing of the local h-polynomial is implied by the nonvanishing of a specific element in the highest degree cohomology of an associated complete toric variety, expressed as a non-square free monomial in the invariant divisors. We then compute an explicit nonzero formula for this element, completing the proof.

4.2. **The commutative algebra of local h-polynomials.** Let Δ be a rational simplicial fan in $\mathbb{R}^n$ with support $\mathbb{R}^n_{\geq 0}$. For each ray of Δ, choose a rational, nonzero point v. Consider the unique piecewise $\mathbb{Q}$-linear function $\psi\colon \mathbb{R}^n_{\geq 0} \to \mathbb{R}$ defined by $\psi(v) = 1$ for all such v, and let $\mathcal{S} = \{x \in \mathbb{R}^n_{\geq 0} : \psi(x) = 1\}$. Then $\mathcal{S}$ is a simplicial complex with vertices $\{v\}$, and $\mathcal{S}$ induces a geometric triangulation $\sigma\colon \mathcal{S} \to 2^{[n]}$ of a simplex by projecting onto a transverse hyperplane. The combinatorial type of this triangulation is independent of both the choice of $\{v\}$, and the choice of transverse hyperplane. Explicitly, if F is a face of $\mathcal{S}$, then $\mathbb{R}^{\sigma(F)}$ is the smallest coordinate hyperplane containing F. Conversely, given a geometric triangulation of a simplex, we may deform the vertices without changing the combinatorial type to assume that the triangulation is rational, and then the triangulation is realized by some such $\mathcal{S}$.

If F is a face of $\mathcal{S}$, let C_F denote the cone over F. For example, when $F = \emptyset$, then $C_F = \{0\}$. Then $\Delta = \{C_F : F \in \mathcal{S}\}$. Fix a face E of $\mathcal{S}$. Then the collection of cones Δ_E given by the images of $\{C_F : F \in \mathrm{lk}_{\mathcal{S}}(E)\}$ in $\mathbb{R}^n/\operatorname{span}(E)$, forms a fan. For example, $\Delta_\emptyset = \Delta$, and Δ_E is complete if E is an interior face of $\mathcal{S}$. Consider the standard lattice $\mathbb{Z}^n \subset \mathbb{R}^n$, and let X_E denote the toric variety associated to Δ_E. The torus orbits in X_E are in inclusion reversing bijection with the faces in $\mathrm{lk}_{\mathcal{S}}(E)$. If $E \subset E'$, then X'_E is the closure in X_E of the torus orbit corresponding to the face $E' \smallsetminus E$ of $\mathrm{lk}_{\mathcal{S}}(E)$.

Given a finite simplicial complex $\mathcal{T}$, let $\mathbb{Q}[\mathcal{T}]$ denote the *face ring* of $\mathcal{T}$ over $\mathbb{Q}$, i.e., the quotient of the polynomial ring over $\mathbb{Q}$ with variables corresponding to the vertices of $\mathcal{T}$ by the ideal generated by monomials corresponding to non-faces. For a face $F \in \mathcal{T}$, let $x^F \in \mathbb{Q}[\mathcal{T}]$ denote the product of the variables corresponding to the vertices of F. We write $|G|$ for the number of vertices in a face G. In particular, x^F is a squarefree monomial of degree $|F|$.

Note that $\mathbb{Q}[\mathcal{T}]$ is graded by degree. A *linear system of parameters* (l.s.o.p.) for a finitely generated graded $\mathbb{Q}$-algebra R of Krull dimension d is a sequence of elements $\theta_1, \ldots, \theta_d$ in R_1 such that $R/(\theta_1, \ldots, \theta_d)$ is a finite-dimensional $\mathbb{Q}$-vector space. If $\mathcal{T}$ has dimension $d-1$, then $\mathbb{Q}[\mathcal{T}]$ has Krull dimension d.

Let $c = n - |E|$. Note that c is the Krull dimension of $\mathbb{Q}[\mathrm{lk}_{\mathcal{S}}(E)]$. The *support* of an element $\theta = \sum a_v x^v \in \mathbb{Q}[\mathrm{lk}_{\mathcal{S}}(E)]_1$ is $\operatorname{supp}(\theta) := \{v \colon a_v \neq 0\}$. A linear system of parameters $\theta_1, \ldots, \theta_c$ for $\mathbb{Q}[\mathrm{lk}_{\mathcal{S}}(E)]$ is *special*, as defined in [Sta92, Ath12a] if, for each vertex $v \in [n] \smallsetminus \sigma(E)$, there is an element θ_v of the l.s.o.p. such that $\operatorname{supp}(\theta_v)$ consists of vertices in $\mathrm{lk}_{\mathcal{S}}(E)$ whose carrier contains v, and such that $\theta_v \neq \theta_{v'}$ for $v \neq v'$.

Proposition 4.2.1. [Ath12a, Ath12b]*, see also* [LPS22, Proof of Theorem 1.2] *Let I be the ideal in $\mathbb{Q}[\mathrm{lk}_{\mathcal{S}}(E)]$ generated by $\{x^F : F \sqcup E$ is interior$\}$. Let $L(\mathcal{S}, E)$ be the image of I in $\mathbb{Q}[\mathrm{lk}_{\mathcal{S}}(E)]/(\theta_1, \ldots, \theta_c)$, where $\theta_1, \ldots, \theta_c$ is a special l.s.o.p. Then the Hilbert series of $L(\mathcal{S}, E)$ is $\ell(\mathcal{S}, E; t)$.*

We call $L(\mathcal{S}, E)$ the *local face module*. Note that the local face module depends on the choice of a special l.s.o.p. In this paper, we will consider a particular special l.s.o.p. that is defined in terms of Δ.

Below we view elements of $(\mathbb{Q}^n/\operatorname{span}(E))^* \hookrightarrow (\mathbb{Q}^n)^*$ as $\mathbb{Q}$-linear functions vanishing on $\operatorname{span}(E)$. For $u \in (\mathbb{Q}^n/\operatorname{span}(E))^*$, let $\theta_u = \sum_{v \in \mathrm{lk}_{\mathcal{S}}(E)} \langle u, v \rangle x^v \in \mathbb{Q}[\mathrm{lk}_{\mathcal{S}}(E)]$. Consider the ideal $J_E = (\theta_u : u \in (\mathbb{Q}^n/\operatorname{span}(E))^*)$ in $\mathbb{Q}[\mathrm{lk}_{\mathcal{S}}(E)]$. Note that J_E is generated by a special l.s.o.p., obtained by extending $\{e_i^* \colon i \in [n] \smallsetminus \sigma(E)\}$ to a basis for $(\mathbb{Q}^n/\operatorname{span}(E))^*$.

Let $H^*(E) = \mathbb{Q}[\mathrm{lk}_{\mathcal{S}}(E)]/J_E$. Then $H^*(E)$ is isomorphic to the rational cohomology ring $H^*(X_E, \mathbb{Q})$ of X_E. The ideal in $H^*(E)$ generated by $\{x^F \colon F \in \mathrm{lk}_{\mathcal{S}}(E), F \sqcup E \text{ interior}\}$ is $L(\mathcal{S}, E)$. We will show nonvanishing of $\ell(\mathcal{S},E;t)$ by showing that a certain element of $L(\mathcal{S},E)$ is nonzero. To achieve this, we require the three constructions.

First, if $E \subset E'$ is an inclusion of faces in $\mathcal{S}$, then there is a graded $\mathbb{Q}$-algebra homomorphism $\iota^* = \iota^*_{E,E'} \colon H^*(E) \to H^*(E')$ corresponding to the pullback map on cohomology. The *closed star* $\mathrm{Star}(E' \smallsetminus E)$ of $E' \smallsetminus E$ is the subcomplex of $\mathrm{lk}_{\mathcal{S}}(E)$ that consists of faces H such that $H \cup (E' \smallsetminus E)$ is a face of $\mathrm{lk}_{\mathcal{S}}(E)$. Then ι^* may be characterized as follows: let $v \in \mathrm{lk}_{\mathcal{S}}(E)$. Then

(1) $\iota^*(x^v) = 0$ if $v \notin \mathrm{Star}(E' \smallsetminus E)$,
(2) $\iota^*(x^v) = x^v$ if $v \in \mathrm{lk}_{\mathcal{S}}(E')$.

Note that $\mathrm{Star}(E' \smallsetminus E)$ is the join of $\mathrm{lk}_{\mathcal{S}}(E')$ with $E' \smallsetminus E$. If $v \in E' \smallsetminus E$, then there exists a linear form u_v in $(\mathbb{Q}^n/\operatorname{span}(E))^*$ that takes value 1 on v and vanishes on all other $v' \in E'$, and the above properties imply that $\iota^*(x^v) = -\sum_{v' \in \mathrm{lk}_{\mathcal{S}}(E')} \langle u_v, v' \rangle x^{v'}$.

Second, let $j_* = j_{E',E,*} \colon H^*(E') \to H^*(E)$ be defined by $j_*(x^G) = x^G x^{E' \smallsetminus E}$ for all $G \in \mathrm{lk}_{\mathcal{S}}(E')$, corresponding to the Gysin pushforward map on cohomology. It then follows from the characterization of ι^* via (1) and (2), that $j_* \circ \iota^* \colon H^*(E) \to H^*(E)$ is multiplication by $x^{E' \smallsetminus E}$.

Finally, assume that E is interior. Then X_E is a complete toric variety, and the degree map on top cohomology gives rise to a $\mathbb{Q}$-vector space isomorphism $\deg_E \colon H^c(E) \to \mathbb{Q}$. We have the following explicit description. For a facet H of $\mathcal{S}$, let $m(H)$ be the absolute value of the determinant of the matrix whose columns are the coordinates of the vertices of H in $\mathbb{R}^n$. If G is a facet of $\mathrm{lk}_{\mathcal{S}}(E)$, then $\deg_E(x^G) = 1/m(G \sqcup E)$. If $E \subset E'$, then it follows from the description of the degree map that $\deg_{E'} = \deg_E \circ j_*$. Let $c' = n - |E'|$. Then for any element $z \in H^{c'}(E)$, we compute

$$\deg_E(z x^{E' \smallsetminus E}) = \deg_E(j_* \circ \iota^*(z)) = \deg_{E'}(\iota^*(z)). \tag{10}$$

Remark 4.2.2. If we replace $\{e_i^*\}$ by $\{\lambda_i e_i^*\}$ for some $\lambda_i \in \mathbb{Q}_{>0}$, then the definitions of $H^*(E)$, ι^* and j_* are unaffected, while $\deg_E$ is composed with multiplication by $\prod_{i=1}^n \lambda_i$.

4.3. **Reduction steps.** In this subsection, we reduce Theorem 4.1.3 to an explicit calculation in the top cohomology group of a complete toric variety. We continue with the notation above.

Lemma 4.3.1. *Let $F \in \mathrm{lk}_{\mathcal{S}}(E)$ be a face with a full partition $F = F_1 \sqcup F_2 \sqcup \mathcal{A}_F$. Let $G \in \mathrm{lk}_{\mathcal{S}}(E \sqcup F)$. Consider $F' = F \sqcup G$ in $\mathrm{lk}_{\mathcal{S}}(E)$. Then*

$$F' = F_1' \sqcup F_2' \sqcup \mathcal{A}_{F'}$$

is a full partition, where $\widehat{\mathcal{A}}_F = \{A \in \mathcal{A}_F : V_A \subset \sigma(G)\}$, $F_1' = F_1 \sqcup \widehat{\mathcal{A}}_F$, $F_2' = F_2 \sqcup G$ and $\mathcal{A}_{F'} = \mathcal{A}_F \smallsetminus \widehat{\mathcal{A}}_F$.

Proof. As $F \sqcup E$ is interior and $F \subset F'$, if F' is a pyramid with apex A, then $A \in \mathcal{A}_F$. For any $A \in \mathcal{A}_F$, F' is a pyramid with apex A if and only if $A \notin \widehat{\mathcal{A}}_F$, so $\mathcal{A}_{F'} = \mathcal{A}_F \smallsetminus \widehat{\mathcal{A}}_F$. We have that $(F_1 \sqcup \widehat{\mathcal{A}}_F) \sqcup (\mathcal{A}_F \smallsetminus \widehat{\mathcal{A}}_F) \sqcup E = F_1 \sqcup \mathcal{A}_F \sqcup E$ is interior by the full partition condition on F. The definition of $\widehat{\mathcal{A}}_F$ and the full partition condition on F imply that $\sigma(F_2 \sqcup G \sqcup E) = [n] \smallsetminus \cup_{A \in \mathcal{A}_{F'}} V_A$. □

Remark 4.3.2. Let F be a face of $\mathrm{lk}_{\mathcal{S}}(E)$ such that $F \sqcup E$ is interior. Assume that $F \in \mathrm{lk}_E(\mathcal{S})$ is not a U-pyramid. Then $\operatorname{codim}(F \sqcup E) \geq |\mathcal{A}_F|$, and equality implies that $|V_A| = 2$ for all $A \in \mathcal{A}_F$.

Definition 4.3.3. *Let F be a face of $\mathrm{lk}_{\mathcal{S}}(E)$. We say that F is a* maximal non-U-pyramid *if $F \sqcup E$ is interior, F is not a U-pyramid, and for any $F \subset F' \in \mathrm{lk}_{\mathcal{S}}(E)$ with $F \neq F'$, F' is a U-pyramid.*

Proposition 4.3.4. *Let F be a face of $\mathrm{lk}_{\mathcal{S}}(E)$ that is a maximal non-U-pyramid. Then* $\operatorname{codim}(F \sqcup E) = |\mathcal{A}_F|$.

Proof. By Remark 4.3.2, $\operatorname{codim}(F \sqcup E) \geq |\mathcal{A}_F|$. Assume that $\operatorname{codim}(F \sqcup E) > |\mathcal{A}_F|$. We need to show that there exists $F \subset F'$ with $F \neq F'$, and F' is not a U-pyramid.

We first describe a basis for $(\mathbb{Q}^n/\operatorname{span}(F \sqcup E))^*$. For each $A \in \mathcal{A}_F$, choose an ordering on $V_A = \{i_{A,1}, \ldots, i_{A,|V_A|}\}$. For each $1 \leq j \leq |V_A| - 1$, let $u_{A,j} = \langle e^*_{i_{A,j+1}}, A\rangle e^*_{i_{A,j}} - \langle e^*_{i_{A,j}}, A\rangle e^*_{i_{A,j+1}}$. Then we can find $\{t_k\}$ such that $\{u_{A,j}\} \cup \{t_k\}$ is a basis for $(\mathbb{Q}^n/\operatorname{span}(F \sqcup E))^*$, and $\langle t_k, e_{i_{A,j}}\rangle = 0$ for all choices of A, j, k. We consider the corresponding isomorphism $\phi\colon \mathbb{R}^n/\operatorname{span}(F \sqcup E) \to (\mathbb{R}^n/\operatorname{span}(F \sqcup E))^*$, given by $\phi(V) = \sum_{A,j}\langle u_{A,j}, V\rangle u_{A,j} + \sum_k \langle t_k, V\rangle t_k$.

Consider the complete fan $\Delta_{F\sqcup E}$ given by the images of $\{C_G : G \in \operatorname{lk}_{\mathcal{S}}(F \sqcup E)\}$ in $\mathbb{R}^n/\operatorname{span}(F \sqcup E)$. Given a ray R in $(\mathbb{R}^n/\operatorname{span}(F \sqcup E))^*$, there is a unique element $V \in \mathbb{R}^n$ in the subcomplex $\operatorname{lk}_{\mathcal{S}}(F \sqcup E)$ of $\mathcal{S}$ such that the image of V in $\mathbb{R}^n/\operatorname{span}(F \sqcup E)$ maps via ϕ to an element of R. Let G be the unique nonempty face of $\operatorname{lk}_{\mathcal{S}}(F \sqcup E)$ containing V in its relative interior. Then for each $1 \leq i \leq n$, $\langle e_i^*, V\rangle > 0$ if and only if there exists a vertex W of G such that $\langle e_i^*, W\rangle > 0$. For any choice of R, we may consider the face $F' = F \sqcup G$ of $\operatorname{lk}_{\mathcal{S}}(E)$. It remains to choose R such that F' is not a U-pyramid. There are two cases.

First, suppose that $|V_A| > 2$ for some A. Consider the ray $R := \{\lambda u_{A,|V_A|-1} \colon \lambda \leq 0\}$. For each $A' \neq A$, $\langle u_{A',j}, V\rangle = 0$ for $1 \leq j \leq |V_{A'}|-1$, and hence $\{\langle e^*_{i_{A',j}}, V\rangle\}_{1\leq j\leq |V_{A'}|-1}$ are either all nonzero or all zero. In the latter case, $A' \notin \mathcal{A}_{F'}$. In the former case, $A' \in \mathcal{A}_{F'}$, and $|V_{A'}(F')| = |V_{A'}(F)| \geq 2$. Also, $\langle u_{A,|V_A|-1}, V\rangle < 0$ implies that $\langle e^*_{i_{A,|V_A|}}, V\rangle > 0$, and $\langle u_{A,j}, V\rangle = 0$, for $1 \leq j \leq |V_A| - 2$, implies that $\{\langle e^*_{i_{A,j}}, V\rangle\}_{1\leq j\leq |V_A|-2}$ are either all nonzero or all zero. In the latter case, $A \notin \mathcal{A}_{F'}$. In the former case, $A \in \mathcal{A}_{F'}$, and $|V_A(F')| = |V_A(F)| - 1 \geq 2$. We conclude that F' is not a U-pyramid.

Second, suppose that $|V_A| = 2$ for all A. Then we may consider the ray $R := \{\lambda t_1 \colon \lambda > 0\}$. For each A, $\langle u_{A,j}, V\rangle = 0$ for $1 \leq j \leq |V_A| - 1$, and hence $\{\langle e^*_{i_{A,j}}, V\rangle\}_{1\leq j\leq |V_A|-1}$ are either all nonzero or all zero. In the latter case, $A \notin \mathcal{A}_{F'}$. In the former case, $A \in \mathcal{A}_{F'}$, and $|V_A(F')| = |V_A(F)| \geq 2$. We again conclude that F' is not a U-pyramid. □

The key computation used to prove Theorem 4.1.3 is the following proposition. We explain how it implies Theorem 4.1.3, and then we prove it in the following section.

Proposition 4.3.5. *Let F be a face of $\operatorname{lk}_{\mathcal{S}}(E)$ that is a maximal non-U-pyramid. Then*

$$(-1)^{|\mathcal{A}_F|}\deg_{F\sqcup E}(\iota^*_{E,F\sqcup E}(x^{\mathcal{A}_F})) > 0.$$

Proof of Theorem 4.1.3. Suppose $F \in \operatorname{lk}_{\mathcal{S}}(E)$ admits a full partition $F = F_1 \sqcup F_2 \sqcup \mathcal{A}_F$ and is not a U-pyramid. There exists $G \in \operatorname{lk}_{\mathcal{S}}(F \sqcup E)$ such that $F' = F \sqcup G$ is a maximal non-U-pyramid. By Lemma 4.3.1, F' admits a full partition $F_1' \sqcup F_2' \sqcup \mathcal{A}_{F'}$ with $F_1' = F_1 \sqcup \widehat{\mathcal{A}}_F$, $F_2' = F_2 \sqcup G$, and $\mathcal{A}_{F'} = \mathcal{A}_F \smallsetminus \widehat{\mathcal{A}}_F$ for some subface $\widehat{\mathcal{A}}_F$ of $\mathcal{A}_F$. By the definition of a full partition, $F_1 \sqcup \mathcal{A}_F \sqcup E$ is interior, so $x^{F_1\sqcup\mathcal{A}_F} = x^{F_1'\sqcup\mathcal{A}_{F'}} \in H^*(E)$ is contained in $L(\mathcal{S}, E)$. If we can show that $x^{F_1'\sqcup\mathcal{A}_{F'}} \in H^*(E)$ is nonzero, then $L(\mathcal{S}, E)$ is nonzero in degree $|F_1| + |\mathcal{A}_F|$, as desired.

We apply (10) with E replaced by $F_2' \sqcup \mathcal{A}_F' \sqcup E$, E' replaced by $F' \sqcup E$, and $z = \iota^*_{E,F_2'\sqcup\mathcal{A}_{F'}\sqcup E}(x^{\mathcal{A}_{F'}})$, to obtain

$$\deg_{F_2'\sqcup\mathcal{A}_F'\sqcup E}(\iota^*_{E,F_2'\sqcup\mathcal{A}_{F'}\sqcup E}(x^{\mathcal{A}_{F'}})x^{F_1'}) = \deg_{F_2'\sqcup\mathcal{A}_F'\sqcup E}(\iota^*_{E,F_2'\sqcup\mathcal{A}_{F'}\sqcup E}(x^{F_1'\sqcup\mathcal{A}_{F'}})) = \deg_{F'\sqcup E}(\iota^*_{E,F'\sqcup E}(x^{\mathcal{A}_{F'}})).$$

By Proposition 4.3.5, the right-hand side of the above equation is nonzero, and hence $x^{F_1'\sqcup\mathcal{A}_{F'}}$ is nonzero. □

Remark 4.3.6. The statement of Proposition 4.3.5 has the following geometric interpretation. The refined self-intersection of the compact T-invariant subvariety $X_{F\sqcup E}$ of half-dimension in the toric variety $X_{(F\sqcup E)\smallsetminus\mathcal{A}_F}$ is not zero, and its sign is determined by the number of apices $|\mathcal{A}_F|$.

4.4. **Positivity for maximal non-U-pyramids.** In this section, we prove Proposition 4.3.5. We first fix our setup. Let F be a face of $\operatorname{lk}_{\mathcal{S}}(E)$ that is a maximal non-U-pyramid. Let $r = \operatorname{codim}(F \sqcup E)$. By Remark 4.3.2 and Proposition 4.3.4, we may let $\mathcal{A}_F = \{A_1, \ldots, A_r\}$ and assume that $V_{A_i}(F) = \{2i-1, 2i\}$.

For a vector $V \in \mathbb{R}^n$, let $V_j = \langle e_j^*, V\rangle$. By Remark 4.2.2, we may rescale the first $2r$ coordinates, and assume that $(A_i)_{2i-1} = (A_i)_{2i} = 1$. Recall that for $u \in (\mathbb{Q}^n/\operatorname{span}(E))^*$, we have the equality in $H^*(E)$

$$\theta_u := \sum_{v \in \operatorname{lk}_{\mathcal{S}}(E)} \langle u, v\rangle x^v = 0. \tag{11}$$

We introduce the following notation: given $F \in \operatorname{lk}_{\mathcal{S}}(E)$, we write $y^F := \iota^*_{E,F\sqcup E}(x^F) \in H^*(F \sqcup E)$. Then our goal is to prove that $(-1)^r \deg_{F\sqcup E}(y^{A_1}\cdots y^{A_r}) > 0$. For G a face in $\operatorname{lk}_{\mathcal{S}}(F\sqcup E)$, we define $\operatorname{Supp}_G := \{1 \le j \le 2r : V_j \neq 0 \text{ for some vertex } V \in G\}$. Let $\pi\colon \mathbb{R}^n \to \mathbb{R}^r$ be defined by $\pi(x_1,\dots,x_n) = (x_1 - x_2, x_3 - x_4, \dots, x_{2r-1} - x_{2r})$ (cf. the proof of Proposition 4.3.4). Then π induces an isomorphism $\mathbb{R}^n/\operatorname{span}(F\sqcup E) \cong \mathbb{R}^r$, mapping the complete fan $\Delta_{F\sqcup E}$ to the complete fan with cones $\{\pi(C_G) : G \in \operatorname{lk}_{\mathcal{S}}(F \sqcup E)\}$.

4.4.1. *Motivation for the proof.* In order to help motivate and guide the reader, we give a quick explanation of the proof when $r = 2$, and then explain some of the key ideas for the general case.

Assume that $r = 2$. To compute the degree of $y^{A_1}y^{A_2} \in H^2(F\sqcup E)$, we want to show it is equivalent to a sum $\sum_G \lambda_G y^G$, where G varies over the facets of $\operatorname{lk}_{\mathcal{S}}(F\sqcup E)$, for some $\lambda_G \in \mathbb{Q}$. Then $\deg_{F\sqcup E}(y^{A_1}y^{A_2}) = \sum_G \lambda_G \deg_{F\sqcup E}(y^G)$. This is always possible, but, in general, it will lead to a large sum with positive and negative contributions. Instead, our goal is to arrange that the sum consists of a single term.

Let $\mu = e_1^*$. Applying (11) with $u = \mu$, implies that

$$-y^{A_1}y^{A_2} = \sum_{v \in \operatorname{lk}_{\mathcal{S}}(F\sqcup E)} v_1 y^v y^{A_2}.$$

Consider $v \in \operatorname{lk}_{\mathcal{S}}(F\sqcup E)$ such that $v_1 > 0$. Suppose that $v_2 > 0$. We claim that $y^v y^{A_2} = 0$. Indeed, maximality implies that $F \sqcup \{v\}$ is a U-pyramid. Hence, without loss of generality, we may assume that $v_3 = 0$ and $v_4 > 0$. Applying (11) with $u = e_3^*$, implies that:

$$-y^v y^{A_2} = \sum_{w \in \operatorname{lk}_{\mathcal{S}}(F\sqcup v\sqcup E)} w_3 y^v y^w.$$

If $w_3 > 0$ in any term above, then $F \sqcup v \sqcup w$ is not a U-pyramid, contradicting maximality. Therefore $v_2 = 0$, so $\pi_1(v) = v_1 > 0$. We deduce that

$$-y^{A_1}y^{A_2} = \sum_{\substack{v \in \operatorname{lk}_{\mathcal{S}}(F\sqcup E)\\ \pi_1(v) > 0}} \pi_1(v) y^v y^{A_2}. \tag{12}$$

For $x, y \in \mathbb{R}^n$, let $A(x,y) = \begin{bmatrix} \pi_1(x) & \pi_1(y) \\ x_3 & y_3 \end{bmatrix}$, and $A'(x,y) = \begin{bmatrix} \pi_1(x) & \pi_1(y) \\ \pi_2(x) & \pi_2(y) \end{bmatrix}$. Fix $v \in \operatorname{lk}_{\mathcal{S}}(F\sqcup E)$, and consider the linear function $\mu = \mu_v\colon \mathbb{R}^n \to \mathbb{R}$, defined by $\mu(x) = \det(A(v,x))$. Applying (11) with $u = \mu$ to each term in the right-hand side of (12) implies that

$$y^{A_1}y^{A_2} = \sum_{\substack{v\sqcup w \in \operatorname{lk}_{\mathcal{S}}(F\sqcup E)\\ \pi_1(v) > 0}} \det A(v,w) y^v y^w,$$

where the sum is over ordered pairs (v,w) of distinct vertices such that $v \sqcup w$ is a face in $\operatorname{lk}_{\mathcal{S}}(F\sqcup E)$. Switching the roles of v and w cancels the contributions where $\pi_1(w) > 0$, leaving:

$$y^{A_1}y^{A_2} = \sum_{\substack{v\sqcup w \in \operatorname{lk}_{\mathcal{S}}(F\sqcup E)\\ \pi_1(v) > 0, \pi_1(w) \le 0}} \det A(v,w) y^v y^w.$$

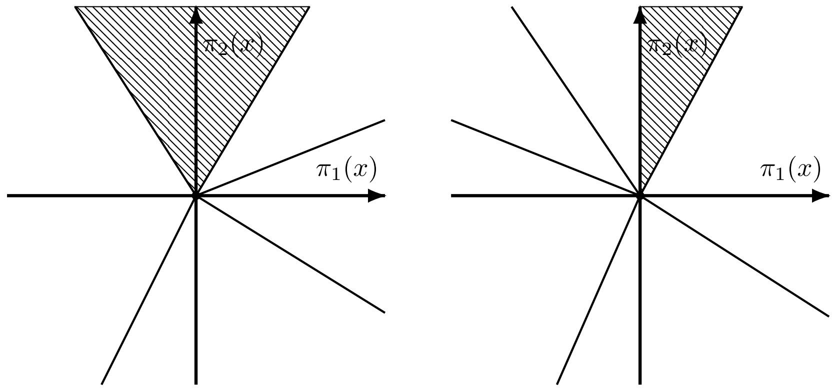

FIGURE 1. Examples of fans and their unique positive maximal cones.

Consider a nonzero term above. Then $\det A(v,w) > 0$. We claim that $\det A(v,w) = \det A'(v,w)$. Indeed, if $\pi_1(w) < 0$, then since $F \sqcup v \sqcup w$ is not a U-pyramid, $v_4 = w_4 = 0$ because v_3 and w_3 cannot both vanish. If $\pi_1(w) = 0$, then $\det A(v,w) = \pi_1(v)w_3$. Since $F \sqcup w$ is not a U-pyramid, $w_4 = 0$. In both cases, the claim follows. We then deduce:

$$y^{A_1}y^{A_2} = \sum_{\substack{v\sqcup w \in \mathrm{lk}_{\mathcal{S}}(F\sqcup E)\\ \pi_1(v)>0, \pi_1(w)\le 0\\ \det A'(v,w)>0}} \det A'(v,w) y^v y^w.$$

There is a unique facet G in $\mathrm{lk}(F \sqcup E)$ such that $G = v \sqcup w$ appears in the summation above. Explicitly, G is the unique facet such that the cone $\pi(C_G)$ contains $(0,1) \in \mathbb{R}^2$, and the projection of $\pi(C_G)$ onto the first coordinate contains $1 \in \mathbb{R}$. Equivalently, G is the unique facet such that $\pi(C_G)$ contains $(\epsilon, 1)$ for $0 < \epsilon \ll 1$. We call a cone *positive* if it satisfies these equivalent conditions. See Figure 1. We deduce that $y^{A_1}y^{A_2} = \mathrm{mult}(G)y^G$, where $\mathrm{mult}(G) = \det A'(v,w) = |\det A'(v,w)| > 0$ is the multiplicity of C_G. Since $\deg_{F\sqcup E}(y^G) > 0$ by definition, this completes the proof of Proposition 4.3.5.

We next discuss the general case. Below, we will generalize and extend both the notion of positive cones, and the expansion techniques above. We aim to show that $(-1)^r \deg_{F\sqcup E}(y^{A_1}\cdots y^{A_r}) = \mathrm{mult}(G)y^G$, where G is the unique facet in $\mathrm{lk}_{\mathcal{S}}(F \sqcup E)$ such that $\pi(C_G)$ is a positive cone. As above, this will complete the proof of Proposition 4.3.5.

The final obstruction is not visible in the $r = 2$ case. Namely, if one follows the techniques described above when $r > 2$, error terms naturally appear that must be shown to vanish. One can apply similar techniques to expand the error terms, and secondary error terms naturally appear that also must be shown to vanish. Continuing in this way, we obtain an infinite series of error terms that we must show vanish. This leads one to set up a more general problem involving all the terms we wish to analyze. More specifically, we introduce the elements of interest in Definition 4.4.25, and our main result is to compute them in Proposition 4.4.31. As a special case, $(-1)^r \deg_{F\sqcup E}(y^{A_1}\cdots y^{A_r}) = \mathrm{mult}(G)y^G$, as desired.

Additional notation. We introduce the following notation for use in the proof. Let S, T be (possibly empty) ordered sets that are subsets of $[r]$. Suppose we have an inclusion of sets $S \subset T$. Given an ordered set $U = \{u_{i_j}\}_{i_j \in T}$ indexed by T, we write $U|_S := \{v_{i_j}\}_{i_j \in S}$. If f is a function defined on the elements of U, we write $f(U) = \{f(u_{i_j})\}_{i_j \in T}$. Let $\mathbb{R}^S, \mathbb{R}^T$ denote real vector spaces with coordinates indexed by S, T respectively, and let $\mathrm{pr}_{T,S} \colon \mathbb{R}^T \to \mathbb{R}^S$ denote the associated projection map. If $S = \emptyset$, then $\mathbb{R}^S = \{0\}$. If U

is an ordered set of elements of $\mathbb{R}^S$, indexed by T, then let A_U to be the $|S| \times |T|$ matrix with columns given by the elements of U, ordered by T, and rows indexed by S.

4.4.2. *Positive cones.* We now study the combinatorics of certain cones which will play an important role in the sequel. Throughout this section, $S = \{i_1 < \cdots < i_s\}$ will be a (possibly empty) ordered set of $s = |S|$ positive integers, and $i_0 = 0$.

Definition 4.4.1. *We say that C is* positive *if $(0, \ldots, 0, 1) \in \mathrm{pr}_{S,S\cap[i_k]}(C) \subset \mathbb{R}^{S\cap[i_k]}$ for $1 \leq k \leq s$.*

For example, when $S = \emptyset$, $C = \mathbb{R}^S = \{0\}$ is positive. For $\epsilon > 0$, let $\vec{\epsilon}_S := (\epsilon^{-i_1}, \epsilon^{-i_2}, \ldots, \epsilon^{-i_s}) \in \mathbb{R}^S$. If $S = \emptyset$, let $\vec{\epsilon}_S = 0 \in \mathbb{R}^S = \{0\}$.

Remark 4.4.2. Let $W \subset \mathbb{R}^S$ be a finite union of proper affine subspaces. Since $\{\epsilon_S : \epsilon > 0\} \cap W$ is finite, it follows that $\epsilon_S \cap W = \emptyset$ for $\epsilon > 0$ sufficiently small.

Lemma 4.4.3. *A cone C is positive if and only if $\vec{\epsilon}_S \in C$ for all $\epsilon > 0$ sufficiently small.*

Proof. Suppose $\vec{\epsilon}_S \in C$ for all $\epsilon > 0$ sufficiently small. For $1 \leq k \leq s$, we compute in $\mathbb{R}^{S\cap[i_k]}$:

$$(0, \ldots, 0, 1) = \lim_{\epsilon\to 0} \epsilon^{i_k} \vec{\epsilon}_{S\cap[i_k]} = \lim_{\epsilon\to 0} \mathrm{pr}_{S,S\cap[i_k]}(\epsilon^{i_k}\vec{\epsilon}_S) \in \mathrm{pr}_{S,S\cap[i_k]}(C).$$

Hence, C is positive. Conversely, assume that C is positive. By hypothesis, there exists an ordered set $U = \{u_{i_1}, \ldots, u_{i_s}\}$ of elements of C such that $\mathrm{pr}_{S,S\cap[i_k]}(u_{i_k}) = (0, \ldots, 0, 1)$ for $1 \leq k \leq s$. The associated matrix A_U is lower triangular with 1's on the diagonal. Since A_U^{-1} is also lower diagonal with 1's on the diagonal, $\lim_{\epsilon\to 0} \epsilon^{i_k}(A_U^{-1}\vec{\epsilon}_S)_{i_k} = 1$, and hence, for ϵ sufficiently small, $A_U^{-1}\vec{\epsilon}_S$ has positive entries, and $\vec{\epsilon}_S = A_U(A_U^{-1}\vec{\epsilon}_S) \in C$. □

In particular, positive cones are full-dimensional, and every complete polyhedral fan contains a unique positive cone. The goal for the remainder of this section is to develop an alternative inductive criterion for C to be positive, Proposition 4.4.14. We will use the following linear algebra lemma.

Lemma 4.4.4. *Let $m, n \in \mathbb{Z}_{\geq 0}$ with $p\colon \mathbb{R}^m \to \mathbb{R}^n$ a linear map. Let $C \subset \mathbb{R}^m$ be a simplicial cone, and let $\widehat{C}$ be the smallest face of C containing $C \cap \ker(p)$. Then $p(\widehat{C})$ is the largest linear subspace contained in $p(C)$.*

Moreover, if $\mathrm{span}(C \cap \ker(p)) = \mathrm{span}(C) \cap \ker(p)$, then $p(C)/p(\widehat{C}) \subset \mathbb{R}^n/p(\widehat{C})$ is a simplicial cone of dimension $\mathrm{codim}(\widehat{C}, \mathrm{span}(C))$. The faces of $p(C)$ are $\{p(C') : \widehat{C} \subset C' \subset C\}$, and $\mathrm{codim}(p(C'), \mathrm{span}(p(C))) = \mathrm{codim}(C', \mathrm{span}(C))$, for $\widehat{C} \subset C' \subset C$.

Proof. Replacing $\mathbb{R}^m$ with $\mathrm{span}(C)$ and $\mathbb{R}^n$ with $\mathrm{span}(p(C))$, we reduce to the case when C and $p(C)$ are full-dimensional. The result holds when $m = 0$. Assume that $m > 0$.

Let $u_1, \ldots, u_m$ be generators of the rays of C. Let L be the largest linear subspace contained in $p(C)$. Suppose $x \in L$. Then $x = \sum_{i=1}^m \alpha_i p(u_i)$ for some $\alpha_i \geq 0$, and $-x = \sum_{i=1}^m \beta_i p(u_i)$ for some $\beta_i \geq 0$. Hence $\sum_{i=1}^m (\alpha_i + \beta_i)u_i \in C \cap \ker(p)$. If $\alpha_i > 0$, then $u_i \in \widehat{C}$, and hence $x \in p(\widehat{C})$.

Conversely, suppose $u_j \in \widehat{C}$. Then there exists $y = \sum_{i=1}^m \alpha_i u_i \in C \cap \ker(p)$, for some $\alpha_i \geq 0$ such that $\alpha_j > 0$. Then $-p(u_j) = (1/\alpha_j)\sum_{i\neq j} \alpha_i p(u_i) \in p(C)$, and hence $p(u_j) \in L$. We conclude that $p(\widehat{C}) \subset L$.

Assume that $\mathrm{span}(C \cap \ker(p)) = \ker(p)$. Then the restriction $p|_{\mathrm{span}(\widehat{C})}\colon \mathrm{span}(\widehat{C}) \to \mathrm{span}(p(\widehat{C}))$ is surjective with kernel $\ker(p)$, and hence $\dim p(\widehat{C}) = \dim \widehat{C} - \dim \ker(p)$. Then $\mathrm{codim}(\widehat{C}, \mathbb{R}^m) = \mathrm{codim}(p(\widehat{C}), \mathbb{R}^n)$. Since $p(C)/p(\widehat{C})$ is spanned by the images of the $\mathrm{codim}(\widehat{C}, \mathbb{R}^m)$ rays in $C \smallsetminus \widehat{C}$, we deduce that $p(C)/p(\widehat{C})$ is simplicial. The final statement about the faces of $p(C)$ follows. □

Definition 4.4.5. *Assume that C is simplicial. For $0 \leq k \leq s$, let $C^{(i_k)}$ be the smallest face of C containing $C \cap \ker(\mathrm{pr}_{S,S\cap[i_k]})$.*

For example, $\ker(\mathrm{pr}_{S,S\cap[i_0]}) = \mathbb{R}^S$ and $C^{(i_0)} = C$. Also, $C^{(i_s)} = \{0\}$.

Remark 4.4.6. It follows from Lemma 4.4.4 that if C is simplicial, then $\mathrm{pr}_{S,S\cap[i_k]}(C^{(i_k)})$ is the largest linear subspace contained in $\mathrm{pr}_{S,S\cap[i_k]}(C)$.

Corollary 4.4.7. *Assume C is simplicial. For $1 \leq j \leq k \leq s$, $\mathrm{pr}_{S,\{i_j\}}(C^{(i_k)}) \subset \mathbb{R}^{\{i_j\}}$ equals $\{0\}$ or $\mathbb{R}^{\{i_j\}}$.*

Proof. Note that $\mathrm{pr}_{S,\{i_j\}}$ factors through $\mathrm{pr}_{S,S\cap[i_k]}$, and, by Remark 4.4.6, $\mathrm{pr}_{S,S\cap[i_k]}(C^{(i_k)})$ is a linear space. This implies that $\mathrm{pr}_{S,\{i_j\}}(C^{(i_k)})$ is a linear space. □

Lemma 4.4.8. *Assume that C is full-dimensional and simplicial. Fix $1 \leq k \leq s$. Assume that $(0, \dots, 0, 1) \in \mathrm{pr}_{S,S\cap[i_j]}(C) \subset \mathbb{R}^{S\cap[i_j]}$ for $j > k$. Then $\mathrm{pr}_{S,S\cap[i_k]}(C)/\mathrm{pr}_{S,S\cap[i_k]}(C^{(i_k)})$ is a simplicial cone of dimension $\mathrm{codim}(C^{(i_k)}, \mathbb{R}^S)$. The faces of $\mathrm{pr}_{S,S\cap[i_k]}(C)$ are*

$$\{\mathrm{pr}_{S,S\cap[i_k]}(C') : C^{(i_k)} \subset C' \subset C\},$$

and $\mathrm{codim}(\mathrm{pr}_{S,S\cap[i_k]}(C'), \mathbb{R}^{S\cap[i_k]}) = \mathrm{codim}(C', \mathbb{R}^S)$, for $C^{(i_k)} \subset C' \subset C$.

Proof. Consider an ordered set $U = \{u_{i_{k+1}}, \dots, u_{i_s}\}$ of elements of C such that $\mathrm{pr}_{S,S\cap[i_j]}(u_{i_j}) = (0, \dots, 0, 1)$ for $k < j \leq s$. Then U is a basis of $\ker(\mathrm{pr}_{S,S\cap[i_k]})$ contained in C, so $\mathrm{span}(C \cap \ker(\mathrm{pr}_{S,S\cap[i_k]})) = \mathrm{span}(C) \cap \ker(\mathrm{pr}_{S,S\cap[i_k]})$. The result then follows from Lemma 4.4.4. □

Lemma 4.4.9. *Assume that C is full-dimensional and simplicial. Fix $1 < k \leq s$. Assume that $(0, \dots, 0, 1) \in \mathrm{pr}_{S,S\cap[i_j]}(C) \subset \mathbb{R}^{S\cap[i_j]}$ for $j > k$. Consider the projection map $p_k := \mathrm{pr}_{S\cap[i_k],S\cap[i_{k-1}]} : \mathbb{R}^{S\cap[i_k]} \to \mathbb{R}^{S\cap[i_{k-1}]}$. Then, for $\epsilon > 0$ sufficiently small, there is a bijection*

$$\psi_\epsilon \colon \partial(\mathrm{pr}_{S,S\cap[i_k]}(C)) \cap p_k^{-1}(p_k(\vec{\epsilon}_{S\cap[i_k]})) \to \{C^{(i_k)} \subset C' \not\subset C : \mathrm{codim}(C', \mathbb{R}^S) = 1, \mathrm{pr}_{S,S\smallsetminus\{i_k\}}(C') \text{ is positive}\},$$

such that $x \in \partial(\mathrm{pr}_{S,S\cap[i_k]}(C)) \cap p_k^{-1}(p_k(\vec{\epsilon}_{S\cap[i_k]}))$ lies in the relative interior of the face $\mathrm{pr}_{S,S\cap[i_k]}(\psi_\epsilon(x)) \subset \mathrm{pr}_{S,S\cap[i_k]}(C)$, and $\{\psi_\epsilon^{-1}(C')\} = \mathrm{pr}_{S,S\cap[i_k]}(C') \cap p_k^{-1}(p_k(\vec{\epsilon}_{S\cap[i_k]}))$.

Proof. Applying Lemma 4.4.8, we have that the faces of $\mathrm{pr}_{S,S\cap[i_k]}(C)$ are $\{\mathrm{pr}_{S,S\cap[i_k]}(C') : C^{(i_k)} \subset C' \subset C\}$, and $\mathrm{codim}(\mathrm{pr}_{S,S\cap[i_k]}(C'), \mathbb{R}^{S\cap[i_k]}) = \mathrm{codim}(C', \mathbb{R}^S)$, for $C^{(i_k)} \subset C' \subset C$. It follows from the assumption that $(0, \dots, 0, 1) \in \mathrm{pr}_{S,S\cap[i_j]}(C^{(i_k)}) \subset \mathbb{R}^{S\cap[i_j]}$ for $j > k$. If $C^{(i_k)} \subset C'$, then it follows from Definition 4.4.1 that $\mathrm{pr}_{S,S\smallsetminus\{i_k\}}(C')$ is positive if and only if $\mathrm{pr}_{S,S\cap[i_{k-1}]}(C') = p_k(\mathrm{pr}_{S,S\cap[i_k]}(C'))$ is positive.

Suppose $C^{(i_k)} \subset C' \not\subset C$, and $x \in \mathrm{pr}_{S,S\cap[i_k]}(C')$. Then $x \in p_k^{-1}(p_k(\vec{\epsilon}_{S\cap[i_k]}))$ if and only if $p_k(x) = p_k(\vec{\epsilon}_{S\cap[i_k]}) = \vec{\epsilon}_{S\cap[i_{k-1}]}$. By Remark 4.4.2, for $\epsilon > 0$ sufficiently small, $\vec{\epsilon}_{S\cap[i_{k-1}]} \in p_k(\mathrm{pr}_{S,S\cap[i_k]}(C'))$ implies that $p_k(\mathrm{pr}_{S,S\cap[i_k]}(C')) \subset \mathbb{R}^{S\cap[i_{k-1}]}$ is full-dimensional, and hence $\mathrm{codim}(\mathrm{pr}_{S,S\cap[i_k]}(C'), \mathbb{R}^{S\cap[i_k]}) = \mathrm{codim}(C', \mathbb{R}^S) = 1$, and $\mathrm{pr}_{S,S\cap[i_k]}(C') \cap p_k^{-1}(p_k(\vec{\epsilon}_{S\cap[i_k]}))$ is a single point. By Lemma 4.4.3, we deduce that, for $\epsilon > 0$ sufficiently small, $\partial(\mathrm{pr}_{S,S\cap[i_k]}(C)) \cap p_k^{-1}(p_k(\vec{\epsilon}_{S\cap[i_k]}))$ is a union of points $\{x_{\epsilon,C'}\}$ indexed by the codimension 1 faces $C' \subset C$ such that $p_k(\mathrm{pr}_{S,S\cap[i_k]}(C'))$ is positive, where $x_{\epsilon,C'}$ lies in the relative interior of $\mathrm{pr}_{S,S\cap[i_k]}(C')$. □

We continue with the assumptions and notation of Lemma 4.4.9. More specifically, $S = \{i_1 < i_2 < \cdots < i_s\}$ is an ordered set, $C \subset \mathbb{R}^S$ is a full dimensional, simplicial cone, $1 < k \leq s$, and $(0, \dots, 0, 1) \in \mathrm{pr}_{S,S\cap[i_j]}(C) \subset \mathbb{R}^{S\cap[i_j]}$ for $j > k$. Then for $\epsilon > 0$ sufficiently small, there are 5 possibilities for $\mathrm{pr}_{S,S\cap[i_k]}(C) \cap p_k^{-1}(p_k(\vec{\epsilon}_{S\cap[i_k]}))$:

(1) $\emptyset$,
(2) $x_\epsilon + \mathbb{R}_{\geq 0}(0, \dots, 0, 1)$,

(3) $x_\epsilon + \mathbb{R}_{\geq 0}(0,\ldots,0,-1)$,
(4) $[x_\epsilon, y_\epsilon]$,
(5) $\mathrm{pr}_{S,S\cap[i_k]}(C) \cap p_k^{-1}(p_k(\vec{\epsilon}_{S\cap[i_k]})) \cong \mathbb{R}$,

for some $x_\epsilon, y_\epsilon \in \mathbb{R}^{S\cap[i_k]}$. The corresponding form of $\partial(\mathrm{pr}_{S,S\cap[i_k]}(C))\cap p_k^{-1}(p_k(\vec{\epsilon}_{S\cap[i_k]}))$ is $\emptyset$,$\{x_\epsilon\}$,$\{x_\epsilon\}$,$\{x_\epsilon, y_\epsilon\}$, and $\emptyset$ respectively. Using the bijection ψ_ϵ, the number of codimension 1 faces $C^{(i_k)} \subset C' \not\subset C$ such that $\mathrm{pr}_{S,S\smallsetminus\{i_k\}}(C')$ is positive equals 0,1,1,2, and 0 respectively.

In particular, given a codimension 1 face $C^{(i_k)} \subset C' \not\subset C$ such that $\mathrm{pr}_{S,S\smallsetminus\{i_k\}}(C')$ is positive, we can define an 'orientation' $\sigma(C') \in \{-1,1\}$ by the condition that $(\psi_\epsilon^{-1}(C') + \mathbb{R}_{>0}(0,\ldots,0,\sigma(C'))) \cap \mathrm{pr}_{S,S\cap[i_k]}(C) \neq \emptyset$. For any other proper face $C^{(i_k)} \subset C' \not\subset C$, let $\sigma(C') = 0$. Let $\zeta_{C,k} := \sum_{C^{(i_k)}\subset C'\not\subset C} \sigma(C')$.

Lemma 4.4.10. *Assume that C is full-dimensional and simplicial. Fix $1 < k \leq s$. Assume that $(0,\ldots,0,1) \in \mathrm{pr}_{S,S\cap[i_j]}(C) \subset \mathbb{R}^{S\cap[i_j]}$ for $j > k$. Then $\zeta_{C,k} \in \{-1,0,1\}$, and $\zeta_{C,k} \neq 0$ if and only if there is a unique $C^{(i_k)} \subset C' \not\subset C$ such that $\mathrm{pr}_{S,S\smallsetminus\{i_k\}}(C')$ is positive.*

Moreover, the following are equivalent:

(1) $\zeta_{C,k} = 1$.
(2) $\zeta_{C,k} \neq 0$ and $(0,\ldots,0,1) \in \mathrm{pr}_{S,S\cap[i_k]}(C)$.
(3) $\zeta_{C,k} \neq 0$ and $(0,\ldots,0,-1) \notin \mathrm{pr}_{S,S\cap[i_k]}(C)$.
(4) C is positive and $(0,\ldots,0,-1) \notin \mathrm{pr}_{S,S\cap[i_k]}(C)$.

Proof. From the above discussion and the definition of σ, $\zeta_{C,k} \neq 0$ if and only if $\mathrm{pr}_{S,S\cap[i_k]}(C)\cap p_k^{-1}(p_k(\vec{\epsilon}_{S\cap[i_k]}))$ has the form either $x_\epsilon + \mathbb{R}_{\geq 0}(0,\ldots,0,1)$ or $x_\epsilon + \mathbb{R}_{\geq 0}(0,\ldots,0,-1)$ for some x_ϵ, which holds if and only if the number of codimension 1 faces $C^{(i_k)} \subset C' \not\subset C$ such that $\mathrm{pr}_{S,S\smallsetminus\{i_k\}}(C')$ is positive equals 1. Note also that, since $(0,\ldots,0,1) \in \mathrm{pr}_{S,S\cap[i_j]}(C) \subset \mathbb{R}^{S\cap[i_j]}$ for $j > k$, C is positive if and only if $\mathrm{pr}_{S,S\cap[i_k]}(C)$ is positive.

In particular, $\zeta_{C,k} = 1$ if and only if $\mathrm{pr}_{S,S\cap[i_k]}(C) \cap p_k^{-1}(p_k(\vec{\epsilon}_{S\cap[i_k]}))$ has the form $x_\epsilon + \mathbb{R}_{\geq 0}(0,\ldots,0,1)$, for some x_ϵ. If $\zeta_{C,k} \neq 0$, then the latter condition holds if and only if $(0,\ldots,0,1) \in \mathrm{pr}_{S,S\cap[i_k]}(C)$ if and only if $(0,\ldots,0,-1) \notin \mathrm{pr}_{S,S\cap[i_k]}(C)$. That is, conditions (1)-(3) are equivalent.

Assume that (4) holds. Then $\mathrm{pr}_{S,S\cap[i_k]}(C)$ is positive, so, by Lemma 4.4.3, $\epsilon_{S\cap[i_k]} \in \mathrm{pr}_{S,S\cap[i_k]}(C) \cap p_k^{-1}(p_k(\vec{\epsilon}_{S\cap[i_k]}))$, for $\epsilon > 0$ sufficiently small. By the definition of positive, $(0,\ldots,0,1) \in \mathrm{pr}_{S,S\cap[i_k]}(C)$. Hence $\mathrm{pr}_{S,S\cap[i_k]}(C) \cap p_k^{-1}(p_k(\vec{\epsilon}_{S\cap[i_k]}))$ contains $\epsilon_{S\cap[i_k]} + \mathbb{R}_{\geq 0}(0,\ldots,0,1)$. Since $(0,\ldots,0,-1) \notin \mathrm{pr}_{S,S\cap[i_k]}(C)$, $\mathrm{pr}_{S,S\cap[i_k]}(C) \cap p_k^{-1}(p_k(\vec{\epsilon}_{S\cap[i_k]}))$ has the form $x_\epsilon + \mathbb{R}_{\geq 0}(0,\ldots,0,1)$, for some x_ϵ, and (1)-(3) hold.

Conversely, suppose (1)-(3) hold. By assumption, $(0,\ldots,0,1) \in \mathrm{pr}_{S,S\cap[i_k]}(C)$. There is a unique codimension 1 face $C^{(i_k)} \subset C' \not\subset C$ such that $\mathrm{pr}_{S,S\smallsetminus\{i_k\}}(C')$ is positive. Since $\mathrm{pr}_{S,S\smallsetminus\{i_k\}}(C')$ is positive,

$$(0,\ldots,0,1) \in \mathrm{pr}_{S\smallsetminus\{i_k\},S\cap[i_j]}(\mathrm{pr}_{S,S\smallsetminus\{i_k\}}(C')) \subset \mathrm{pr}_{S,S\cap[i_j]}(C) \subset \mathbb{R}^{S\cap[i_j]}$$

for $1 \leq j \leq k-1$. We conclude that $\mathrm{pr}_{S,S\cap[i_k]}(C)$ is positive, and (4) holds. □

Let $U = \{u_{i_1},\ldots,u_{i_s}\}$ be an ordered set of elements of $\mathbb{R}^S$. Then U is *positively ordered* if, for all $1 \leq k \leq s$, $\{\mathrm{pr}_{S,S\cap[i_k]}(u_j) : 1 \leq j \leq k\}$ span a positive cone in $\mathbb{R}^{S\cap[i_k]}$. Given a permutation $w \in \mathrm{Sym}_s$ and an ordered set $U = \{u_{i_1},\ldots,u_{i_s}\}$, we write $w(U) = \{u_{i_{w(1)}},\ldots,u_{i_{w(s)}}\}$.

Lemma 4.4.11. *Assume that C is simplicial and positive. Let $U = \{u_{i_1},\ldots,u_{i_s}\}$ be an ordered set of generators of the rays of C. Then there exists a unique permutation $w \in \mathrm{Sym}_s$ such that $w(U)$ is positively ordered.*

Proof. We may assume that $s > 1$. By Lemma 4.4.10, there exists a unique codimension 1 face $C' \subset C$ such that $\mathrm{pr}_{S,S\cap[i_{s-1}]}(C')$ is positive. Let $u_{i_{w(s)}}$ be the unique element of U in $C \smallsetminus C'$. Then C' is also simplicial, so the result follows by induction. □

Lemma 4.4.12. *Assume that C is simplicial and positive. Fix a positively ordered set of generators $U = \{u_{i_1}, \dots, u_{i_s}\}$ of the rays of C. Then $\det(A_U) > 0$.*

Proof. The assumptions imply that there are $|S| \times |S|$ matrices $\mathcal{U}_U$ and $\mathcal{L}_U$ such that $A_U \mathcal{U}_U^{-1} = \mathcal{L}_U$, $\mathcal{U}_U^{-1}$ has nonnegative entries and is upper triangular, and $\mathcal{L}_U$ is lower triangular with all 1s on the diagonal. □

Assume that C is full-dimensional and simplicial. Let $U = \{u_{i_1}, \dots, u_{i_s}\}$ be an ordered set of generators of the rays of C. Fix $1 \leq k \leq s$. Then U is *i_k-weakly positively ordered* or i_k-WPO if the ordered set $\mathrm{pr}_{S,S\smallsetminus\{i_k\}}(U|_{S\smallsetminus\{i_k\}})$ of elements in $\mathbb{R}^{S\smallsetminus\{i_k\}}$ is positively ordered, and $u_{i_k} \notin C^{(i_k)}$. By Lemma 4.4.11, there is a bijection between permutations $w \in \mathrm{Sym}_s$ such that $w(U)$ is i_k-WPO, and codimension 1 faces $C^{(i_k)} \subset C' \not\subset C$ such that $\mathrm{pr}_{S,S\smallsetminus\{i_k\}}(C')$ is positive, where $w(U)|_{S\smallsetminus\{i_k\}}$ is a positively ordered set of generators of the rays of C', and $u_{i_{w(k)}}$ generates the unique ray in $C \smallsetminus C'$.

We say that a nonzero element $x \in \mathbb{R}$ has sign $x/|x| \in \{-1, 1\}$.

Lemma 4.4.13. *Assume that C is full-dimensional and simplicial. Fix $1 < k \leq s$. Assume that $(0, \dots, 0, 1) \in \mathrm{pr}_{S,S\cap[i_j]}(C) \subset \mathbb{R}^{S\cap[i_j]}$ for $j > k$. Let $U = \{u_{i_1}, \dots, u_{i_s}\}$ be an ordered set of generators of the rays of C. Assume that U is i_k-WPO, and $U|_{S\smallsetminus\{i_k\}}$ are a set of generators of the rays of $C^{(i_k)} \subset C' \not\subset C$. Then $\det(A_U)$ has sign $\sigma(C') \in \{-1, 1\}$.*

Proof. Let $U' = \mathrm{pr}_{S,S\smallsetminus\{i_k\}}(U|_{S\smallsetminus\{i_k\}})$. Since U' is positively ordered, $\det(A_{U'}) > 0$ by Remark 4.4.12. Since $(0, \dots, 0, 1) \in \mathrm{pr}_{S,S\cap[i_j]}(C) \subset \mathbb{R}^{S\cap[i_j]}$ for $j > k$, we can replace u_{i_k} by adding an element of $\ker(\mathrm{pr}_{S,S\cap[i_k]}(C))$ without changing $\det(A_U)$. By Lemma 4.4.8, $\mathrm{pr}_{S,S\cap[i_k]}(C')$ has codimension 1 in $\mathbb{R}^{S\cap[i_k]}$. It follows that we can write

$$\mathrm{pr}_{S,S\cap[i_k]}(u_{i_k}) = \alpha(0, \dots, 0, 1) \in \mathbb{R}^{S\cap[i_k]}/\operatorname{span}(\mathrm{pr}_{S,S\cap[i_k]}(C')),$$

for some $\alpha \neq 0$ with sign $\sigma(C')$. The result follows since $\det(A_U) = \alpha \det(A_{U'})$. □

The following result is the main conclusion of our study of positive cones.

Proposition 4.4.14. *Assume C is full-dimensional and simplicial. Let $U = \{u_{i_1}, \dots, u_{i_s}\}$ be an ordered set of generators of the rays of C. Fix $1 < k \leq s$, and assume $(0, \dots, 0, 1) \in \mathrm{pr}_{S,S\cap[i_j]}(C)$ for $j > k$. Let*

$$\lambda(U) := \sum_{\substack{w \in \mathrm{Sym}_s \\ w(U) \text{ is } i_k-WPO}} \det(A_{w(U)})$$

Then $\lambda(U) \neq 0$ if and only if there is a unique permutation $w \in \mathrm{Sym}_s$ such that $w(U)$ is i_k-WPO. Moreover, the following are equivalent:

(1) $\lambda(U) > 0$
(2) $\lambda(U) \neq 0$ and $(0, \dots, 0, 1) \in \mathrm{pr}_{S,S\cap[i_k]}(C)$.
(3) $\lambda(U) \neq 0$ and $(0, \dots, 0, -1) \notin \mathrm{pr}_{S,S\cap[i_k]}(C)$.
(4) C is positive and $(0, \dots, 0, -1) \notin \mathrm{pr}_{S,S\cap[i_k]}(C)$.

Proof. By the above discussion, there is a bijection between permutations $w \in \mathrm{Sym}_s$ such that $w(U)$ is i_k-WPO, and codimension 1 faces $C^{(i_k)} \subset C' \not\subset C$ such that $\mathrm{pr}_{S,S\smallsetminus\{i_k\}}(C')$ is positive. By Lemma 4.4.13, if w corresponds to C' under this bijection, then $\det(A_{w(U)}) = \sigma(C')|\det(A_U)|$. Hence $\lambda(U) = |\det(A_U)|\zeta_{C,k}$. The result now follows from Lemma 4.4.10. □

4.4.3. *Expansion.* We now develop some tools that will help us expand $y^{A_1}y^{A_2}\cdots y^{A_r} \in H^*(F \sqcup E)$ as a sum of squarefree products of vertices of $\mathrm{lk}_{\mathcal{S}}(F \sqcup E)$.

Lemma 4.4.15. *Let $S \subset [r]$ be a nonempty subset, and let G be a face of $\mathrm{lk}_{\mathcal{S}}(F \sqcup E)$. Assume that $\{2j-1, 2j\} \subset \mathrm{Supp}_G$ for all $j \in S$. Then $y^G(\prod_{j \notin S} y^{A_j}) = 0$ in $H^*(F \sqcup E)$.*

Proof. The proof proceeds by induction on $r - |S|$. We first show that the assumption that F is a maximal non-U-pyramid means that $r - |S| = 0$ is impossible. Indeed, there exists $1 \leq i \leq r$ such that precisely one of $\{2i-1, 2i\}$ lies in Supp_G. In particular, $i \notin S$, and hence $S \neq [r]$ and $r - |S| > 0$. Without loss of generality, assume that $2i \in \mathrm{Supp}_G$ and $2i - 1 \notin \mathrm{Supp}_G$. We apply (11) with the global linear function e_{2i-1} to get the relation in $H^*(E)$:

$$x^{A_i} + \sum_{\substack{v \in \mathrm{lk}_{\mathcal{S}}(E) \\ v \notin \{A_1, \dots, A_r\}}} v_{2i-1} x^v = 0.$$

Let $v \in \mathrm{lk}_{\mathcal{S}}(E)$ be a vertex with $v_{2i-1} \neq 0$ appearing in the above sum. Since A_i is an apex of $F \sqcup G \sqcup E$ with base direction e^*_{2i-1}, it follows that $v \notin F \sqcup G \sqcup E$, and $y^v y^G = 0$ in $H^*(F \sqcup E)$ unless $G \sqcup v$ is a face in $\mathrm{link}_{\mathcal{S}}(F \sqcup E)$. In the latter case, $\{2i-1, 2i\} \in \mathrm{Supp}_{G \sqcup v}$. We now compute in $H^*(F \sqcup E)$:

$$y^G(\prod_{j \notin S} y^{A_j}) = -\sum_{v \in \mathrm{lk}_{\mathcal{S}}(F \sqcup E)} v_{2i-1} y^v y^G(\prod_{j \notin S \cup \{i\}} y^{A_j}).$$

Finally, the induction hypothesis implies that if $v_{2i-1} \neq 0$, then $y^{G \sqcup v}(\prod_{j \notin S \cup \{i\}} y^{A_j}) = 0$. □

Definition 4.4.16. *Let $R \subset [r]$. Let $\pi_R \colon \mathbb{R}^n \to \mathbb{R}^r$ be the linear function defined by*

$$\pi_R(x)_i = \begin{cases} \pi(x)_i = x_{2i-1} - x_{2i} & \text{if } i \notin R, \\ x_{2i-1} & \text{if } i \in R. \end{cases}$$

Definition 4.4.17. *Let $R, S, T \subset [r]$. Consider an ordered set V of elements of $\mathbb{R}^n$ indexed by T. If S and T are nonempty, we may consider the ordered set $\mathrm{pr}_{[r],S}(\pi_R(V))$ of elements of $\mathbb{R}^S$ indexed by T, and we define the $|S| \times |T|$ matrix $A_{V,S,R} := A_{\mathrm{pr}_{[r],S}(\pi_R(V))}$.*

Let G be a nonempty face in $\mathrm{lk}_{\mathcal{S}}(F \sqcup E)$, and assume that $S \neq \emptyset$. Let V_G be an ordered set consisting of the vertices of G, indexed by T. Let $C_{G,S,R} \subset \mathbb{R}^S$ be the cone spanned by the columns of $A_{V_G,S,R}$. If $G = \emptyset$ or $S = \emptyset$, then $C_{G,S,R} = \{0\}$. If $|G| = |S|$, we define the multiplicity *of (G, S, R) to be $\mathrm{mult}(G, S, R) := |\det(A_{V_G,S,R})|$. If $G = S = \emptyset$, then $\mathrm{mult}(G, S, R) = 1$.*

For example, if G is a face in $\mathrm{lk}_{\mathcal{S}}(F \sqcup E)$, $S = [r]$, and $R = \emptyset$, then $C_{G,S,R} = \pi(C_G)$, where $C_G \in \Delta$ is the cone over G in $\mathbb{R}^n$. Note that the definitions of $C_{G,S,R}$ and $\mathrm{mult}(G, S, R)$ above are independent of the choice of an ordering V_G of the vertices of G.

Proposition 4.4.18. *Let H be a face in $\mathrm{lk}_{\mathcal{S}}(F \sqcup E)$. Let $R \subset S = \{i_1 < \cdots < i_s\} \subset [r]$, for some $s > 1$. Fix $1 \leq k \leq s$, and assume that $k < j$ for all $i_j \in R$. Assume that $|H| = s - 1$, and $(0, \dots, 0, 1) \in C_{H, S \cap [i_j], R \cup \{i_k\}} \subset \mathbb{R}^{S \cap [i_j]}$ for $j > k$. Consider an ordering V_H of the vertices of H, indexed by $S \smallsetminus \{i_k\}$. If $H \subset G$ and $|G| = s$, consider the unique ordering $V_{H,G}$ of the vertices of G indexed by S such that $V_{H,G}|_{S \smallsetminus \{i_k\}} = V_H$. Then we have the following equality in $H^*(F \sqcup E)$:*

$$\det(A_{V_H, S \smallsetminus \{i_k\}, R}) y^H \Big(\prod_{i \notin S \smallsetminus \{i_k\}} y^{A_i} \Big) = -\sum_{\substack{H \subset G \\ |G| = s}} \det(A_{V_{H,G}, S, R \cup \{i_k\}}) y^G \Big(\prod_{i \notin S} y^{A_i} \Big).$$

Proof. Our goal is to expand $y^{A_{i_k}}$ on the left hand side by constructing a linear function and applying (11). We define a linear function $\mu = \mu_{H,S,R,i_k}\colon \mathbb{R}^n \to \mathbb{R}$ as follows: if $x \in \mathbb{R}^n$, then

$$\mu(x) = \det(A_{V_x,S,R\cup\{i_k\}}),$$

where V_x is the ordered set indexed by S defined by $V_x|_{S\smallsetminus\{i_k\}} = V_H$ and $V_x|_{\{i_k\}} = \{x\}$. By definition, $\mu(x) = 0$ if x is a vertex of H. If x is a vertex of $F \sqcup E$ and $j \leq k$, then

$$(A_{V_x,S,R\cup\{i_k\}})_{i_j,i_k} = \pi_R(x)_{i_j} = \begin{cases} 1 & \text{if } j = k, x = A_{i_k}, \\ 0 & \text{otherwise.} \end{cases}$$

Since $(0,\dots,0,1) \in C_{H,S\cap[i_j],R\cup\{i_k\}} \subset \mathbb{R}^{S\cap[i_j]}$ for $j > k$, it follows that we can apply column operations to ensure that $(A_{V_x,S,R\cup\{i_k\}})_{i_j,i_k} = 0$ for $j > k$, without affecting $\mu(x)$ or the values of any other elements of $A_{V_x,S,R\cup\{i_k\}}$. We deduce that if x is a vertex of $F \sqcup E$, then $\mu(x) = \det(A_{V_H,S\smallsetminus\{i_k\},R})$ if $x = A_{i_k}$, and $\mu(x) = 0$ otherwise.

We apply (11) with the global linear function μ to get the relation in $H^*(E)$:

$$\mu(A_{i_k})x^{A_{i_k}} + \sum_{\substack{v\in \mathrm{lk}_{\mathcal{S}}(E)\\ v\notin F\sqcup H}} \mu(v)x^v = 0.$$

Note that if $v \notin F \sqcup H$, then $y^v y^H = 0$ in $H^*(F \sqcup E)$ unless $H \sqcup v$ is a face in $\mathrm{lk}_{\mathcal{S}}(F \sqcup E)$. Considering the image of the above equality in $H^*(F \sqcup E)$ and multiplying both sides by $y^H(\prod_{i\notin S} y^{A_i})$ gives the result. □

We will use the formula in Proposition 4.4.18 to repeatedly to expand $y^{A_1}\cdots y^{A_r}$. The terms that appear at some point in this expansion depend on a tuple $(\widehat{H}, S, R, i_k)$, where $\widehat{H}$ is a face of $\mathrm{lk}_{\mathcal{S}}(F \sqcup E)$, $R \subset S$, and $0 \leq k \leq s$. We introduce two concepts that will be used to control terms in this expansion.

Definition 4.4.19. *Let $\widehat{H}$ be a face in $\mathrm{lk}_{\mathcal{S}}(F \sqcup E)$. Let $R \subset S = \{i_1 < \cdots < i_s\} \subset [r]$. Fix $0 \leq k \leq s$. Then $(\widehat{H}, S, R, i_k)$ is* material *if $C_{\widehat{H},S,R}$ is simplicial, $C^{(i_k)}_{\widehat{H},S,R} = C_{\widehat{H},S,R}$, and $(0,\dots,0,1) \in \mathrm{pr}_{S,S\cap[i_j]}(C_{\widehat{H},S,R}) \subset \mathbb{R}^{S\cap[i_j]}$ for $j > k$.*

Remark 4.4.20. When $k = 0$, the condition $C^{(i_k)}_{\widehat{H},S,R} = C_{\widehat{H},S,R}$ holds immediately. When $k = s$, the condition $C^{(i_k)}_{\widehat{H},S,R} = C_{\widehat{H},S,R}$ holds if and only if $C_{\widehat{H},S,R} = \{0\}$. It follows that if $\hat{H} = \emptyset$ and $k = s$, then $(\widehat{H}, S, R, i_k)$ is material.

Remark 4.4.21. Note that if $(\widehat{H}, S, R, i_k)$ is material, then $\ker(\mathrm{pr}_{S,S\cap[i_k]}) \subset \mathrm{span}(C_{\widehat{H},S,R})$. Indeed, by assumption there exists $u_1,\dots,u_{s-k} \in C_{\widehat{H},S,R}$ such that $\mathrm{pr}_{S,S\cap[i_{k+j}]}(u_j) = (0,\dots,0,1)$ for $1 \leq j \leq s-k$. Since $\dim(\ker(\mathrm{pr}_{S,S\cap[i_k]})) = s-k$, it follows that $\ker(\mathrm{pr}_{S,S\cap[i_k]}) = \mathrm{span}(u_1,\dots,u_{s-k}) \subset \mathrm{span}(C_{\widehat{H},S,R})$.

Lemma 4.4.22. *Let $\widehat{H}$ be a face in $\mathrm{lk}_{\mathcal{S}}(F \sqcup E)$. Let $R \subset S = \{i_1 < \cdots < i_s\} \subset [r]$. Fix $1 \leq k \leq s$. Assume that $(v_{2i_k-1}, v_{2i_k}) = (0,0)$ for all vertices $v \in \widehat{H}$. Then $(\widehat{H}, S, R, i_k)$ is material if and only if $(\widehat{H}, S, R\cup\{i_k\}, i_k)$ is material. Moreover, if $(\widehat{H}, S, R, i_k)$ is material, then $(\widehat{H}, S\smallsetminus\{i_k\}, R, i_{k-1})$ is material.*

Proof. For all vertices $v \in \widehat{H}$, $(v_{2i_k-1}, v_{2i_k}) = (0,0)$, and hence $\pi_R(v)_{i_k} = \pi_{R\cup\{i_k\}}(v)_{i_k} = 0$. It follows that $C_{\widehat{H},S,R} = C_{\widehat{H},S,R\cup\{i_k\}}$. This establishes the first statement, so we now prove the second statement. When $k = 1$, the claim is immediate, so we may assume that $1 < k \leq s$. Let $\psi\colon C_{\widehat{H},S,R} \to C_{\widehat{H},S\smallsetminus\{i_k\},R}$ denote the restriction of $\mathrm{pr}_{S,S\smallsetminus\{i_k\}}$. Then ψ is surjective by definition. Since $\pi_R(v)_{i_k} = 0$ for all vertices $v \in \widehat{H}$, $\ker(\mathrm{pr}_{S,S\smallsetminus\{i_k\}}) \cap C_{\widehat{H},S,R} = \{0\}$, and hence ψ is an isomorphism of cones. Also, $\mathrm{pr}_{S,S\smallsetminus\{i_k\}}$ induces an isomorphism from $\ker(\mathrm{pr}_{S,S\cap[i_k]})$ to $\ker(\mathrm{pr}_{S\smallsetminus\{i_k\},S\cap[i_{k-1}]})$. It follows that $\psi(C^{(i_k)}_{\widehat{H},S,R}) = C^{(i_{k-1})}_{\widehat{H},S\smallsetminus\{i_k\},R}$, which implies the result. □

Lemma 4.4.23. *Let $\widehat{H}$ be a face in $\mathrm{lk}_{\mathcal{S}}(F \sqcup E)$. Let $R \subset S = \{i_1 < \cdots < i_s\} \subset [r]$. Fix $1 \leq k \leq s$. Assume that $i_k \notin R$ and $(\widehat{H}, S, R, i_k)$ is material. Then either $\{2i_k - 1, 2i_k\} \subset \mathrm{Supp}_{\widehat{H}}$ or $(v_{2i_k-1}, v_{2i_k}) = (0,0)$ for all vertices $v \in \widehat{H}$.*

Proof. By Corollary 4.4.7, $\mathrm{pr}_{S,\{i_k\}}(C_{\widehat{H},S,R})$ is $\{0\}$ or $\mathbb{R}^{\{i_k\}}$. Assume there is a vertex $v \in \widehat{H}$ such that $(v_{2i_k-1}, v_{2i_k}) \neq (0,0)$. If $\pi(v)_{i_k} = 0$, then $v_{2i_k-1} = v_{2i_k} \neq 0$. If $\pi(v)_{i_k} \neq 0$, then $\mathrm{pr}_{S,\{i_k\}}(C_{\widehat{H},S,R}) = \mathbb{R}^{\{i_k\}}$, and there is $v' \in \widehat{H}$ such that $\pi(v')_{i_k} \neq 0$ has the opposite sign to $\pi(v)_{i_k}$. In either case, $2i_k - 1, 2i_k \in \mathrm{Supp}_{\widehat{H}}$. □

Let G be a face in $\mathrm{lk}_{\mathcal{S}}(F \sqcup E)$. Let $R \subset S = \{i_1 < \cdots < i_s\} \subset [r]$. Assume that $|G| = s$. Then we say that (G, S, R) is *positive* if $C_{G,S,R} \subset \mathbb{R}^S$ is positive. An ordering V of the vertices of G indexed by S is *positively ordered* if $\mathrm{pr}_{[r],S}(\pi_R(V))$ is positively ordered. For example, if $G = S = R = \emptyset$, then $C_{G,S,R} = \mathbb{R}^S = \{0\}$ and (G, S, R) is positive.

Remark 4.4.24. If (G, S, R) is positive then $C_{G,S,R} \subset \mathbb{R}^S$ is a full-dimensional simplicial cone, with faces $\{C_{H,S,R} : H \subset G\}$. Also, in this case, Lemma 4.4.11 implies that there is a unique positive ordering of the vertices of G.

4.4.4. *Computing the degree.* We now define an element of $H^*(F \sqcup E)$ that depends on a tuple $(\widehat{H}, S, R, i_k)$, which we call $\theta(\widehat{H}, S, R, i_k)$. For some tuples, the formula for $\theta(\widehat{H}, S, R, i_k)$ is very simple, and it behaves well with respect to the formula in Proposition 4.4.18. This allows us to compute the degree of $y^{A_1} \cdots y^{A_r}$.

Definition 4.4.25. *Let $\widehat{H}$ be a face in $\mathrm{lk}_{\mathcal{S}}(F \sqcup E)$. Let $R \subset S = \{i_1 < \cdots < i_s\} \subset [r]$. Fix $0 \leq k \leq s$. We define an element $\theta(\widehat{H}, S, R, i_k) \in H^*(F \sqcup E)$ by*

$$\theta(\widehat{H}, S, R, i_k) := \sum_{\substack{\widehat{H} \subset G, |G| = s \\ C^{(i_k)}_{G,S,R} = C_{\widehat{H},S,R} \\ (G,S,R) \text{ positive}}} \mathrm{mult}(G, S, R) y^G \Big(\prod_{i \notin S} y^{A_i} \Big).$$

Remark 4.4.26. We highlight a special case of the above definition. We have that

$$\theta(\widehat{H}, S, R, i_0) = \begin{cases} \mathrm{mult}(\widehat{H}, S, R) y^{\widehat{H}} (\prod_{i \notin S} y^{A_i}) & \text{if } |\widehat{H}| = s \text{ and } (\widehat{H}, S, R) \text{ is positive,} \\ 0 & \text{otherwise.} \end{cases}$$

In particular, if $S = \emptyset$ (and therefore $R = \emptyset$ and $k = 0$),

$$\theta(\widehat{H}, S, R, i_k) = \begin{cases} y^{A_1} \cdots y^{A_r} & \text{if } \widehat{H} = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

We now state four properties of θ. They will allow us to compute $\theta(\emptyset, \emptyset, \emptyset, i_0) = y^{A_1} \cdots y^{A_r}$ via a recursion, which we use to complete the proof of Proposition 4.3.5. Afterwards, we will prove the four properties.

Lemma 4.4.27. *Let $\widehat{H}$ be a face in $\mathrm{lk}_{\mathcal{S}}(F \sqcup E)$. Let $R \subset S = \{i_1 < \cdots < i_s\} \subset [r]$. Fix $0 \leq k \leq s$. Then $\theta(\widehat{H}, S, R, i_k) \neq 0$ implies that $(\widehat{H}, S, R, i_k)$ is material. If $(\widehat{H}, S, R, i_k)$ is material and $\widehat{H} \subset G$ such that $|G| = s$ and (G, S, R) is positive, then $C^{(i_k)}_{G,S,R} = C_{\widehat{H},S,R}$. In particular, if $(\widehat{H}, S, R, i_k)$ is material, then*

$$\theta(\widehat{H}, S, R, i_k) = \sum_{\substack{\widehat{H} \subset G, |G| = s \\ (G,S,R) \text{ positive}}} \mathrm{mult}(G, S, R) y^G \Big(\prod_{i \notin S} y^{A_i} \Big).$$

Lemma 4.4.28. *Let $\widehat{H}$ be a face in $\mathrm{lk}_{\mathcal{S}}(F \sqcup E)$. Let $R \subset S = \{i_1 < \cdots < i_s\} \subset [r]$. Fix $1 \le k \le s$. If $(\widehat{H}, S, R, i_k)$ is material, then*

$$\theta(\widehat{H}, S, R, i_k) = \sum_{\widehat{H} \subset \widehat{H}'} \theta(\widehat{H}', S, R, i_{k-1}).$$

Lemma 4.4.29. *Let $\widehat{H}$ be a face in $\mathrm{lk}_{\mathcal{S}}(F \sqcup E)$. Let $R \subset S = \{i_1 < \cdots < i_s\} \subset [r]$. Fix $1 \le k \le s$. Assume that $\{2i_k - 1, 2i_k\} \not\subset \mathrm{Supp}_{\widehat{H}}$. Then*

$$\theta(\widehat{H}, S, R, i_{k-1}) = \theta(\widehat{H}, S, R \cup \{i_k\}, i_{k-1}).$$

Lemma 4.4.30. *Let $\widehat{H}$ be a face in $\mathrm{lk}_{\mathcal{S}}(F \sqcup E)$. Let $R \subset S = \{i_1 < \cdots < i_s\} \subset [r]$. Fix $1 \le k \le s$. Assume that $k < j \le s$ for all $i_j \in R$. Assume that $(\widehat{H}, S, R \cup \{i_k\}, i_k)$ and $(\widehat{H}, S \smallsetminus \{i_k\}, R, i_{k-1})$ are material. Then*

$$\theta(\widehat{H}, S, R \cup \{i_k\}, i_k) = -\theta(\widehat{H}, S \smallsetminus \{i_k\}, R, i_{k-1}).$$

Proposition 4.4.31. *Let $\widehat{H}$ be a face in $\mathrm{lk}_{\mathcal{S}}(F \sqcup E)$. Let $R \subset S = \{i_1 < \cdots < i_s\} \subset [r]$. Fix $0 \le k \le s$. Assume that $k < j \le s$ for all $i_j \in R$. Assume that $2i_j - 1, 2i_j \in \mathrm{Supp}_{\widehat{H}}$ for $k < j \le s$. Then*

$$\theta(\widehat{H}, S, R, i_k) = \begin{cases} (-1)^s y^{A_1} \cdots y^{A_r} & \text{if } \widehat{H} = R = \emptyset, k = s \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We proceed by induction on k. Assume first that $k = 0$. If $s = 0$, then the result follows from Remark 4.4.26. Assume that $s > 0$. By Remark 4.4.26,

$$\theta(\widehat{H}, S, R, i_0) = \begin{cases} \mathrm{mult}(\widehat{H}, S, R) y^{\widehat{H}} \left(\prod_{i \notin S} y^{A_i} \right) & \text{if } |\widehat{H}| = s, (\widehat{H}, S, R) \text{ is positive} \\ 0 & \text{otherwise.} \end{cases}$$

By assumption, $2i_j - 1, 2i_j \in \mathrm{Supp}_{\widehat{H}}$ for all $1 \le j \le s$. Then Lemma 4.4.15 implies that $\theta(\widehat{H}, S, R, i_0) = 0$.

Assume that $k > 0$. In particular, $s > 0$. By Lemma 4.4.27 and Remark 4.4.20, we may assume that $(\widehat{H}, S, R, i_k)$ is material. By Lemma 4.4.28,

$$\theta(\widehat{H}, S, R, i_k) = \sum_{\widehat{H} \subset \widehat{H}'} \theta(\widehat{H}', S, R, i_{k-1}).$$

By induction, this simplifies to

$$\theta(\widehat{H}, S, R, i_k) = \sum_{\substack{\widehat{H} \subset \widehat{H}' \\ \{2i_k - 1, 2i_k\} \not\subset \mathrm{Supp}_{\widehat{H}'}}} \theta(\widehat{H}', S, R, i_{k-1}). \tag{13}$$

In particular, we may assume that $\{2i_k - 1, 2i_k\} \not\subset \mathrm{Supp}_{\widehat{H}}$, else the right-hand side of (13) is zero, and the result holds. Since $i_k \notin R$ by assumption, Lemma 4.4.23 implies that $(v_{2i_k - 1}, v_{2i_k}) = (0, 0)$ for all vertices $v \in \widehat{H}$. By Lemma 4.4.22, $(\widehat{H}, S, R \cup \{i_k\}, i_k)$ and $(\widehat{H}, S \smallsetminus \{i_k\}, R, i_{k-1})$ are material. As above, by Lemma 4.4.28 and the induction hypothesis,

$$\theta(\widehat{H}, S, R \cup \{i_k\}, i_k) = \sum_{\substack{\widehat{H} \subset \widehat{H}' \\ \{2i_k - 1, 2i_k\} \not\subset \mathrm{Supp}_{\widehat{H}'}}} \theta(\widehat{H}', S, R \cup \{i_k\}, i_{k-1}).$$

Comparing with (13), Lemma 4.4.29 implies that

$$\theta(\widehat{H}, S, R, i_k) = \theta(\widehat{H}, S, R \cup \{i_k\}, i_k).$$

By Lemma 4.4.30,

$$\theta(\widehat{H}, S, R \cup \{i_k\}, i_k) = -\theta(\widehat{H}, S \smallsetminus \{i_k\}, R, i_{k-1}).$$

The result now follows by induction. □

Proof of Proposition 4.3.5. Recall that for each face $G \in \mathrm{lk}_{\mathcal{S}}(F \sqcup E)$, $C_{G,[r],R} = \pi(C_G)$, where $C_G \in \Delta$ is the cone over G in $\mathbb{R}^n$. Moreover, the cones $\{\pi(C_G) : \mathrm{lk}_{\mathcal{S}}(F \sqcup E)\}$ form a complete fan. Therefore, there is a unique facet $\widehat{G} \in \mathrm{lk}_{\mathcal{S}}(F \sqcup E)$ such that $\pi(C_{\widehat{G}})$ is positive. By Definition 4.4.25, $\theta(\emptyset, [r], \emptyset, i_r) = \mathrm{mult}(\widehat{G}, [r], \emptyset) y^{\widehat{G}}$. By Proposition 4.4.31, we have the following equality in $H^*(F \sqcup E)$:

$$(-1)^r y^{A_1} \cdots y^{A_r} = \mathrm{mult}(\widehat{G}, [r], \emptyset) y^{\widehat{G}}.$$

Since $\mathrm{mult}(\widehat{G}, [r], \emptyset)$ and $\deg_{F \sqcup E}(y^{\widehat{G}})$ are both strictly positive, we deduce that $(-1)^r \deg_{F \sqcup E}(y^{A_1} \cdots y^{A_r}) > 0$. □

Proof of Lemma 4.4.27. If $\theta(\widehat{H}, S, R, i_k) \neq 0$, then there is a face $G \supset \widehat{H}$ with $|G| = s$, $C^{(i_k)}_{G,S,R} = C_{\widehat{H},S,E}$ and (G, S, R) positive. As $C^{(i_k)}_{G,S,R}$ is a face of the simplicial cone $C_{G,S,R}$, $C_{\widehat{H},S,E} = C^{(i_k)}_{G,S,R}$ is simplicial. We also see that $C^{(i_k)}_{\widehat{H},S,E} = C_{\widehat{H},S,E}$. The positivity of (G, S, R) implies that $(0, \ldots, 0, 1) \in \mathrm{pr}_{S,S\cap[i_j]}(C_{\widehat{H},S,E})$ for $j > k$, so $(\widehat{H}, S, R, i_k)$ is material.

Suppose $(\widehat{H}, S, R, i_k)$ is material and $G \supset \widehat{H}$ with $|G| = s$ and (G, S, R) positive. By Remark 4.4.24, $C_{G,S,R} \subset \mathbb{R}^S$ is a full-dimensional simplicial cone with faces $\{C_{H,S,R} : H \subset G\}$. Then $C_{\widehat{H},S,E}$ is a face of $C_{G,S,R}$, and it follows from Remark 4.4.21 that $C^{(i_k)}_{G,S,R} = C^{(i_k)}_{\widehat{H},S,R} = C_{\widehat{H},S,R}$. Then the formula for $\theta(\widehat{H}, S, R, i_k)$ is immediate. □

Proof of Lemma 4.4.28. Let G be a face of $\mathrm{lk}_{\mathcal{S}}(F \sqcup E)$ such that $|G| = s$ and (G, S, R) is positive. Let $\widehat{H}' \subset G$ be the unique face such that $C^{(i_{k-1})}_{G,S,R} = C_{\widehat{H}',S,R}$. By Lemma 4.4.27, if $\widehat{H} \subset G$, then $C^{(i_k)}_{G,S,R} = C_{\widehat{H},S,R}$. It is enough to show that $\widehat{H} \subset G$ if and only if $\widehat{H} \subset \widehat{H}'$. Clearly, if $\widehat{H} \subset \widehat{H}'$ then $\widehat{H} \subset G$. If $\widehat{H} \subset G$, then $C^{(i_k)}_{G,S,R} = C_{\widehat{H},S,R}$ which is a face of $C^{(i_{k-1})}_{G,S,R} = C_{\widehat{H}',S,R}$, and the result follows from Remark 4.4.24. □

Proof of Lemma 4.4.29. Let R_1, R_2 be sets such that $\{R_1, R_2\} = \{R, R \cup \{i_k\}\}$. Consider a face $G \supset \widehat{H}$ such that $|G| = s$. Assume that $C^{(i_{k-1})}_{G,S,R_1} \subset C_{\widehat{H},S,R_1}$ and (G, S, R_1) is positive. By definition, $(0, \ldots, 0, 1) \in \mathrm{pr}_{S,S\cap[i_j]}(C^{(i_{k-1})}_{G,S,R_1})$ for $k \leq j \leq s$. In particular, $\mathrm{pr}_{S,\{i_k\}}(C^{(i_{k-1})}_{G,S,R_1}) \not\subset \mathbb{R}_{\leq 0}$. Since $C^{(i_{k-1})}_{G,S,R_1} \subset C_{\widehat{H},S,R_1}$ and $\pi_{R_1}(v)_{i_k} \in \{\pi(v)_{i_k}, v_{2i_k-1}\}$ for $v \in \widehat{H}$, we deduce that $2i_k - 1 \in \mathrm{Supp}_{\widehat{H}}$. Since $\{2i_k - 1, 2i_k\} \not\subset \mathrm{Supp}_{\widehat{H}}$, we have $2i_k \notin \mathrm{Supp}_{\widehat{H}}$, and hence $\pi_{R_1}(v)_{i_k} = \pi_{R_2}(v)_{i_k}$ for $v \in \widehat{H}$. We deduce that $C^{(i_{k-1})}_{G,S,R_1} \subset C_{\widehat{H},S,R_1} = C_{\widehat{H},S,R_2} \subset C_{G,S,R_2}$. In particular, $(0, \ldots, 0, 1) \in \mathrm{pr}_{S,S\cap[i_j]}(C_{G,S,R_2})$ for $k \leq j \leq s$, and $C^{(i_{k-1})}_{G,S,R_2} \subset C^{(i_{k-1})}_{G,S,R_1} \subset C_{\widehat{H},S,R_2}$. For $1 \leq j < k$, $(0, \ldots, 0, 1) \in \mathrm{pr}_{S,S\cap[i_j]}(C_{G,S,R_1}) = \mathrm{pr}_{S,S\cap[i_j]}(C_{G,S,R_2})$ by definition. We deduce that (G, S, R_2) is positive.

We conclude that the condition that both $C^{(i_{k-1})}_{G,S,R_1} \subset C_{\widehat{H},S,R_1}$ and (G, S, R_1) is positive is independent of the choice of R_1. Moreover, if this condition holds, then the above argument shows that both $C^{(i_{k-1})}_{G,S,R_1}$ and $C_{\widehat{H},S,R_1}$ are independent of the choice of R_1. It follows that the condition that both $C^{(i_{k-1})}_{G,S,R_1} = C_{\widehat{H},S,R_1}$ and (G, S, R_1) is positive is also independent of the choice of R_1, as desired. □

Proof of Lemma 4.4.30. By Lemma 4.4.27, $\theta(\widehat{H}, S \smallsetminus \{i_k\}, R, i_{k-1})$ equals

$$\sum_{\substack{\widehat{H} \subset H, |H| = s-1 \\ (H, S \smallsetminus \{i_k\}, R) \text{ positive}}} \mathrm{mult}(H, S \smallsetminus \{i_k\}, R) y^H \Big(\prod_{i \notin S \smallsetminus \{i_k\}} y^{A_i} \Big).$$

Consider a face $\widehat{H} \subset H$ with $|H| = s - 1$ and $(H, S \smallsetminus \{i_k\}, R)$ positive. Let V_H be the unique positive ordering of the vertices of H, indexed by $S \smallsetminus \{i_k\}$. If $H \subset G$ and $|G| = s$, consider the unique ordering $V_{H,G}$ of the vertices of G indexed by S such that $V_{H,G}|_{S \smallsetminus \{i_k\}} = V_H$. The above expression equals

$$\sum_{\substack{\widehat{H} \subset H, |H| = s-1 \\ (H, S \smallsetminus \{i_k\}, R) \text{ positive}}} \det(A_{V_H, S \smallsetminus \{i_k\}, R}) y^H \Big(\prod_{i \notin S \smallsetminus \{i_k\}} y^{A_i} \Big).$$

Since $(\widehat{H}, S, R \cup \{i_k\}, i_k)$ is material, $(0, \dots, 0, 1) \in C_{\widehat{H}, S \cap [i_j], R \cup \{i_k\}} \subset \mathbb{R}^{S \cap [i_j]}$ for $j > k$. Hence we can apply Proposition 4.4.18 to obtain that the above expression is equal to

$$- \sum_{\substack{\widehat{H} \subset H, |H| = s-1 \\ (H, S \smallsetminus \{i_k\}, R) \text{ positive}}} \sum_{\substack{H \subset G \\ |G| = s}} \det(A_{V_{H,G}, S, R \cup \{i_k\}}) y^G \Big(\prod_{i \notin S} y^{A_i} \Big)$$

$$= - \sum_{\substack{\widehat{H} \subset G \\ |G| = s}} \sum_{\substack{\widehat{H} \subset H \subset G, |H| = s-1 \\ (H, S \smallsetminus \{i_k\}, R) \text{ positive}}} \det(A_{V_{H,G}, S, R \cup \{i_k\}}) y^G \Big(\prod_{i \notin S} y^{A_i} \Big).$$

Consider a face $\widehat{H} \subset G$ with $|G| = s$. Let $C = C_{G, S, R \cup \{i_k\}}$. If C is not full-dimensional, then we have $\det(A_{V_{H,G}, S, R \cup \{i_k\}}) = 0$ for all choices of H. We may assume that C is full-dimensional, and hence simplicial. As above, $(0, \dots, 0, 1) \in C_{\widehat{H}, S \cap [i_j], R \cup \{i_k\}} \subset \mathrm{pr}_{S, S \cap [i_j]}(C) \subset \mathbb{R}^{S \cap [i_j]}$ for $j > k$. Also, $\pi_{R \cup \{i_k\}}(v)_{i_k} = v_{2i_k - 1} \geq 0$ for all $v \in G$, and hence $(0, \dots, 0, -1) \notin \mathrm{pr}_{S, S \cap [i_k]}(C)$. Then Proposition 4.4.14 implies that the above expression simplifies to give

$$= - \sum_{\substack{\widehat{H} \subset G, |G| = s \\ (G, S, R \cup \{i_k\}) \text{ positive}}} \mathrm{mult}(G, S, R \cup \{i_k\}) y^G \Big(\prod_{i \notin S} y^{A_i} \Big).$$

Since $(\widehat{H}, S, R \cup \{i_k\}, i_k)$ is material, Lemma 4.4.27 implies that the latter equals $-\theta(\widehat{H}, S, R \cup \{i_k\}, i_k)$. □

5. The local formal zeta function and its candidate poles

5.1. **Overview.** In this section, we prove a mild generalization of Theorem 1.4.7, with a hypothesis that is weaker than $\mathrm{Newt}(f)$ being simplicial. In Section 5.2, we recall the formula for $Z_{\mathrm{mot}}(T)$ in [BN20, Theorem 8.3.5], and then in Section 5.3 and 5.4 we introduce the local formal zeta and develop some of its properties.

After some combinatorial preliminaries, in Section 5.6 we construct a small polyhedral neighborhood $N_{M, \leq \delta}$ of the dual cone to each minimal face M of $\mathrm{Contrib}(\alpha)$, which allows us to consider each minimal face separately. In Section 5.7, we prove our main theorem on poles assuming the existence of an *α-compatible pair*. In Sections 5.8 and 5.9, we construct α-compatible pairs.

We first introduce some notation for use in this section. Let $\Gamma = \Gamma(\mathrm{Newt}(f))$ be the union of the proper interior faces of $\mathrm{Newt}(f)$ and their subfaces. That is, Γ is the union of faces F of $\mathrm{Newt}(f)$ that are visible from the origin in the sense that for every $W \in F$, the intersection of $\mathrm{Newt}(f)$ with the interval from the origin to W equals $\{W\}$. Let $\Sigma = \Sigma_f$ be the dual fan of $\mathrm{Newt}(f)$. For each face F of $\mathrm{Newt}(f)$, let σ_F be the cone of Σ dual to F. For $F \in \Gamma$, we may consider the unbounded directions of F:

$$I_F := \{e_\ell \mid 1 \leq \ell \leq n, F + \mathbb{R}_{\geq 0} e_\ell \subset F\}.$$

We let $\mathrm{Vert}(F)$ denote the set of vertices of F. Given a face F of Γ, recall that C_F is the closure of the cone over F, with distinguished generators $\mathrm{Gen}(C_F)$. Then $\mathrm{span}(F) = \mathrm{span}(C_F)$ and $\mathrm{Gen}(C_F) = \mathrm{Vert}(F) \cup I_F$.

Given an inclusion of faces $M \subset F$, let $\operatorname{Gen}(C_F \smallsetminus C_M) = \operatorname{Gen}(C_F) \smallsetminus \operatorname{Gen}(C_M)$. For a face B_1-face M, let $\mathcal{A}_M$ be the set of all apices in M.

Definition 5.1.1. *We say that the Newton polyhedron* $\operatorname{Newt}(f)$ *is* α-simplicial *if for any minimal element* M *in* $\operatorname{Contrib}(\alpha)$ *and any face* $F \supset M$*, the image of* C_F *in* $\mathbb{R}^n/\operatorname{span}(C_M)$ *is simplicial, i.e., the set* $\operatorname{Gen}(C_F \smallsetminus C_M)$ *is linearly independent in* $\mathbb{R}^n/\operatorname{span}(C_M)$*.*

Equivalently, the link of the cone corresponding to any minimal face of $\operatorname{Contrib}(\alpha)$ is simplicial. For example, if $\operatorname{Newt}(f)$ is simplicial, then $\operatorname{Newt}(f)$ is α-simplicial. If all minimal elements in $\operatorname{Contrib}(\alpha)$ are facets, then $\operatorname{Newt}(f)$ is α-simplicial.

We now state our main theorem.

Theorem 5.1.2. *Suppose* f *is nondegenerate. Let*

$$\mathcal{P} = \{\alpha \in \mathbb{R} : \operatorname{Contrib}(\alpha) \neq \emptyset\} \cup \{-1\}, \text{ and}$$

$$\mathcal{P}' = \{\alpha \in \mathcal{P} : \alpha \notin \mathbb{Z}_{<0}, \text{every face in } \operatorname{Contrib}(\alpha) \text{ is } UB_1 \text{ and } \operatorname{Newt}(f) \text{ is } \alpha\text{-simplicial}\}.$$

Then $\mathcal{P} \smallsetminus \mathcal{P}'$ *is a set of candidate poles for* $Z_{\mathrm{mot}}(T)$*.*

5.2. **Formula for the local motivic zeta function.** We will now discuss the formula of Bultot and Nicaise for the local motivic zeta function of a nondegenerate polynomial f in terms of $\operatorname{Newt}(f)$.

Consider a nonempty compact face K of Γ. Following [BN20], we associate two classes in $\mathcal{M}_{\Bbbk}^{\hat{\mu}}$ to K. For $i \in \{0,1\}$, let $Y_K(i)$ be the closed subscheme of $\operatorname{Spec} \Bbbk[\operatorname{span}(K) \cap \mathbb{Z}^n]$ cut out by $f_K = i$. When $i = 0$, we endow $Y_K(0)$ with the trivial $\hat{\mu}$-action, and obtain a class $[Y_K(0)] \in \mathcal{M}_{\Bbbk}^{\hat{\mu}}$. We define a $\hat{\mu}$-action on $Y_K(1)$ as follows. Let ρ_K be the lattice distance of K to the origin, and let $w = w_K := \rho_K \psi_K \in \operatorname{Hom}(\operatorname{span}(K) \cap \mathbb{Z}^n, \mathbb{Z})$. Then w determines a cocharacter $\operatorname{Spec} \Bbbk[\mathbb{Z}] \to \operatorname{Spec} \Bbbk[\operatorname{span}(K) \cap \mathbb{Z}^n]$, which we can restrict via $\operatorname{Spec} \Bbbk[T]/(T^\rho - 1) \to \operatorname{Spec} \Bbbk[\mathbb{Z}]$ to determine a μ_ρ-action on $\operatorname{Spec} \Bbbk[\operatorname{span}(K) \cap \mathbb{Z}^n]$. This induces an action of μ_ρ on $Y_K(1)$. Explicitly, choose a basis for $\operatorname{span}(K) \cap \mathbb{Z}^n$ and write $w = (w_1, \ldots, w_r)$ and $f = \sum_{a \in \mathbb{Z}^r} \lambda_a x^a$. Then for each $a = (a_1, \ldots, a_r)$ with $\lambda_a \neq 0$, $\sum_{i=1}^r a_i w_i$ is divisible by ρ, and the action is

$$\zeta \cdot (x_1, \ldots, x_r) = (\zeta^{w_1} x_1, \ldots, \zeta^{w_r} x_r).$$

This gives a class $[Y_K(1)]$ in $\mathcal{M}_{\Bbbk}^{\hat{\mu}}$. When K is the empty compact face of Γ, $\operatorname{span}(K) \cap \mathbb{Z}^n = \{0\}$, and we let $Y_K(0)$ be the point $\operatorname{Spec} \Bbbk[\operatorname{span}(K) \cap \mathbb{Z}^n]$ and let $Y_K(1) = \emptyset$. Then $[Y_K(0)] = 1$ and $[Y_K(1)] = 0$.

Remark 5.2.1. The above construction differs slightly from that in [BN20]. Explicitly, for $i \in \{0,1\}$, [BN20] let $X_K(i)$ be the closed subscheme of $\operatorname{Spec} \Bbbk[\mathbb{Z}^n]$ cut out by $f_K = i$. Consider $X_K(0)$ with the trivial $\hat{\mu}$-action. Let w be any linear function in $\mathbb{Z}^n$ that restricts to $\rho\psi_K$, and, as above, consider the corresponding μ_ρ-action on $X_K(1) \subset \operatorname{Spec} \Bbbk[\mathbb{Z}^n]$. They consider the classes $[X_K(i)]$ in $\mathcal{M}_{\Bbbk}^{\hat{\mu}}$. It follows from [BN20, Proposition 7.1.1] that $[X_K(i)] = [Y_K(i)](\mathbb{L} - 1)^{n-1-\dim K}$.

The following lemma will be important in the proof of Theorem 5.1.2.

Lemma 5.2.2. *Let* $G \subset F$ *be an inclusion of compact faces of* Γ*. Suppose there exists a vertex* A *of* F *such that* $F = \operatorname{Conv}\{G, A\}$ *and* $\operatorname{span}(F) \cap \mathbb{Z}^n = \operatorname{span}(G) \cap \mathbb{Z}^n + \mathbb{Z} \cdot A$*. Then for* $i \in \{0,1\}$*,* $[Y_G(i)] + [Y_F(i)] = (\mathbb{L} - 1)^{\dim F} \in \mathcal{M}_{\Bbbk}^{\hat{\mu}}$*.*

Proof. The second statement follows immediately from the first statement. Let $r = \dim F$. Let ρ_G and ρ_F be the smallest positive integers such that $w_G = \rho_G \psi_G$ and $w_F = \rho_F \psi_F$ lie in $\operatorname{Hom}(\operatorname{span}(G) \cap \mathbb{Z}^n, \mathbb{Z})$ and $\operatorname{Hom}(\operatorname{span}(F) \cap \mathbb{Z}^n, \mathbb{Z})$ respectively.

Then, we may choose coordinates such that $Y_F(i)$ is defined by $\{f_F(x_0, \ldots, x_r) = i\}$ in $\operatorname{Spec} \Bbbk[\operatorname{span}(F) \cap \mathbb{Z}^n]$, $Y_G(i)$ is defined by $\{f_G(x_1, \ldots, x_r) = i\}$ in $\operatorname{Spec} \Bbbk[\operatorname{span}(G) \cap \mathbb{Z}^n]$, and $f_F(x_0, \ldots, x_r) = x_0 + f_G(x_1, \ldots, x_r)$. Also, we may set $\rho = \rho_F = \rho_G$, and write $w_G = (w_1, \ldots, w_r)$ and $w_F = (1, w_1, \ldots, w_r)$. As above, w_G and w_F

induce μ_ρ-actions on $\operatorname{Spec}\Bbbk[\operatorname{span}(G)\cap\mathbb{Z}^n]$ and $\operatorname{Spec}\Bbbk[\operatorname{span}(F)\cap\mathbb{Z}^n]$ respectively. Consider the μ_ρ-equivariant map

$$\phi : \operatorname{Spec}\Bbbk[\operatorname{span}(G)\cap\mathbb{Z}^n] \smallsetminus Y_G(i) \to Y_F(i),$$
$$\phi(x_1,\ldots,x_r) = (i - f_G(x_1,\ldots,x_r), x_1,\ldots,x_r).$$

Then ϕ is an isomorphism, with inverse $\phi^{-1}(x_0,x_1,\ldots,x_r) = (x_1,\ldots,x_r)$. By [BN20, Lemma 7.1.1], the class of any r-dimensional torus in $\mathcal{M}_{\Bbbk}^{\hat{\mu}}$ is $(\mathbb{L}-1)^r$, and the result follows. □

Example 5.2.3. Let A be a primitive vertex of Γ. Then Lemma 5.2.2, with $F = \{A\}$ and $G = \emptyset$, implies that $[Y_A(0)] = 0$ and $[Y_A(1)] = 1$.

Example 5.2.4. Let F be a compact B_1-face of Γ with nonempty base G. Then Lemma 5.2.2 implies that $[Y_G(0)]\frac{\mathbb{L}^{-1}T}{1-\mathbb{L}^{-1}T} + [Y_G(1)] + [Y_F(0)]\frac{\mathbb{L}^{-1}T}{1-\mathbb{L}^{-1}T} + [Y_F(1)] = \frac{(\mathbb{L}-1)^{\dim F}}{1-\mathbb{L}^{-1}T}$.

We now discuss two results on lattice point enumeration. The first result is standard.

Lemma 5.2.5. *Let C be a nonzero rational polyhedral cone in $\mathbb{R}^n_{\geq 0}$ with rays spanned by primitive integer vectors $u_1,\ldots,u_r$. Let $\operatorname{Box}^+(C) = \{u \in \mathbb{N}^n : u = \sum_{i=1}^r \lambda_i u_i$ for some $0 < \lambda_i \leq 1\}$. Then*

$$\sum_{u\in C^\circ\cap\mathbb{N}^n} x^u = \frac{\sum_{u\in\operatorname{Box}^+(C)} x^u}{\prod_{i=1}^r (1-x^{u_i})} \in \mathbb{Z}[\![x_1,\ldots,x_n]\!].$$

Lemma 5.2.6. *Let C be a rational polyhedral cone of dimension d contained in $\mathbb{R}^n_{\geq 0}$, and let Y be a $\mathbb{Z}$-linear function that takes nonnegative values on C and is not identically zero on C. Let $u_1,\ldots,u_r$ be the primitive generators of the rays of C. Let $I = \{i \in [r] \colon \langle u_i, Y\rangle \neq 0\}$. Assume that $\langle u_j, \mathbf{1}\rangle = 1$ for $j \notin I$. Then*

$$(L-1)^{d-1} \sum_{u\in C^\circ\cap\mathbb{N}^n} L^{-\langle u,\mathbf{1}\rangle} T^{\langle u,Y\rangle}$$

lies in the subring

$$\mathbb{Z}[L,L^{-1},T]\Big[\frac{1}{1-L^{-\langle u_i,\mathbf{1}\rangle}T^{\langle u_i,Y\rangle}}\Big]_{i\in I} \subset \mathbb{Z}[L][\![L^{-1},T]\!].$$

Proof. This is essentially [BN20, Lemma 5.1.1]; the point is that we may reduce to the case when C is simplicial and then apply Lemma 5.2.5. The $1/(1-L^{-1})$ terms that arise from $j \notin I$ are cancelled by the $(L-1)^{d-1}$ factor. □

Define a piecewise linear function N on Σ by

$$N(u) = \min\{\langle u, W\rangle \colon W \in \operatorname{Newt}(f)\}.$$

Remark 5.2.7. If u is a primitive generator of a ray in the dual fan, corresponding to a facet F of $\operatorname{Newt}(f)$, then $N(u)$ is the lattice distance of F to the origin. If $N(u) = 0$, then $u = e_i^*$, for some $1 \leq i \leq n$, and hence $\langle u, \mathbf{1}\rangle = 1$.

Lemma 5.2.8. [BN20, proof of Theorem 8.3.5] *Let $u_1,\ldots,u_r$ be the primitive generators of the rays of σ_K. The element*

$$(L-1)^{n-\dim K} \sum_{u\in\sigma_K^\circ\cap\mathbb{N}^n} L^{-\langle u,\mathbf{1}\rangle} T^{N(u)}$$

lies in the subring $\mathbb{Z}[L,L^{-1},T]\Big[\frac{1}{1-L^{-\langle u_i,\mathbf{1}\rangle}T^{N(u_i)}}\Big]_{\{i\in[r]:N(u_i)\neq 0\}}$ of $\mathbb{Z}[L][\![L^{-1},T]\!]$.

Proof. Observe that the restriction of N to σ_K is a nonnegative linear function. The result follows from Lemma 5.2.6 and Remark 5.2.7. □

In [BN20], they define

$$(\mathbb{L}-1)^{n-\dim K}\sum_{u\in\sigma_K^\circ\cap\mathbb{N}^n}\mathbb{L}^{-\langle u,\mathbf{1}\rangle}T^{N(u)}\in\mathcal{M}_{\Bbbk}^{\hat\mu}[\![T]\!]$$

to be the image of the expression in Lemma 5.2.8 under the specialization map $\mathbb{Z}[L,L^{-1}][\![T]\!]\to\mathcal{M}_{k}^{\hat\mu}[\![T]\!]$ that sends L to $\mathbb{L}$.

Theorem 5.2.9. [BN20, Theorem 8.3.5] *Suppose f is nondegenerate. Then*

$$Z_{\mathrm{mot}}(T)=\sum_K\left([Y_K(0)]\frac{\mathbb{L}^{-1}T}{1-\mathbb{L}^{-1}T}+[Y_K(1)]\right)\left((\mathbb{L}-1)^{n-\dim K}\sum_{u\in\sigma_K^\circ\cap\mathbb{N}^n}\mathbb{L}^{-\langle u,\mathbf{1}\rangle}T^{N(u)}\right)\in\mathcal{M}_{\Bbbk}^{\hat\mu}[\![T]\!],\tag{14}$$

where the sum is over nonempty compact faces $K\in\Gamma$.

Remark 5.2.10. There is an extra factor of $(\mathbb{L}-1)$ in (14) that does not appear in [BN20, Theorem 8.3.5], for consistency with our choice of normalization of the local motivic zeta function, cf. Remark 1.2.6.

5.3. **The local formal zeta function.** We now introduce the *local formal zeta function* of f, denoted $Z_{\mathrm{for}}(T)$, which is a power series over a polynomial ring which specializes to $Z_{\mathrm{mot}}(T)$. The key advantage of $Z_{\mathrm{for}}(T)$ is that it lies in a power series ring over an integral domain, so it easier to understand sets of candidate poles of $Z_{\mathrm{for}}(T)$.

Let D be a ring containing $\mathbb{Z}[L,L^{-1},T,\frac{1}{1-L^{-1}T}]$ as a subring. Let $R_D=D[Y_K:\emptyset\neq K\in\Gamma, K \text{ compact}]/(\mathcal{I}_1+\mathcal{I}_2)$, where

$$\mathcal{I}_1=(Y_V-1\colon V\text{ primitive vertex of }\Gamma),$$
$$\mathcal{I}_2=(Y_G+Y_F-\frac{(L-1)^{\dim F}}{1-L^{-1}T}\colon F\text{ compact }B_1\text{-face with nonempty base }G).$$

When $D=\mathbb{Z}[L,L^{-1}][\![T]\!]$, we write $R:=R_D$. It follows from Example 5.2.3 and Example 5.2.4 that we have a well-defined $\mathbb{Z}[\![T]\!]$-algebra homomorphism

$$\mathrm{sp}\colon R\to\mathcal{M}_{\Bbbk}^{\hat\mu}[\![T]\!],$$
$$\mathrm{sp}(L)=\mathbb{L},\ \mathrm{sp}(Y_K)=[Y_K(0)]\frac{\mathbb{L}^{-1}T}{1-\mathbb{L}^{-1}T}+[Y_K(1)].$$

Definition 5.3.1. *With the notation above,*

$$Z_{\mathrm{for}}(T):=\sum_{\substack{\emptyset\neq K\in\Gamma\\ K\ \text{compact}}}Y_K\left((L-1)^{n-\dim K}\sum_{u\in\sigma_K^\circ\cap\mathbb{N}^n}L^{-\langle u,\mathbf{1}\rangle}T^{N(u)}\right)\in R.$$

Above, the fact that the right-hand side lies in R follows from Lemma 5.2.8. By Theorem 5.2.9, $\mathrm{sp}(Z_{\mathrm{for}}(T))=Z_{\mathrm{mot}}(T)$.

Lemma 5.3.2. *The ring R_D is isomorphic to a polynomial ring over D. Moreover, if D is a subring of D', then R_D is naturally a subring of $R_{D'}$.*

Proof. Consider the following change of variables. If K is a nonempty compact face of Γ, then let

$$Z_K:=(-1)^{\dim K}\left(Y_K-\frac{(L-1)^{\dim F+1}}{1-L^{-1}T}\sum_{i=0}^{n-\dim F-1}(1-L)^i\right).$$

Then $R_D=D[Z_K:\emptyset\neq K\in\Gamma, K\text{ compact}]/(\mathcal{I}_1+\mathcal{I}_2)$, where

$$\mathcal{I}_1=(Z_V-1+\frac{L-1}{1-L^{-1}T}\sum_{i=0}^{n-1}(1-L)^i\colon V\text{ primitive vertex of }\Gamma),$$

$$\mathcal{I}_2 = (Z_G - Z_F \colon F \text{ compact } B_1\text{-face with nonempty base } G).$$

Consider the equivalence relation on nonempty compact faces in Γ generated by $G \sim F$ whenever F is a compact B_1-face with nonempty base G. Then R is isomorphic to a polynomial ring over D with variables indexed by all equivalence classes that do not contain a primitive vertex of Γ. If D is a subring of D', then $R_{D'}$ is a polynomial ring over D' in the same variables as above. It follows that the natural map $R_D \to R_{D'}$ is injective. □

When $D = \mathbb{Z}[L][\![L^{-1}, T]\!]$, we let $\tilde{R} := R_D$. By Lemma 5.3.2, R is a subring of $\tilde{R}$. In what follows, we will freely view $Z_{\text{for}}(T)$ as an element of $\tilde{R}$, in order to ensure relevant infinite sums in L^{-1} and T are well-defined.

We next define the notion of a set of candidate poles for the local formal zeta function. Let $\mathcal{P}$ be a finite set of rational numbers containing -1. Then $\mathcal{P}$ is a *set of candidate poles* for some power series $Z(T) \in R$ if $Z(T)$ belongs to the subring R_D of R, where

$$D = \mathbb{Z}[L, L^{-1}, T]\left[\frac{1}{1 - L^a T^b}\right]_{(a,b) \in \mathbb{Z} \times \mathbb{Z}_{>0}, a/b \in \mathcal{P}}.$$

By Lemma 5.2.8, $\{\alpha \in \mathbb{R} : \mathrm{Contrib}(\alpha) \neq \emptyset\} \cup \{-1\}$ is a set of candidate poles for $Z_{\text{for}}(T)$.

Remark 5.3.3. Since $\mathrm{sp}(Z_{\text{for}}(T)) = Z_{\text{mot}}(T)$, any set of candidate poles for $Z_{\text{for}}(T)$ is a set of candidate poles for $Z_{\text{mot}}(T)$.

Remark 5.3.4. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be sets of candidate poles for elements $Z_1(T)$ and $Z_2(T)$ in R respectively. It follows from the definition that $\mathcal{P}_1 \cup \mathcal{P}_2$ is a set of candidate poles for $Z_1(T) + Z_2(T)$.

The main benefit of working with candidate poles of $Z_{\text{for}}(T)$ is that they satisfy the following lemma.

Lemma 5.3.5. *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be sets of candidate poles for $Z(T) \in R$. Then $\mathcal{P}_1 \cap \mathcal{P}_2$ is a set of candidate poles for $Z(T)$.*

Proof. Let $D' = \mathbb{Z}[L, L^{-1}, T, \frac{1}{1-L^{-1}T}]$ and $R' = R_{D'}$. We can write $Z(T) = \frac{F_i(T)}{G_i(t)}$ for $i \in \{1, 2\}$, for some $F_i(T) \in R$ and some $G_i(t)$ a finite product of terms of the form $\{1 - L^a T^b : (a, b) \in \mathbb{Z} \times \mathbb{Z}_{>0}, a/b \in \mathcal{P}_i\}$. Suppose that $G_1(T) = (1 - L^c T^d) G_1'(T)$ for some $(c, d) \in \mathbb{Z} \times \mathbb{Z}_{>0}, c/d \in \mathcal{P}_1 \smallsetminus \mathcal{P}_2$. By induction on $\deg(G_1(T))$, the result will follow if we can show that $F_1(T) = (1 - L^c T^d) F_1'(T)$, for some $F_1'(T) \in R'$.

Since the leading coefficient of $1 - L^c T^d$ is a unit in R', we may apply the division algorithm to write $F_1(T) = (1 - L^c T^d) F_1'(T) + \tilde{F}_1(T)$, for some $F_1'(T), \tilde{F}_1(T) \in R'$, with $\deg \tilde{F}_1(T) < d$. The equality $F_1(T) G_2(T) = F_2(T)(1 - L^c T^d) G_1'(T)$ in R' implies that $1 - L^c T^d$ divides $\tilde{F}_1(T) G_2(T)$. Over an appropriate choice of ring R_D containing R as a subring, $1 - L^c T^d$ has roots $\{\exp(\frac{2\pi i j}{d}) L^{-c/d} : 0 \leq j < d\}$. Similarly, $G_2(T)$ has roots contained in $\{\exp(\frac{2\pi i j}{b}) L^{-a/b} : 0 \leq j < b, (a, b) \in \mathbb{Z} \times \mathbb{Z}_{>0}, a/b \in \mathcal{P}_2\}$. Since $c/d \notin \mathcal{P}_2$ and $\deg \tilde{F}_1(T) < d$, we deduce that $\tilde{F}_1(T) = 0$, and $Z(T) = \frac{F_1'(T)}{G_1'(T)}$, as desired. □

Remark 5.3.6. By Lemma 5.3.5, if $Z(T) \in R$ admits a set of candidate poles, then there exists a minimal set of candidate poles. In particular, there exists a minimal set of candidate poles of $Z_{\text{for}}(T)$.

5.4. **Simplifying the local formal zeta function.** We now develop some tools to manipulate the local formal zeta function.

Given a subset $C \subset \mathbb{R}^n_{\geq 0}$, we write

$$Z_{\text{for}}(T)|_C := \sum_{\substack{\emptyset \neq K \in \Gamma \\ K \text{ compact}}} Y_K \left((L-1)^{n - \dim K} \sum_{u \in \sigma_K^\circ \cap C \cap \mathbb{N}^n} L^{-\langle u, \mathbf{1} \rangle} T^{N(u)} \right) \in \tilde{R}.$$

We will now prove a key technical tool we will use to manipulate $Z_{\text{for}}(T)$. Lemma 5.4.1 is analogous to [ELT22, Lemma 3.3].

Lemma 5.4.1. *Let F be a compact B_1-face with nonempty base G and apex A in the direction e_ℓ^*. Let C' be a nonzero rational polyhedral cone with $(C')^\circ \subset \sigma_F^\circ$, and let $C \subset \sigma_G$ be the convex hull of C' and $\mathbb{R}_{\geq} e_\ell^*$. Then*

$$Z_{\text{for}}(T)|_{(C^\circ \cup (C')^\circ)} = (L-1)^n \Bigg(\sum_{u \in (C^\circ \cup (C')^\circ) \cap \mathbb{N}^n} L^{-\langle u, \mathbf{1} \rangle} T^{\langle u, A \rangle} \Bigg) \in \tilde{R}.$$

Proof. A simplicial refinement of C' induces a simplicial refinement of C, so we may reduce to the case that C' is simplicial. Let $u_1, \dots, u_r$ be the primitive integer vectors spanning the rays of C'. Then $u_0 = e_\ell^*, u_1, \dots, u_r$ are the primitive integer vectors spanning the rays of C. With the notation of Lemma 5.2.5, $\mathrm{Box}^+(C') = \{u \in \mathbb{N}^n : u = \sum_{i=1}^r \lambda_i u_i \text{ for some } 0 < \lambda_i \leq 1\}$ and $\mathrm{Box}^+(C) = \{u \in \mathbb{N}^n : u = \sum_{i=0}^r \lambda_i u_i \text{ for some } 0 < \lambda_i \leq 1\}$. We claim that $\mathrm{Box}^+(C) = \{u + e_\ell^* : u \in \mathrm{Box}^+(C')\}$.

Clearly, $\{u + e_\ell^* : u \in \mathrm{Box}^+(C')\} \subset \mathrm{Box}^+(C)$. Conversely, consider an element $u' = \sum_{i=0}^r \lambda_i u_i \in \mathrm{Box}^+(C)$, for some $0 < \lambda_i \leq 1$. Let X be a point in G. For $1 \leq i \leq r$, since $u_i \in \sigma_F$, we have $\langle u_i, A - X \rangle = 0$. Also, $N(e_\ell^*) = \langle e_\ell^*, X \rangle = 0$ and $\langle e_\ell^*, A \rangle = 1$. We compute:

$$\langle u', A - X \rangle = \lambda_0 + \sum_{i=1}^r \lambda_i \langle u_i, A - X \rangle = \lambda_0 \in \mathbb{Z}.$$

Hence $\lambda_0 \in \mathbb{Z} \cap (0, 1] = \{1\}$. Then $u' = \sum_{i=1}^r \lambda_i u_i + e_\ell^* \in \{u + e_\ell^* : u \in \mathrm{Box}^+(C')\}$, which proves the claim.

Observe that if $u \in \mathrm{Box}^+(C')$, then $N(u + e_\ell^*) = N(u) = \langle u, A \rangle$ and $\langle u + e_\ell^*, \mathbf{1} \rangle = \langle u, \mathbf{1} \rangle + 1$. Also, $N(u_i) = \langle u_i, A \rangle$ for $1 \leq i \leq r$. Then using Lemma 5.2.5 and the relations in $\mathcal{I}_2$, we compute the left hand side of the equality in Lemma 5.4.1:

$$\begin{aligned}
&(L-1) \left(Y_F (L-1)^{n-1-\dim F} + Y_G (L-1)^{n-1-\dim G} \frac{L^{-1}}{1 - L^{-1}} \right) \frac{\sum_{u \in \mathrm{Box}^+(C')} L^{-\langle u, \mathbf{1} \rangle} T^{N(u)}}{\prod_{i=1}^r (1 - L^{-\langle u_i, \mathbf{1} \rangle} T^{N(u_i)})} \\
&= \frac{(L-1)^n}{1 - L^{-1}T} \frac{\sum_{u \in \mathrm{Box}^+(C')} L^{-\langle u, \mathbf{1} \rangle} T^{\langle u, A \rangle}}{\prod_{i=1}^r (1 - L^{-\langle u_i, \mathbf{1} \rangle} T^{\langle u_i, A \rangle})}.
\end{aligned}$$

Similarly, using Lemma 5.2.5, we compute the right-hand side of the equality in Lemma 5.4.1:

$$(L-1)^n \left(1 + \frac{L^{-1}T}{1 - L^{-1}T} \right) \frac{\sum_{u \in \mathrm{Box}^+(C')} L^{-\langle u, \mathbf{1} \rangle} T^{\langle u, A \rangle}}{\prod_{i=1}^r (1 - L^{-\langle u_i, \mathbf{1} \rangle} T^{\langle u_i, A \rangle})}.$$

The result follows. □

Remark 5.4.2. We note that a version of Lemma 5.4.1 holds when G is empty, i.e., $F = \{A\}$ is a vertex with some coordinate 1. Then A is a primitive vertex, so the relations in $\mathcal{I}_1$ imply that for $C \subset \sigma_A^\circ$,

$$Z_{\text{for}}(T)|_C = (L-1)^n \left(\sum_{u \in C \cap \mathbb{N}^n} L^{-\langle u, \mathbf{1} \rangle} T^{\langle u, A \rangle} \right) \in \tilde{R}.$$

5.5. **Combinatorial preliminaries.** In this section, we prepare for the constructions that will take the rest of the paper. First we introduce the notion of an operative labeling, which is a global consequence of the local assumption that every face of $\mathrm{Contrib}(\alpha)$ is UB_1.

Definition 5.5.1. *An* operative labeling *of* $\mathrm{Contrib}(\alpha)$ *is, for each $F \in \mathrm{Contrib}(\alpha)$, a choice (A_F, e_F^*), such that F is a B_1-face with apex A_F and base direction e_F^*, and if $F \subset F'$ and $A_F = A_{F'}$, then $e_F^* = e_{F'}^*$.*

Recall from Definition 1.4.3 that a face G of $\mathrm{Newt}(f)$ is UB_1 if there exists an apex A in G with a unique choice of base direction e_ℓ^*, and $\langle e_\ell^*, A\rangle = 1$.

Lemma 5.5.2. *If every face in* $\mathrm{Contrib}(\alpha)$ *is* UB_1*, then* $\mathrm{Contrib}(\alpha)$ *admits an operative labeling.*

Proof. Suppose that every face F in $\mathrm{Contrib}(\alpha)$ is UB_1. Then it has an apex A_F with a unique choice of base direction e_F^*. We may choose one such apex, and label F by (A_F, e_F^*) to get an operative labeling. Indeed, if $F \subset F'$ and $A_F = A_{F'}$, then $e_{F'}^*$ is a base direction for F with apex A_F. We deduce that $e_F^* = e_{F'}^*$. □

Given an element $V \in \Gamma$, we say that $u \in \mathbb{R}^n$ is *critical with respect to* (α, V) if $\alpha\langle u, V\rangle + \langle u, \mathbf{1}\rangle = 0$. Recall that we have a piecewise linear function $N(u) = \min\{\langle u, W\rangle : W \in \mathrm{Newt}(f)\}$. We say that $u \in \mathbb{R}^n_{\geq 0}$ is *critical with respect to* α if $\alpha N(u) + \langle u, \mathbf{1}\rangle = 0$. Equivalently, for some/any $G \in \Gamma$ such that $u \in \sigma_G$, and some/any element $V \in G$, u is critical with respect to (α, V). A set is critical with respect to α (or (α, V)) if every element of the set is critical with respect to α (or (α, V)).

Remark 5.5.3. Unless $V = -(1/\alpha)\mathbf{1}$, then the points $u \in \mathbb{R}^n$ critical with respect to (α, V) form a hyperplane. A special case of this observation is [ELT22, Lemma 3.4].

We conclude this section with three combinatorial observations.

Lemma 5.5.4. *Let M be an element of* $\mathrm{Contrib}(\alpha)$*, and assume that every face of* $\mathrm{Contrib}(\alpha)$ *is* UB_1*. Assume that $\alpha \neq -|\mathrm{Vert}(M)| \in \mathbb{Z}_{<0}$. Then there exists a vertex V_M of M such that M is not a B_1-face with apex V_M.*

Proof. Suppose every vertex of M is an apex. Since $\mathbf{1} \in \mathrm{span}(M)$, it follows that $\mathbf{1} - \sum_{V \in \mathrm{Vert}(M)} V$ is a linear combination of the unbounded directions of M, and hence $\alpha = -\psi_M(\mathbf{1}) = -|\mathrm{Vert}(M)|$, a contradiction. □

Lemma 5.5.5. *Assume that* $\mathrm{Newt}(f)$ *is α-simplicial. If M_1, M_2 are distinct minimal elements in* $\mathrm{Contrib}(\alpha)$*, then $\sigma_{M_1} \cap \sigma_{M_2} = \{0\}$.*

Proof. We argue by contradiction. Suppose $\sigma_{M_1} \cap \sigma_{M_2} \neq \{0\}$. Then there exists a facet F in $\partial\,\mathrm{Newt}(f)$ such that $\sigma_F \subset \sigma_{M_1} \cap \sigma_{M_2}$. Note that F is interior and hence $F \in \Gamma$. Equivalently, M_1, M_2 are common faces of F. In particular, $M_1 \cap M_2$ is a (possibly empty) face of F, and $C_{M_1 \cap M_2} = C_{M_1} \cap C_{M_2}$. Let $\mathrm{Gen}(C_{M_1} \smallsetminus C_{M_1 \cap M_2}) = \{W_1, \dots, W_s\}$ and $\mathrm{Gen}(C_{M_2} \smallsetminus C_{M_1 \cap M_2}) = \{W_1', \dots, W_{s'}'\}$. By Definition 5.1.1 applied to $M_2 \subset F$, $\mathrm{Gen}(C_{M_1} \smallsetminus C_{M_1 \cap M_2})$ are linearly independent in $\mathbb{R}^n/\mathrm{span}(C_{M_2})$. By Definition 5.1.1 applied to $M_1 \subset F$, $\mathrm{Gen}(C_{M_2} \smallsetminus C_{M_1 \cap M_2})$ are linearly independent in $\mathbb{R}^n/\mathrm{span}(C_{M_1})$. We claim that $W_1, \dots., W_s, W_1', \dots, W_{s'}'$ are linearly independent in $\mathbb{R}^n/\mathrm{span}(C_{M_1 \cap M_2})$. Indeed, if $\sum_i a_i W_i + \sum_j b_j W_j' = 0$ in $\mathbb{R}^n/\mathrm{span}(C_{M_1 \cap M_2})$, then the corresponding equation in $\mathbb{R}^n/\mathrm{span}(C_{M_1})$ implies that $b_j = 0$ for all j, and the corresponding equation in $\mathbb{R}^n/\mathrm{span}(C_{M_2})$ implies that $a_i = 0$ for all i.

By assumption, $\mathbf{1} = a_1 W_1 + \cdots + a_s W_s \in \mathbb{R}^n/\mathrm{span}(C_{M_1 \cap M_2})$, for some $a_1, \dots, a_s \in \mathbb{R}$, and $\mathbf{1} = a_1' W_1' + \cdots + a_{s'}' W_{s'}' \in \mathbb{R}^n/\mathrm{span}(C_{M_1 \cap M_2})$, for some $a_1', .., a_{s'}' \in \mathbb{R}$. Subtracting one equation from the other, and using the linear independence of $W_1, \dots., W_s, W_1', \dots, W_{s'}'$ in $\mathbb{R}^n/\mathrm{span}(C_{M_1 \cap M_2})$, we deduce that $a_i = a_j' = 0$ for all i, j. Hence $\mathbf{1} \in \mathrm{span}(C_{M_1 \cap M_2})$, and $M_1 \cap M_2 \in \mathrm{Contrib}(\alpha)$. This contradicts the minimality of M_1 and M_2. □

Remark 5.5.6. If e_ℓ^* is a base direction of F with apex A, then $e_\ell \notin I_F$. Indeed, if $e_j \in I_F$, then it follows from Definition 1.4.2 that $\langle e_\ell^*, e_j\rangle = 0$.

Assumptions and notation. For the remainder, we will assume that $\alpha \notin \mathbb{Z}_{<0}$, $\mathrm{Newt}(f)$ is α-simplicial and all faces of $\mathrm{Contrib}(\alpha)$ are UB_1. Let M be a minimal element of $\mathrm{Contrib}(\alpha)$. Recall that we have chosen an operative labeling (A_F, e_F^*) of $\mathrm{Contrib}(\alpha)$. By Lemma 5.5.4, we may fix a vertex V_M of M which is not an apex, and hence satisfies

$$\langle e_F^*, V_M\rangle = 0 \text{ for all } F \supset M. \tag{15}$$

We also fix a point W_M in the relative interior of M. Given a nonzero point W and $\epsilon \in \mathbb{R}$, let $\mathrm{H}_{W,\epsilon} = \{u : \langle u, W\rangle = \epsilon\}$, and consider the associated half-spaces $\mathrm{H}_{W,\geq\epsilon}, \mathrm{H}_{W,>\epsilon}, \mathrm{H}_{W,\leq\epsilon}, \mathrm{H}_{W,<\epsilon}$. We let $\mathrm{H}_W := \mathrm{H}_{W,0}$. Let $S' = \{u \in \mathbb{R}^n : \langle u, \mathbf{1}\rangle = 1\}$, and let $S = \mathrm{Conv}\,\{e_1^*, \dots, e_n^*\} \subset S'$ be the standard $(n-1)$-dimensional simplex.

5.6. **Covering the critical locus.** The goal of this section is to build small neighborhoods covering the locus of $u \in \mathbb{R}^n_{\geq 0}$ that is critical with respect to α. This will allow us to concentrate our attention on a single minimal face in $\mathrm{Contrib}(\alpha)$.

Definition 5.6.1. *Let M be a minimal element of* $\mathrm{Contrib}(\alpha)$ *and let $\delta \in \mathbb{Q}_{\geq 0}$. We define $N_{M,\leq\delta}$ to be the cone over $\{u \in S : \langle u, W_M\rangle - N(u) \leq \delta\}$.*

Similarly, we let $N_{M,<\delta}$, $N_{M,\delta}$ and $N_{M,\geq\delta}$ be the cones over $\{u \in S : \langle u, W_M\rangle - N(u) < \delta\}$, $\{u \in S : \langle u, W_M\rangle - N(u) = \delta\}$ and $\{u \in S : \langle u, W_M\rangle - N(u) \geq \delta\}$ respectively.

Here $\langle u, W_M\rangle - N(u)$ is a nonnegative function on $\mathbb{R}^n_{\geq 0}$ that is piecewise linear with respect to Σ. Because $N(u)$ is the support function of a polyhedron and hence convex, $N_{M,\leq\delta}$ is convex. It follows that $N_{M,\leq\delta}$ is a rational polyhedral cone of dimension n. Note that σ_M is the cone over $\{u \in S : \langle u, W_M\rangle - N(u) = 0\}$, and hence $\sigma_M \subset N_{M,\leq\delta}$. Note that we can equivalently write $N_{M,\leq\delta} = \{u \in \mathbb{R}^n_{\geq 0} : \langle u, W_M\rangle - N(u) \leq \delta\langle u, \mathbf{1}\rangle\}$. Also, $N^\circ_{M,\leq\delta} = N_{M,<\delta} \cap \mathbb{R}^n_{>0}$.

Lemma 5.6.2. *Let $C \subset \mathbb{R}^n_{\geq 0}$ be a closed cone such that $C \cap \sigma_M = \{0\}$. Then $C \cap N_{M,\leq\delta} = \{0\}$ for δ sufficiently small.*

Proof. We may assume that $C \cap S \neq \emptyset$. Since $C \cap S$ is compact, we may consider the minimal element $b > 0$ of $\{\langle u, W_M\rangle - N(u) : u \in C \cap S\}$. Then $C \cap N_{M,\leq\delta} \cap S = \emptyset$ for $\delta < b$. □

Lemma 5.6.3. *Let K be a face of $\partial\,\mathrm{Newt}(f)$ and suppose that $K \notin \Gamma$. Let M be a minimal element in* $\mathrm{Contrib}(\alpha)$*. Then $\sigma_K \cap N_{M,\leq\delta} = \{0\}$, for δ sufficiently small.*

Proof. By Lemma 5.6.2, it is enough to show that $\sigma_K \cap \sigma_M = \{0\}$. Suppose that $\sigma_K \cap \sigma_M \neq \{0\}$. Then $\sigma_K \cap \sigma_M = \sigma_{K'}$ for some face K' of $\partial\,\mathrm{Newt}(f)$ containing both K and M. Since $M \subset K'$, $K' \in \mathrm{Contrib}(\alpha)$. Since $K \subset K' \in \Gamma$, we deduce that $K \in \Gamma$, a contradiction. □

The following lemma is immediate from Lemma 5.5.5 and Lemma 5.6.2.

Lemma 5.6.4. *If M_1, M_2 are distinct minimal elements in* $\mathrm{Contrib}(\alpha)$*, then $N_{M_1,\leq\delta} \cap N_{M_2,\leq\delta} = \{0\}$, for δ sufficiently small.*

Lemma 5.6.5. *For δ sufficiently small, if $u \in N_{M,\leq\delta} \smallsetminus \{0\}$, then $N(u) > 0$.*

Proof. After scaling, we may assume that $u \in N_{M,\leq\delta} \cap S$. Since S is compact, we may consider $b = \min\{\langle u, W_M\rangle : u \in S\}$. Since M is interior, $W_M \notin \partial\mathbb{R}^n_{\geq 0}$, and hence $b > 0$. For $\delta < b$ and $u \in N_{M,\leq\delta} \cap S$, $N(u) = \langle u, W_M\rangle - (\langle u, W_M\rangle - N(u)) \geq \langle u, W_M\rangle - \delta \geq b - \delta > 0$. □

Let Δ be a fan supported on $\mathbb{R}^n_{\geq 0}$ that refines Σ. We may consider the fan $\Delta \cap N_{M,\leq\delta}$ supported on $N_{M,\leq\delta}$ given by all cones of the form $\{C \cap C' : C \in \Delta, C' \text{ is a face of } N_{M,\leq\delta}\}$. The lemma below gives a more explicit description.

Lemma 5.6.6. *Let Δ be a fan supported on $\mathbb{R}^n_{\geq 0}$ that refines Σ. Let $C \in \Delta$, and fix δ sufficiently small. Then*

(1) If $C \cap \sigma_M = \{0\}$, then $C \cap N_{M,\leq\delta} = \{0\}$.
(2) If $C \subset \sigma_M$, then C is a cone in $\Delta \cap N_{M,\leq\delta}$.
(3) If $C \cap \sigma_M \neq \{0\}$ and $C \not\subset \sigma_M$, then $\dim(C \cap N_{M,\leq\delta}) = \dim C$*. Moreover,* $\dim(C \cap N_{M,\delta}) = \dim C - 1$*, and $C \cap N_{M,\delta}$ is the only proper face of $C \cap N_{M,\leq\delta}$ that is not contained in a proper face of C.*

Proof. If $C \cap \sigma_M = \{0\}$, then $C \cap N_{M,\leq\delta} = \{0\}$ by Lemma 5.6.2. If $C \subset \sigma_M$, then since $\sigma_M \subset N_{M,\leq\delta}$, $C \cap N_{M,\leq\delta} = C$. This establishes the first two properties. Assume that $C \cap \sigma_M \neq \{0\}$ and $C \not\subset \sigma_M$. Since Δ refines Σ, $C \subset \sigma_K$ for some face K of $\partial\,\mathrm{Newt}(f)$. Fix a vertex V of K, and let $P = C \cap S$. Then $P \cap N_{M,\leq\delta} = P \cap \mathrm{H}_{W_M - V,\leq\delta}$. For any $\delta > 0$, the relative interior of P intersects both $\mathrm{H}_{W_M-V,>0}$, since $P \not\subset \sigma_M$, and $\mathrm{H}_{W_M-V,<\delta}$, since $P \cap \sigma_M \neq \emptyset$. It follows that for δ sufficiently small, $\mathrm{H}_{W_M-V,\delta}$ intersects the relative interior of P. We deduce that $P \cap N_{M,\leq\delta}$ has dimension $\dim P$, and that the only proper face of $P \cap N_{M,\leq\delta}$ that is not contained in a proper face of P is $P \cap \mathrm{H}_{W_M-V,\delta}$, which has dimension $\dim P - 1$. This establishes the result. □

The following two remarks are corollaries of the Lemma 5.6.6 and its proof.

Remark 5.6.7. Let C be a rational polyhedral cone such that $C \subset \sigma_K$ for some face K of $\partial\,\mathrm{Newt}(f)$. Then for δ sufficiently small, $C^\circ \cap N_{M,<\delta} = (C \cap N_{M,\leq\delta})^\circ$, and $C^\circ \cap N_{M,\geq\delta} = (C \cap N_{M,\geq\delta})^\circ \cup (C \cap N_{M,\delta})^\circ$.

Remark 5.6.8. Let K be a nonempty face of $\partial\,\mathrm{Newt}(f)$, and let $\tilde{\sigma}_K = \sigma_K \cap (\cap_i N_{M_i,\geq\delta})$. Assume that δ is chosen sufficiently small. If $K \in \mathrm{Contrib}(\alpha)$, then $M_i \subset K$ for some $1 \leq i \leq r$, and hence $\sigma_K \subset N_{M_i,<\delta}$, and $\tilde{\sigma}_K = \emptyset$. Assume that $K \notin \mathrm{Contrib}(\alpha)$. By Lemma 5.6.4 and Lemma 5.6.6, $\tilde{\sigma}_K$ is a rational polyhedral cone of dimension $\dim \sigma_K$, and $\sigma_K^\circ \cap (\cap_i N_{M_i,\geq\delta}) = \tilde{\sigma}_K^\circ \cup (\cup_i (\tilde{\sigma}_K \cap N_{M_i,\delta})^\circ)$.

Lemma 5.6.9. *Let Δ be a fan supported on $\mathbb{R}^n_{\geq 0}$ that refines Σ. Let γ be a ray of the fan $\Delta \cap N_{M,\leq\delta}$ such that $\gamma \not\subset \sigma_M$, for some δ chosen sufficiently small. Let C be the smallest cone in Δ containing γ. The following properties hold:*

1. *$C \cap \sigma_M \neq \{0\}$,*
2. *the smallest cone in Σ containing C has the form σ_K for some K in Γ,*
3. *if γ is critical with respect to (α, V), for some vertex V of either K or M, then C is critical with respect to (α, V).*

Proof. It follows from Lemma 5.6.6 that $\gamma = C \cap N_{M,\delta}$, $C \cap \sigma_M \neq \{0\}$, and $\dim(C \cap N_{M,\leq\delta}) = \dim C = 2$. This establishes (1). Lemma 5.6.3 implies (2). Since $\sigma_K \cap \sigma_M \neq \{0\}$, $\sigma_K \cap \sigma_M = \sigma_{K'}$ for some K' in Γ containing both M and K. Fix any vertex V of K'. Since $M \in \mathrm{Contrib}(\alpha)$, we have $\sigma_{K'} \subset \mathrm{H}_{\alpha V + \mathbf{1}}$. Let $\gamma' \neq \gamma$ be the other ray spanning $C \cap N_{M,\leq\delta}$. Then $\gamma' \subset \sigma_{K'} \subset \mathrm{H}_{\alpha V+\mathbf{1}}$. Hence, if $\gamma \subset \mathrm{H}_{\alpha V+\mathbf{1}}$, then $\mathrm{span}(C) = \mathrm{span}(C \cap N_{M,\leq\delta}) = \mathbb{R}\gamma + \mathbb{R}\gamma' \subset \mathrm{H}_{\alpha V+\mathbf{1}}$, and $C \subset \mathrm{H}_{\alpha V+\mathbf{1}}$. This establishes (3). □

Lemma 5.6.10. *Let γ be a ray of the fan $\Sigma \cap N_{M,\leq\delta}$ such that $\gamma \not\subset \sigma_M$, for some δ chosen sufficiently small. Then γ is not critical with respect to α.*

Proof. By Lemma 5.6.9, there exists $K \in \Gamma$ such that σ_K is the smallest cone in Σ containing γ. Suppose that γ is critical with respect to α. Then γ is critical with respect to (α, V), for any vertex V of K. By Lemma 5.6.9, σ_K is critical with respect to (α, V), and hence $K \in \mathrm{Contrib}(\alpha)$. Since $\sigma_K \cap N_{M,\leq\delta} \neq \{0\}$, Lemma 5.6.4 implies that $M \subset K$ and hence $\gamma \subset \sigma_M$, a contradiction. □

Lemma 5.6.11. *Let $M_1, \ldots, M_r$ be the minimal elements in* $\mathrm{Contrib}(\alpha)$*, and let K be a nonempty face of $\partial\,\mathrm{Newt}(f)$. Then for δ sufficiently small, $(L-1)^{n-\dim K} \sum_{u \in \sigma_K^\circ \cap (\cap_i N_{M_i,\geq\delta}) \cap \mathbb{N}^n} L^{-\langle u, \mathbf{1}\rangle} T^{N(u)}$ lies in R and admits a set of candidate poles not containing α.*

Proof. Let $\tilde{\sigma}_K = \sigma_K \cap (\cap_i N_{M_i,\geq\delta})$. By Remark 5.6.8, if $K \in \mathrm{Contrib}(\alpha)$, then $\tilde{\sigma}_K = \emptyset$. Assume that $K \notin \mathrm{Contrib}(\alpha)$. By Remark 5.6.8, $\tilde{\sigma}_K$ is a rational polyhedral cone of dimension $\dim \sigma_K$, and $\sigma_K^\circ \cap (\cap_i N_{M_i,\geq\delta}) = \tilde{\sigma}_K^\circ \cup (\cup_i (\tilde{\sigma}_K \cap N_{M_i,\delta})^\circ)$. By Lemma 5.6.2, the rays of $\tilde{\sigma}_K$ are the union of the rays of σ_K that are not critical with respect to α, and the rays of $\sigma_K \cap N_{M_i,\leq\delta}$ that do not lie in σ_{M_i}, for $1 \leq i \leq r$. By Lemma 5.6.10, none of the rays of $\tilde{\sigma}_K$ are critical with respect to α. By Remark 5.2.7 and Lemma 5.6.5, if u is a primitive generator of a ray of $\tilde{\sigma}_K$ and $N(u) = 0$, then $u = e_i^*$ for some $1 \leq i \leq n$, and hence $\langle u, \mathbf{1}\rangle = 1$.

Since the restriction of N to $\tilde{\sigma}_K \subset \sigma_K$ is linear, Lemma 5.2.6 implies that the following elements of $\tilde{R}$ lie in R and admit sets of candidate poles not containing α:

$$(L-1)^{n-\dim K} \sum_{u \in \tilde{\sigma}_K^\circ \cap \mathbb{N}^n} L^{-\langle u, \mathbf{1} \rangle} T^{N(u)}, \text{ and } (L-1)^{n-\dim K} \sum_{u \in (\tilde{\sigma}_K \cap N_{M_i,\delta})^\circ \cap \mathbb{N}^n} L^{-\langle u, \mathbf{1} \rangle} T^{N(u)},$$

for $1 \leq i \leq r$. The result now follows from Remark 5.3.4. □

The lemma below follows immediately from Definition 5.3.1 and Lemma 5.6.11 and will allow us to reduce our study of $Z_{\text{for}}(T)$ to the study of $Z_{\text{for}}(T)|_{N_{M,<\delta}}$, when M is a minimal element of $\text{Contrib}(\alpha)$.

Lemma 5.6.12. *Let $M_1, \ldots, M_r$ be the minimal elements in* $\text{Contrib}(\alpha)$. *Then for δ sufficiently small, $Z_{\text{for}}(T)|_{\cap_i N_{M_i,\geq\delta}}$ lies in R and admits a set of candidate poles not containing α.*

5.7. **Establishing fake poles using α-compatible sets.** The goal of this section is to show that the existence of a fan with certain properties implies that there is a set of candidate poles for the local formal zeta function not containing α.

From this point on, we fix a minimal element M in $\text{Contrib}(\alpha)$. Let $\text{Contrib}(\alpha)_M := \{F \in \text{Contrib}(\alpha) \colon F \supset M\}$. Fix a nonempty finite set $\mathcal{S}$. Given a finite collection $\mathcal{Z} = (Z_s)_{s \in \mathcal{S}}$ of elements in $\mathbb{Q}^n$ indexed by $\mathcal{S}$, we let $Q_{\mathcal{Z}}$ denote the convex hull of $\mathcal{Z}$, and let $\Sigma_{\mathcal{Z}}$ be the corresponding (rational) dual fan supported on $\mathbb{R}^n$. Given a nonempty face J of $Q_{\mathcal{Z}}$, we write τ_J for the corresponding cone in $\Sigma_{\mathcal{Z}}$. Given an element $s \in \mathcal{S}$, we write J_{Z_s} for the smallest face of $Q_{\mathcal{Z}}$ containing Z_s, and write $\tau_{Z_s} := \tau_{J_{Z_s}}$. Explicitly, $\tau_{Z_s} := \{u \in \mathbb{R}^n : \langle u, Z_s \rangle \leq \langle u, Z_{s'} \rangle, \text{ for all } s' \in \mathcal{S}\}$. Observe that $\tau_J = \cap_{Z_s \in J} \tau_{Z_s}$. Also, note that we do not require the elements of $\mathcal{Z}$ to be distinct, and that τ_{Z_s} may equal $\tau_{Z_{s'}}$ even if $s \neq s'$.

Definition 5.7.1. *Consider a pair $(\mathcal{Z}, \mathcal{F})$, where $\mathcal{Z} = (Z_s)_{s \in \mathcal{S}}$ and $\mathcal{F} = (F_s)_{s \in \mathcal{S}}$ are collections of elements of $\mathbb{Q}^n$ and* $\text{Contrib}(\alpha)_M$ *respectively. Then $(\mathcal{Z}, \mathcal{F})$ is* weakly α-compatible *if it satisfies the following property: Suppose that $\sigma_K^\circ \cap \tau_{Z_s} \cap \tau_{Z_{s'}} \neq \emptyset$ for some $K \in$* $\text{Contrib}(\alpha)_M$ *and $s, s' \in \mathcal{S}$, with possibly $s = s'$. Then*

(1) $K \subset F_s$, and
(2) either $F_s \subset F_{s'}$ or $F_{s'} \subset F_s$, and
(3) $\langle e_{F_s}^*, Z_s \rangle = \langle e_{F_s}^*, Z_{s'} \rangle = 0$.

Lemma 5.7.2. *Let $(\mathcal{Z}, \mathcal{F})$ be a weakly α-compatible pair, and let J be a nonempty face of $Q_{\mathcal{Z}}$ such that $\sigma_M \cap \tau_J \neq \{0\}$. Then $\{F_s : Z_s \in J\}$ is the set of elements of a chain of faces in Γ. Moreover, if $\sigma_K^\circ \cap \tau_J \neq \emptyset$ for some $K \in$* $\text{Contrib}(\alpha)_M$*, then $K \subset F_s$ for all $s \in \mathcal{S}$ such that $Z_s \in J$.*

Proof. There exists $K \in \text{Contrib}(\alpha)_M$ such that $\sigma_K^\circ \cap \tau_J \neq \emptyset$. Hence, for any $s, s' \in \mathcal{S}$ such that $Z_s, Z_{s'} \in J$, $\sigma_K^\circ \cap \tau_{Z_s} \cap \tau_{Z_{s'}} \neq \emptyset$. By property (2) of Definition 5.7.1, F_s and $F_{s'}$ are comparable under inclusion. Hence if we consider the set $\{F_s : Z_s \in J\}$ as a poset under inclusion, then all elements are comparable, and the poset is a chain. The second statement follows from property (1) of Definition 5.7.1. □

Definition 5.7.3. *Let $(\mathcal{Z}, \mathcal{F})$ be a weakly α-compatible pair, and let J be a nonempty face of $Q_{\mathcal{Z}}$ such that $\sigma_M \cap \tau_J \neq \{0\}$. Then set $F_J := \max\{F_s : Z_s \in J\}$.*

Lemma 5.7.2 implies that the above is well-defined. Also, it follows from Lemma 5.6.2 that we can replace the condition $\sigma_M \cap \tau_J \neq \{0\}$ with the condition that $\tau_J \cap N_{M,\leq\delta} \neq \{0\}$ for some δ chosen sufficiently small.

Remark 5.7.4. Let J be a nonempty face of $Q_{\mathcal{Z}}$ and $K \in \text{Contrib}(\alpha)_M$ such that $\sigma_K^\circ \cap \tau_J \neq \emptyset$. Then Lemma 5.7.2 implies that $K \subset F_J$.

Remark 5.7.5. If $J \subset J'$ is an inclusion of nonempty faces of $Q_{\mathcal{Z}}$ and $\sigma_M \cap \tau_{J'} \neq \{0\}$, then $\sigma_M \cap \tau_J \neq \{0\}$ and $F_J \subset F_{J'}$.

Lemma 5.7.6. *Let $(\mathcal{Z}, \mathcal{F})$ be a weakly α-compatible pair, and let J be a nonempty face of $Q_{\mathcal{Z}}$ and $K \in \Gamma$ such that $\sigma_K \cap \tau_J \cap N_{M,\leq\delta} \neq \{0\}$ for some δ sufficiently small. Then $K \subset F_J$.*

Proof. Since $\sigma_K \cap \tau_J \cap N_{M,\leq\delta} \neq \{0\}$, Lemma 5.6.2 implies that $\sigma_K \cap \tau_J \cap \sigma_M \neq \{0\}$. In particular, F_J is well-defined. Since $\sigma_K \cap \sigma_M \neq (0)$, we have that $\sigma_K \cap \sigma_M = \sigma_{K'}$ for some $K' \in \mathrm{Contrib}(\alpha)_M$. Since $\sigma_{K'} \cap \tau_J \neq \{0\}$, there exists $K' \subset K'' \in \mathrm{Contrib}(\alpha)_M$ such that $\sigma^\circ_{K''} \cap \tau_J \neq \emptyset$. By Remark 5.7.4, $K'' \subset F_J$. Since $K \subset K' \subset K''$, the result follows. □

Let Σ_1, Σ_2 be fans in $\mathbb{R}^n$ dual to polyhedra P_1, P_2 in $\mathbb{R}^n$ respectively. Then the Minkowski sum $P_1 + P_2$ is dual to the intersection $\Sigma_1 \cap \Sigma_2$ of Σ_1 and Σ_2, where $\Sigma_1 \cap \Sigma_2$ is the fan consisting of all cones $\{\sigma_1 \cap \sigma_2 : \sigma_i \in \Sigma_i\}$. All faces of $P_1 + P_2$ have the form $J_1 + J_2$ for some faces J_i of P_i for $i = 1, 2$. If J_i is dual to σ_i in Σ_i for $i = 1, 2$, then a face of the form $J_1 + J_2$ is dual to $\sigma_1 \cap \sigma_2$. Conversely, every cone C in $\Sigma_1 \cap \Sigma_2$ has the form $C = \sigma_1 \cap \sigma_2$, where σ_i is the smallest face of Σ_i containing C, for $i = 1, 2$. Then $C^\circ = \sigma_1^\circ \cap \sigma_2^\circ$, $\mathrm{span}(C) = \mathrm{span}(\sigma_1) \cap \mathrm{span}(\sigma_2)$, and if σ_i is dual to a face J_i of P_i, for $i = 1, 2$, then C is dual to $J_1 + J_2$. We will be interested in the polyhedron $\mathrm{Newt}(f)_{\mathcal{Z}} := \mathrm{Newt}(f) + Q_{\mathcal{Z}}$ dual to $\Sigma \cap \Sigma_{\mathcal{Z}}$.

Definition 5.7.7. *Let $(\mathcal{Z}, \mathcal{F})$ be a weakly α-compatible pair, let J be a nonempty face of $Q_{\mathcal{Z}}$, and let C be a cone in $\Sigma \cap \Sigma_{\mathcal{Z}}$. Assume that $C \subset \tau_J$ and $C \cap N_{M,\leq\delta} \neq \{0\}$ for some δ chosen sufficiently small. Let $D(C, J) = D(C, J, M, \delta) := \mathrm{Conv}\left\{C \cap \sigma_{A_{F_J}} \cap N_{M,\leq\delta}, \mathbb{R}_{\geq 0} e^*_{F_J}\right\}$.*

The following lemma will allow us to replace contributions to $Z_{\mathrm{for}}(T)$ from certain cones C by contributions from cones $D(C, J)$, whose structure will allow us to apply Lemma 5.4.1.

Lemma 5.7.8. *Let $(\mathcal{Z}, \mathcal{F})$ be a weakly α-compatible pair, let J be a nonempty face of $Q_{\mathcal{Z}}$, and let C be a cone in $\Sigma \cap \Sigma_{\mathcal{Z}}$. Assume that $C \subset \tau_J$ and that $C \cap N_{M,\leq\delta} \neq \{0\}$ for some δ chosen sufficiently small. Then C is dual to a face $K + J'$ of $\mathrm{Newt}(f)_{\mathcal{Z}}$, for some face $K \in \Gamma$ such that $K \subset F_J \subset F_{J'}$, and for some face J' of $Q_{\mathcal{Z}}$ such that $J \subset J'$. Suppose that $C \not\subset \sigma_{A_{F_J}}$. Then $C \cap N_{M,\leq\delta} = D(C, J) \cap N_{M,\leq\delta}$, and $e^*_{F_J} \in \mathrm{span}(C)$.*

Proof. By Lemma 5.6.3, C is dual to a face of $\mathrm{Newt}(f)_{\mathcal{Z}}$ of the form $K + J'$, for some $K \in \Gamma$, and some nonempty face J' of $Q_{\mathcal{Z}}$. Here $C = \sigma_K \cap \tau_{J'}$ and $\tau_{J'}$ is the smallest cone in $\Sigma_{\mathcal{Z}}$ containing C. In particular, $\tau_{J'} \subset \tau_J$ and $J \subset J'$. By Lemma 5.6.2, $\sigma_M \cap \tau_{J'} \neq \{0\}$, and $F_J, F_{J'}$ are well-defined. Also, $\{0\} \neq \sigma_K \cap \tau_J \cap N_{M,\leq\delta}$. Then Lemma 5.7.6 implies that $K \subset F_J$. By Remark 5.7.5, $F_J \subset F_{J'}$.

Fix a vertex V of K. Recall that I_K denotes the unbounded directions of K. Then

$$\sigma_K = \Big(\bigcap_{V \neq V' \in K} \mathrm{H}_{V'-V}\Big) \cap \Big(\bigcap_{V' \notin K} \mathrm{H}_{V'-V,\geq 0}\Big) \cap \Big(\bigcap_{W \in I_K} \mathrm{H}_W\Big) \cap \Big(\bigcap_{W \in \{e_1,\ldots,e_n\} \smallsetminus I_K} \mathrm{H}_{W,\geq 0}\Big), \tag{16}$$

where V' varies over vertices in Γ. Let $\hat{s} \in \mathcal{S}$ such that $Z_{\hat{s}} \in J$ and $F_{\hat{s}} = F_J$. Then $Z_{\hat{s}} \in J'$ and

$$\tau_{J'} = \Big(\bigcap_{\substack{\hat{s} \neq s \in S \\ Z_s \in J'}} \mathrm{H}_{Z_s - Z_{\hat{s}}}\Big) \cap \Big(\bigcap_{\substack{s \in S \\ Z_s \notin J'}} \mathrm{H}_{Z_s - Z_{\hat{s}},\geq 0}\Big). \tag{17}$$

It follows that we may choose a finite collection of nonzero elements $P_C = \{W\} \subset \mathbb{Q}^n \smallsetminus \{0\}$ such that each $W \in P_C$ is of the form either

(1) $W = Z_s - Z_{\hat{s}}$ for some $s \in \mathcal{S}$,
(2) $W = V' - V$ for some vertex V' in Γ,
(3) $W \in \{e_1, \ldots, e_n\}$,

and $C = \sigma_K \cap \tau_{J'}$ is the intersection of half-spaces of the form $\mathrm{H}_{W,\geq 0}$ or hyperplanes of the form H_W for various $W \in P_C$.

Suppose that $C \not\subset \sigma_{A_{F_J}}$. Note that $C \not\subset \sigma_{A_{F_J}}$ if and only if $\sigma_K \not\subset \sigma_{A_{F_J}}$, if and only if $A_{F_J} \notin K$. Let $\tilde{\sigma}$ be the intersection of all such half-spaces and hyperplanes appearing in the description (16) of σ_K such that the $W \in P_C$ that defines the hyperplane or half-space has $C \cap N_{M,\leq\delta} \cap \mathrm{H}_W \neq \{0\}$, and W doesn't have the form $W = V' - V$ with $V' = A_{F_J}$. Similarly, let $\tilde{\tau}$ be the intersection of all such half-spaces and hyperplanes appearing in the description (17) of $\tau_{J'}$ such that $C \cap N_{M,\leq\delta} \cap \mathrm{H}_W \neq \{0\}$. Let $\tilde{C} = \tilde{\sigma} \cap \tilde{\tau}$. Then, by construction, there exists a cone U over a small open neighborhood of $N_{M,\leq\delta} \cap S'$ in $S' = \{u \in \mathbb{R}^n : \langle u, \mathbf{1}\rangle = 1\}$ such that

$$C \cap U = \tilde{C} \cap \mathrm{H}_{A_{F_J}-V,\geq 0} \cap U. \tag{18}$$

We claim that $\mathbb{R}e^*_{F_J} \subset \tilde{C}$. Let $\mathrm{H}_{W,\geq 0}$ or H_W be a defining half-space or hyperplane of $\tilde{C}$. We need to show that $e^*_{F_J} \in \mathrm{H}_W$. Equivalently, we need to show that $\langle e^*_{F_J}, W\rangle = 0$. By assumption, $C \cap N_{M,\leq\delta} \cap \mathrm{H}_W \neq \{0\}$. By Lemma 5.6.2, $C \cap \sigma_M \cap \mathrm{H}_W \neq \{0\}$.

First, assume that $W = V' - V$ for some vertex $V' \neq A_{F_J}$ in Γ. Then $\{0\} \neq C \cap N_{M,\leq\delta} \cap \mathrm{H}_W \subset \sigma_{V'} \cap \tau_J \cap N_{M,\leq\delta}$, and Lemma 5.7.6 implies that $V' \in F_J$. Then V', V are vertices of F_J that are not equal to A_{F_J}, and hence $\langle e^*_{F_J}, V'\rangle = \langle e^*_{F_J}, V\rangle = 0$, so $\langle e^*_{F_J}, W\rangle = 0$.

Second, assume that $W = Z_s - Z_{\hat{s}}$, for some s in $\mathcal{S}$. Since $C \subset \tau_{J'}$, we have $\{0\} \neq C \cap \sigma_M \cap \mathrm{H}_W \subset \tau_{J'} \cap \sigma_M \cap \mathrm{H}_W \subset \sigma_M \cap \tau_{Z_s} \cap \tau_{Z_{\hat{s}}}$. Hence, there exists $K' \in \mathrm{Contrib}(\alpha)_M$ such that $\sigma^\circ_{K'} \cap \tau_{Z_s} \cap \tau_{Z_{\hat{s}}} \neq \emptyset$. By (3) in Definition 5.7.1, $\langle e^*_{F_J}, Z_s\rangle = \langle e^*_{F_J}, Z_{\hat{s}}\rangle = 0$, so $\langle e^*_{F_J}, W\rangle = 0$.

Finally, assume $W \in \{e_1, \dots, e_n\}$. Then $\{0\} \neq C \cap N_{M,\leq\delta} \cap \mathrm{H}_W \subset \sigma_K \cap \tau_J \cap N_{M,\leq\delta} \cap \mathrm{H}_W$. By Lemma 5.6.2, $\sigma_K \cap \tau_J \cap \sigma_M \cap \mathrm{H}_W \neq \{0\}$. It follows that there exists $K' \in \mathrm{Contrib}(\alpha)_M$ such that $K \subset K'$ and $\sigma^\circ_{K'} \cap \tau_J \cap \mathrm{H}_W \neq \{0\}$. By Remark 5.7.4, $K' \subset F_J$. Then $W \in I_{K'} \subset I_{F_J}$, and hence $\langle e^*_{F_J}, W\rangle = 0$ by Remark 5.5.6.

We conclude that $\mathbb{R}e^*_{F_J} \subset \tilde{C}$. Next, we claim that

$$\tilde{C} \cap \mathrm{H}_{A_{F_J}-V,\geq 0} \cap N_{M,\leq\delta} = \mathrm{Conv}\left\{\tilde{C} \cap \mathrm{H}_{A_{F_J}-V} \cap N_{M,\leq\delta}, \mathbb{R}_{\geq 0}e^*_{F_J}\right\} \cap N_{M,\leq\delta}. \tag{19}$$

By (18), the left hand side of (19) is $C \cap N_{M,\leq\delta}$. Let $u \in \tilde{C} \cap \mathrm{H}_{A_{F_J}-V,\geq 0} \cap N_{M,\leq\delta}$. We aim to show that u lies in the right-hand side of (19). It is enough to consider the case when $\langle u, \mathbf{1}\rangle = 1$. Consider the function

$$\phi\colon \mathbb{R}_{\geq 1} \to S' \subset \mathbb{R}^n,$$

$$\phi(\lambda) = \lambda u + (1-\lambda)e^*_{F_J}.$$

Since $\mathbb{R}e^*_{F_J} \subset \tilde{C}$, the image of ϕ is contained in $\tilde{C}$. It is enough to show that $\phi^{-1}(\mathrm{H}_{A_{F_J}-V} \cap N_{M,\leq\delta}) \neq \emptyset$, since if $\lambda \in \phi^{-1}(\mathrm{H}_{A_{F_J}-V} \cap N_{M,\leq\delta})$, then

$$u = (1/\lambda)(\phi(\lambda) + (\lambda - 1)e^*_{F_J}), \tag{20}$$

and u lies in the right-hand side of (19). Moreover, if we choose $u \in C^\circ$, then $u \notin \sigma_{A_{F_J}}$ and hence $\lambda > 1$ in (20). Then $e^*_{F_J} = (1/(\lambda-1))(\lambda u - \phi(\lambda)) \in \mathrm{span}(C)$, which establishes the last statement of the lemma.

Consider the linear function

$$f(\lambda) = \langle \phi(\lambda), W_M - V\rangle.$$

Since $u \in S' \cap N_{M,\leq\delta}$, $f(1) \leq \delta$. Since $\langle e^*_{F_J}, V\rangle = 0$, we compute: $f'(\lambda) = f(1) - \langle e^*_{F_J}, W_M\rangle \leq \delta - \langle e^*_{F_J}, W_M\rangle < 0$. The last inequality follows since M is interior implies that $\langle e^*_{F_J}, W_M\rangle > 0$, and δ is chosen sufficiently small. In particular, since f is a non-constant linear function its image is unbounded. Hence the image of ϕ is unbounded and so $\phi^{-1}(U \smallsetminus N_{M,\leq\delta}) \neq \emptyset$.

We claim that $\phi^{-1}(\mathrm{H}_{A_{F_J}-V,\geq 0} \cap U) \subset \phi^{-1}(N_{M,\leq\delta})$. Indeed, if $\phi(\lambda) \in \mathrm{H}_{A_{F_J}-V,\geq 0} \cap U$, then $\phi(\lambda) \in C$ by (18). Then $\langle \phi(\lambda), W_M\rangle - N(\phi(\lambda)) = f(\lambda) \leq f(1) \leq \delta$. Since $\langle \phi(\lambda), \mathbf{1}\rangle = 1$, we deduce that $\phi(\lambda) \in N_{M,\leq\delta}$.

It follows that $\emptyset \neq \phi^{-1}(U \setminus N_{M,\leq\delta}) \subset \phi^{-1}(\mathrm{H}_{A_{F_J}-V,<0} \cap U)$. Since $1 \in \phi^{-1}(\mathrm{H}_{A_{F_J}-V,\geq 0} \cap U)$, we deduce that $\phi^{-1}(\mathrm{H}_{A_{F_J}-V} \cap N_{M,\leq\delta}) = \phi^{-1}(\mathrm{H}_{A_{F_J}-V} \cap U) \neq \emptyset$, so the left-hand side is contained in the right-hand side of (19).

Conversely, since $\mathbb{R}e^*_{F_J} \subset \tilde{C}$ and $e^*_{F_J} \in \mathrm{H}_{A_{F_J}-V,\geq 0}$, the right-hand side of (19) is contained in $\tilde{C} \cap \mathrm{H}_{A_{F_J}-V,\geq 0} \cap N_{M,\leq\delta}$. This establishes (19). By (18),

$$\tilde{C} \cap \mathrm{H}_{A_{F_J}-V} \cap N_{M,\leq\delta} = (\tilde{C} \cap \mathrm{H}_{A_{F_J}-V,\geq 0} \cap N_{M,\leq\delta}) \cap \sigma_{A_{F_J}} = (C \cap N_{M,\leq\delta}) \cap \sigma_{A_{F_J}}.$$

Substituting this expression into the right-hand side of (19) and combining with (18), we deduce that

$$C \cap N_{M,\leq\delta} = \tilde{C} \cap \mathrm{H}_{A_{F_J}-V,\geq 0} \cap N_{M,\leq\delta} = \mathrm{Conv}\left\{C \cap \sigma_{A_{F_J}} \cap N_{M,\leq\delta}, \mathbb{R}_{\geq 0} e^*_{F_J}\right\} \cap N_{M,\leq\delta}. \qquad \square$$

The following lemma is a corollary of the proof of Lemma 5.7.8.

Lemma 5.7.9. *Let $(\mathcal{Z}, \mathcal{F})$ be a weakly α-compatible pair, and let J be a nonempty face of $Q_{\mathcal{Z}}$ such that $\tau_J \cap N_{M,\leq\delta} \neq \{0\}$ for some δ chosen sufficiently small. Then no ray of $\tau_J \cap N_{M,\leq\delta}$ is contained in σ_M.*

Proof. The proof of Lemma 5.7.8, with σ_K replaced by $\mathbb{R}^n_{\geq 0}$ and $\tau_{J'}$ replaced by τ_J, shows that there exists a polyhedral cone τ' and a cone U over a small open neighborhood of $N_{M,\leq\delta} \cap S'$ in S' such that

(1) $\mathbb{R}e^*_{F_J} \subset \tau'$,
(2) $\mathbb{R}^n_{\geq 0} \cap \tau_J \cap U = \tau' \cap U$.

Let u be a generator of a ray in $\tau_J \cap N_{M,\leq\delta}$. We may assume that $\langle u, \mathbf{1}\rangle = 1$. Suppose that $\langle u, W_M\rangle - N(u) < \delta$. Fix $0 < \epsilon \ll 1$ and let $L_\epsilon = \{u + \lambda e^*_{F_J} : |\lambda| < \epsilon\}$. Then $L_\epsilon \subset \tau' \cap U = \mathbb{R}^n_{\geq 0} \cap \tau_J \cap U$. It follows that $L_\epsilon \subset \tau_J \cap N_{M,\leq\delta}$, contradicting the assumption that u generates a ray. We deduce that $\langle u, W_M\rangle - N(u) = \delta$. In particular, $u \notin \sigma_M$. $\square$

Definition 5.7.10. *Let $(\mathcal{Z}, \mathcal{F})$ be a weakly α-compatible pair, let J be a nonempty face of $Q_{\mathcal{Z}}$, and let C be a cone in $\Sigma \cap \Sigma_{\mathcal{Z}}$. Then we say (C, J) is* α-critical *if the following properties hold:*

(1) $C \cap \sigma_M \neq \{0\}$,
(2) $C \subset \sigma_{A_{F_J}} \cap \tau_J$,
(3) C is critical with respect to (α, A_{F_J}).

Definition 5.7.11. *We say a weakly α-compatible pair $(\mathcal{Z}, \mathcal{F})$ is* α-compatible *if for every α-critical pair (C, J), $C \subset \sigma_M$.*

Note that the notion of an α-compatible pair depends on the choice of a minimal face M. The main technical result required to prove Theorem 5.1.2 is following result on the existence of α-compatible pairs.

Theorem 5.7.12. *Let $\alpha \notin \mathbb{Z}_{<0}$, and assume that all faces of* $\mathrm{Contrib}(\alpha)$ *are UB_1-and* $\mathrm{Newt}(f)$ *is α-simplicial. Then for any minimal face $M \in \mathrm{Contrib}(\alpha)$, there exists an α-compatible pair.*

Lemma 5.7.13. *Consider an α-compatible pair $(\mathcal{Z}, \mathcal{F})$. Let $\Delta = \Sigma \cap \Sigma_{\mathcal{Z}}$, and let γ be a ray of $\Delta \cap N_{M,\leq\delta}$, for some δ chosen sufficiently small. Assume that $\gamma \not\subset \sigma_M$. Then γ is not critical with respect to α. Moreover, if $\gamma \subset \tau_J$ for some face J of $Q_{\mathcal{Z}}$, then γ is not critical with respect to (α, A_{F_J}).*

Proof. There is a unique face J' of $Q_{\mathcal{Z}}$ such that $\gamma \subset \tau^\circ_{J'}$. If $\gamma \subset \tau_J$, then it follows that $J \subset J'$. Let C be the smallest cone in Δ containing γ. Then $\tau_{J'}$ is the smallest cone of $\Sigma_{\mathcal{Z}}$ containing C. By Lemma 5.7.8, C is dual to a face $K + J'$ of $\mathrm{Newt}(f)_{\mathcal{Z}}$, where $K \in \Gamma$ such that $K \subset F_J$. Assume that γ is critical with respect to (α, W) for some vertex W of either K or M. Note that one possible choice for W is A_{F_J}, since $A_{F_J} \in M$. Then Lemma 5.6.9 implies that C is critical with respect to (α, W), and $C \cap \sigma_M \neq \{0\}$.

First, assume that $\gamma \subset \sigma_{A_{F_J}}$. Then $C \subset \sigma_{A_{F_J}}$. If $W = A_{F_J}$, then (C, J) is α-critical, and Definition 5.7.11 implies that $\gamma \subset C \subset \sigma_M$, a contradiction. We conclude that $W \neq A_{F_J}$. That is, γ is not critical with respect to (α, A_{F_J}). Equivalently, in this case, γ is not critical with respect to α.

Second, assume that $\gamma \not\subset \sigma_{A_{F_J}}$. Then $C \not\subset \sigma_{A_{F_J}}$. Equivalently, $A_{F_J} \notin K$. By Lemma 5.7.8, $e^*_{F_J} \in \operatorname{span}(C)$. Let V be a vertex of $K \subset F_J$. Since $V \neq A_{F_J}$, $\langle e^*_{F_J}, V\rangle = 0$, and hence $\langle e^*_{F_J}, \alpha V + \mathbf{1}\rangle = 1 \neq 0$. We deduce that C is not critical with respect to (α, V). Hence $W \neq V$, and γ is not critical with respect to α. Similarly, since $\alpha \neq -1$ by assumption, $\langle e^*_{F_J}, \alpha A_{F_J} + \mathbf{1}\rangle = \alpha + 1 \neq 0$, and hence C is not critical with respect to (α, A_{F_J}). Therefore $W \neq A_{F_J}$, and γ is not critical with respect to (α, A_{F_J}). □

Proof of Theorem 5.1.2. Let $\alpha \notin \mathbb{Z}_{<0}$, and assume that all faces of $\operatorname{Contrib}(\alpha)$ are UB_1 and $\operatorname{Newt}(f)$ is α-simplicial. By Remark 5.3.3 and Remark 5.3.6, it is enough to show that there exists a set of candidate poles for $Z_{\text{for}}(T)$ not containing α. Let $M_1, \dots, M_r$ be the minimal elements in $\operatorname{Contrib}(\alpha)$. Assume that δ is chosen sufficiently small. By Lemma 5.6.4,

$$Z_{\text{for}}(T) = Z_{\text{for}}(T)|_{\cap_i N_{M_i,\geq\delta}} + \sum_{i=1}^{r} Z_{\text{for}}(T)|_{N_{M_i,<\delta}}.$$

By Lemma 5.6.12, $Z_{\text{for}}(T)|_{\cap_i N_{M_i,\geq\delta}}$ lies in R and admits a set of candidate poles not containing α. Choose a minimal face $M \in \operatorname{Contrib}(\alpha)$. By Theorem 5.7.12, we may fix an α-compatible pair $(\mathcal{Z}, \mathcal{F})$. By definition, $Z_{\text{for}}(T)|_{\partial\mathbb{R}^n_{\geq 0}} = 0$. Recall that $N^\circ_{M,\leq\delta} = N_{M,<\delta} \cap \mathbb{R}^n_{>0}$. We have

$$Z_{\text{for}}(T)|_{N_{M,<\delta}} = Z_{\text{for}}(T)|_{N^\circ_{M,\leq\delta}} = \sum_{\substack{\emptyset \neq J \subset Q_{\mathcal{Z}} \\ \tau_J \cap N^\circ_{M,\leq\delta} \neq \emptyset}} Z_{\text{for}}(T)|_{\tau^\circ_J \cap N^\circ_{M,\leq\delta}}.$$

Let J be a nonempty face of $Q_{\mathcal{Z}}$ such that $\tau_J \cap N^\circ_{M,\leq\delta} \neq \emptyset$. By Remark 5.3.4, to show that $Z_{\text{for}}(T)$ lies in R and admits a set of candidate poles not containing α, it is enough to show that $Z_{\text{for}}(T)|_{\tau^\circ_J \cap N^\circ_{M,\leq\delta}}$ lies in R and admits a set of candidate poles not containing α for every J.

Let $\Delta = \Sigma \cap \Sigma_{\mathcal{Z}}$. Let C be a cone in Δ such that $C^\circ \subset \tau^\circ_J$ and $C \cap N^\circ_{M,\leq\delta} \neq \emptyset$. By Lemma 5.7.8, $C = \sigma_G \cap \tau_J$ and $C^\circ = \sigma^\circ_G \cap \tau^\circ_J$, for some face $G \in \Gamma$ with $G \subset F_J$. Since $C \not\subset \partial\mathbb{R}^n_{\geq 0}$, G is compact.

Suppose that $C \not\subset \sigma_{A_{F_J}}$. Then $A_{F_J} \notin G$, and G is contained in the base of the (possibly unbounded) B_1-face F_J. Since F_J is a B_1-face, we may consider the face $F = \operatorname{Conv}\{G, A_{F_J}\}$ of F_J. Then F is a compact B_1-face with apex A_{F_J} and base G in the direction $e^*_{F_J}$. Consider the face $C' = C \cap \sigma_{A_{F_J}} \cap N_{M,\leq\delta}$ of $C \cap N_{M,\leq\delta}$. Note that $C' \subset C \cap \sigma_{A_{F_J}} \subset \sigma_F$. By Lemma 5.7.8,

$$C \cap N_{M,\leq\delta} = D \cap N_{M,\leq\delta}, \tag{21}$$

where $D = D(C, J) = \operatorname{Conv}\left\{C', \mathbb{R}_{\geq 0} e^*_{F_J}\right\} \subset \sigma_G$. By Lemma 5.7.13, the rays of $C \cap N_{M,\leq\delta}$ that are critical with respect to either α or (α, A_{F_J}) are contained σ_M. If V is a vertex of G, then $\langle e^*_{F_J}, V\rangle = 0$, and hence $\langle e^*_{F_J}, \alpha V + \mathbf{1}\rangle = 1 \neq 0$. Also, $\langle e^*_{F_J}, \alpha A_{F_J} + \mathbf{1}\rangle = \alpha + 1 \neq 0$, by assumption. We deduce that $e^*_{F_J}$ is not critical with respect to α or (α, A_{F_J}). This implies that no ray of $D \cap N_{M,\geq\delta}$ is critical with respect to α or (α, A_{F_J}). Since $\sigma^\circ_G, \sigma^\circ_F \subset \mathbb{R}^n_{>0}$, Remark 5.6.7 gives the following equalities:

$$(C \cap \sigma_{A_{F_J}})^\circ \cap N^\circ_{M,\leq\delta} = (C')^\circ,$$

$$C^\circ \cap N^\circ_{M,\leq\delta} = (C \cap N_{M,\leq\delta})^\circ = (D \cap N_{M,\leq\delta})^\circ = D^\circ \cap N^\circ_{M,\leq\delta},$$

$$D^\circ \cap N_{M,\geq\delta} = (D \cap N_{M,\geq\delta})^\circ \cup (D \cap N_{M,\delta})^\circ.$$

Since the restriction of N to σ_G is linear, Lemma 5.2.6 and Lemma 5.6.5, imply that the following elements of $\tilde{R}$ lie in R and admit sets of candidate poles not containing α:

$$(L-1)^{n-\dim G}\sum_{u\in D^\circ\cap N_{M,\geq\delta}\cap\mathbb{N}^n} L^{-\langle u,\mathbf{1}\rangle}T^{N(u)},\ (L-1)^n\sum_{u\in D^\circ\cap N_{M,\geq\delta}\cap\mathbb{N}^n} L^{-\langle u,\mathbf{1}\rangle}T^{\langle u,A_{F_J}\rangle}$$

We claim that $(C')^\circ\subset\sigma_F^\circ$. We have $(C')^\circ\subset\sigma_{F'}^\circ$ for some $F\subset F'$. By Lemma 5.7.6, $F'\subset F_J$. If $\sigma_{F'}\subset\partial\mathbb{R}^n_{\geq 0}$, then (21) implies that $C\cap N_{M,\leq\delta}\subset\partial\mathbb{R}^n_{\geq 0}$, a contradiction. Hence F' is a compact face of F_J containing A_{F_J}. It follows that F' is a B_1-face with apex A_{F_J} and base $G'\supset G$. Then (21) implies that $C\cap N_{M,\leq\delta}\subset\sigma_{G'}$. Then $\emptyset\neq(C\cap N_{M,\leq\delta})^\circ=C^\circ\cap N^\circ_{M,\leq\delta}\subset\sigma_{G'}$. Since $C^\circ\subset\sigma_G^\circ$, we conclude that $G=G'$, and hence $F=F'$, completing the proof of the claim. We may then apply Lemma 5.4.1 to obtain

$$Z_{\text{for}}(T)|_{(D^\circ\cup(C')^\circ)}=(L-1)^n\left(\sum_{u\in(D^\circ\cup(C')^\circ)\cap\mathbb{N}^n} L^{-\langle u,\mathbf{1}\rangle}T^{\langle u,A_{F_J}\rangle}\right)\in\tilde{R}.$$

Since $D^\circ=(D^\circ\cap N_{M,\geq\delta})\cup(D^\circ\cap N^\circ_{M,\leq\delta})$, using the above calculations and Remark 5.3.4, we deduce that

$$(22)\qquad Z_{\text{for}}(T)|_{((C\cap\sigma_{A_{F_J}})^\circ\cup C^\circ)\cap N^\circ_{M,\leq\delta}}-(L-1)^n\left(\sum_{u\in((C\cap\sigma_{A_{F_J}})^\circ\cup C^\circ)\cap N^\circ_{M,\leq\delta})\cap\mathbb{N}^n} L^{-\langle u,\mathbf{1}\rangle}T^{\langle u,A_{F_J}\rangle}\right)$$

lies in R and admits a set of candidate poles not containing α.

Since Δ refines $\Sigma_{\mathcal{Z}}$, we may consider the subfan $\Delta\cap N_{M,\leq\delta}|_{\tau_J}$ of $\Delta\cap N_{M,\leq\delta}$. It follows from the description of $\Delta\cap N_{M,\leq\delta}$ in Lemma 5.6.6 and the fact that $\tau_J\cap\mathbb{R}^n_{>0}\neq\emptyset$ that

$$(23)\qquad (\tau_J\cap N_{M,\leq\delta})^\circ=\bigcup_{\substack{C\in\Delta\\ C^\circ\subset\tau_J^\circ\\ C\cap N^\circ_{M,\leq\delta}\neq\emptyset}}(C\cap N_{M,\leq\delta})^\circ=\bigcup_{\substack{C\in\Delta\\ C^\circ\subset\tau_J^\circ\\ C\cap N^\circ_{M,\leq\delta}\neq\emptyset}} C^\circ\cap N^\circ_{M,\leq\delta}=\tau_J^\circ\cap N^\circ_{M,\leq\delta}.$$

It follows from (21) that we may rewrite this as:

$$\tau_J^\circ\cap N^\circ_{M,\leq\delta}=(\sigma^\circ_{A_{F_J}}\cap\tau_J^\circ\cap N^\circ_{M,\leq\delta})\cup\bigcup_{\substack{C\in\Delta\\ C^\circ\subset\tau_J^\circ\\ C\cap N^\circ_{M,\leq\delta}\neq\emptyset\\ C\not\subset\sigma_{A_{F_J}}}}((C\cap\sigma_{A_{F_J}})^\circ\cup C^\circ)\cap N^\circ_{M,\leq\delta}.$$

We deduce from Remark 5.4.2 and (22) that

$$Z_{\text{for}}(T)|_{\tau_J^\circ\cap N^\circ_{M,\leq\delta}}-(L-1)^n\left(\sum_{u\in\tau_J^\circ\cap N^\circ_{M,\leq\delta}\cap\mathbb{N}^n} L^{-\langle u,\mathbf{1}\rangle}T^{\langle u,A_{F_J}\rangle}\right)$$

lies in R and admits a set of candidate poles not containing α. By Remark 5.3.4 and (23), it is enough to show that

$$(L-1)^n\left(\sum_{u\in(\tau_J\cap N_{M,\leq\delta})^\circ\cap\mathbb{N}^n} L^{-\langle u,\mathbf{1}\rangle}T^{\langle u,A_{F_J}\rangle}\right)$$

lies in R and admits a set of candidate poles not containing α. By Lemma 5.7.9 and Lemma 5.7.13, no rays of $\tau_J\cap N_{M,\leq\delta}$ are critical with respect to (α,A_{F_J}). The result now follows from Lemma 5.2.6 and Lemma 5.6.5. $\square$

5.8. **Existence of α-compatible sets.** We continue with the notation of the previous section. Recall that we consider pairs $(\mathcal{Z},\mathcal{F})$, where $\mathcal{Z}=(Z_s)_{s\in\mathcal{S}}$ and $\mathcal{F}=(F_s)_{s\in\mathcal{S}}$ are collections of elements of $\mathbb{Q}^n$ and $\mathrm{Contrib}(\alpha)_M$ respectively.

Definition 5.8.1. *Consider a pair $(\mathcal{Z},\mathcal{F})$. Then $(\mathcal{Z},\mathcal{F})$ is* restricted *if $Z_s\in\mathrm{span}(\{V_M\}\cup\mathrm{Gen}(C_{F_s}\smallsetminus C_M)\cup\mathcal{A}_M)$, for every $s\in\mathcal{S}$.*

Our goal is to reduce the existence of an α-compatible set to the existence of a restricted, weakly α-compatible set. We will consider sets $\epsilon=\{\epsilon_s\}_{s\in\mathcal{S}}\in\mathbb{R}^{\mathcal{S}}$; note that we allow ϵ_s to be negative. We say that ϵ is chosen to be sufficiently small if $|\epsilon_s|$ is chosen to be sufficiently small for all $s\in\mathcal{S}$. Explicitly, a property holds for ϵ sufficiently small if there exists $\delta>0$ such that the property holds for all ϵ such that $|\epsilon_s|<\delta$ for all $s\in\mathcal{S}$. Given a sequence of sets $\{\epsilon_m\}_{m\in\mathbb{Z}_{\geq 0}}$, where $\epsilon_m=\{\epsilon_{m,s}\}_{s\in\mathcal{S}}$, for some $\epsilon_{m,s}\in\mathbb{R}$, we write $\lim_{m\to\infty}\epsilon_m=0$ if $\lim_{m\to\infty}\epsilon_{m,s}=0$ for all $s\in\mathcal{S}$. Given a set $\mathcal{Z}=\{Z_s\}_{s\in\mathcal{S}}$ of elements in $\mathbb{R}^n$, we let $Z_s(\epsilon_s):=Z_s+\epsilon_s V_M$, and let $\mathcal{Z}(\epsilon):=\{Z_s(\epsilon_s)\}_{s\in\mathcal{S}}$. Then $Q_{\mathcal{Z}(\epsilon)}$ is the convex hull of the elements of $\mathcal{Z}(\epsilon)$, and is dual to the fan $\Sigma_{\mathcal{Z}(\epsilon)}$. Also, $\mathrm{Newt}(f)_{\mathcal{Z}(\epsilon)}=\mathrm{Newt}(f)+Q_{\mathcal{Z}(\epsilon)}$ is dual to $\Sigma\cap\Sigma_{\mathcal{Z}(\epsilon)}$.

Lemma 5.8.2. *Consider a pair $(\mathcal{Z},\mathcal{F})$. For $\epsilon\in\mathbb{Q}^{\mathcal{S}}$ sufficiently small, $(\mathcal{Z}(\epsilon),\mathcal{F})$ is restricted if $(\mathcal{Z},\mathcal{F})$ is restricted, and $(\mathcal{Z}(\epsilon),\mathcal{F})$ is weakly α-compatible if $(\mathcal{Z},\mathcal{F})$ is weakly α-compatible.*

Proof. Assume that $(\mathcal{Z},\mathcal{F})$ is restricted. Since $Z_s(\epsilon_s)$ is a linear combination of Z_s and V_M, it follows that $(\mathcal{Z}(\epsilon),\mathcal{F})$ is restricted.

Assume that $(\mathcal{Z},\mathcal{F})$ is weakly α-compatible. We want to show that $(\mathcal{Z}(\epsilon),\mathcal{F})$ is weakly α-compatible. Fix a face $K\in\mathrm{Contrib}(\alpha)_M$ and $s,s'\in\mathcal{S}$. There is nothing to show if, after possibly shrinking ϵ, $\sigma_K^\circ\cap\tau_{\mathcal{Z}(\epsilon),s}\cap\tau_{\mathcal{Z}(\epsilon),s'}=\emptyset$. Hence we may assume that there exists a sequence of sets $\{\epsilon_m\}_{,\in\mathbb{Z}_{\geq 0}}$ such that $\lim_{m\to\infty}\epsilon_m=0$ and $\sigma_K^\circ\cap\tau_{\mathcal{Z}(\epsilon_m),s}\cap\tau_{\mathcal{Z}(\epsilon_m),s'}\neq\emptyset$. Then Bolzano–Weierstrass implies that $\sigma_K\cap\tau_{\mathcal{Z},s}\cap\tau_{\mathcal{Z},s'}\neq\{0\}$. Hence there exists $K\subset K'$ such that $\sigma_{K'}^\circ\cap\tau_{\mathcal{Z},s}\cap\tau_{\mathcal{Z},s'}\neq\emptyset$. Since $\mathcal{Z}$ is weakly α-compatible, we deduce that $K'\subset F_s$, either $F_{s'}\subset F_s$ or $F_s\subset F_{s'}$, and $\langle e^*_{F_s},Z_s\rangle=\langle e^*_{F_s},Z_{s'}\rangle=0$. Then $K\subset K'\subset F_s$. By (15), $\langle e^*_{F_s},Z_s(\epsilon_s)\rangle=\langle e^*_{F_s},Z_s\rangle=0$ and $\langle e^*_{F_s},Z_{s'}(\epsilon_{s'})\rangle=\langle e^*_{F_s},Z_{s'}\rangle=0$. ☐

Before proceeding, we need a series of basic lemmas on deforming polyhedra. Let $\mathrm{rec}(P)$ denote the recession cone of a polyhedron P. Let $\sigma^\vee$ denote the dual cone to a cone σ. Given a face K of P, let τ_K denote the corresponding cone in the dual fan to P. In particular, $\mathrm{rec}(K)$ is a face of $\mathrm{rec}(P)$, $\tau_{\mathrm{rec}(K)}$ is a face of $\mathrm{rec}(P)^\vee$, and $\tau_K^\circ\subset\tau_{\mathrm{rec}(K)}^\circ$.

Fix a nonempty finite set T. Let $P=\mathrm{Conv}\{V_t:t\in T\}+\sigma\subset\mathbb{R}^n$ be a polyhedron, for some $V_t\in\mathbb{R}^n$, and some pointed (polyhedral) recession cone $\sigma=\mathrm{rec}(P)$. Let $\{P(\epsilon)=\mathrm{Conv}\{V_t(\epsilon):t\in T\}+\sigma\}_\epsilon$ be a set of polyhedra with the same recession fan indexed by $\epsilon\in\mathbb{R}^\ell$, for some $\ell\geq 1$. Assume that $V_t(\epsilon)\in\mathbb{R}^n$ is a continuous function of $\epsilon\in\mathbb{R}^\ell$, for all t in T. If $J(\epsilon)$ is a nonempty face of $P(\epsilon)$ and J is a nonempty face of P, we write $T(J(\epsilon)):=\{t\in T:V_t(\epsilon)\in J(\epsilon)\}$ and $T(J):=\{t\in T:V_t\in J\}$. We may also consider the recession cones $\mathrm{rec}(J(\epsilon))$ and $\mathrm{rec}(J)$, which are both faces of σ.

Definition 5.8.3. *For fixed $\epsilon\in\mathbb{R}^\ell$, we say $P(\epsilon)$* refines *P if for any proper nonempty face $J(\epsilon)$ of $P(\epsilon)$, there exists a proper nonempty face J of P such that $T(J(\epsilon))\subset T(J)$ and $\mathrm{rec}(J(\epsilon))\subset\mathrm{rec}(J)$.*

Lemma 5.8.4. *For $\epsilon\in\mathbb{R}^\ell$ sufficiently small, $P(\epsilon)$ refines P.*

Proof. Assume the conclusion fails. Then there exists a sequence $\{\epsilon_m\}_{m\in\mathbb{Z}_{\geq 0}}$ such that $\lim_{m\to\infty}\epsilon_m=0$, and a sequence of proper nonempty faces $J(\epsilon_m)$ of $P(\epsilon_m)$, such that, for any m and any proper nonempty face J of P, either $T(J(\epsilon_m))\not\subset T(J)$ or $\mathrm{rec}(J(\epsilon_m))\not\subset\mathrm{rec}(J)$. Since T is finite and σ has finitely many faces, after possibly replacing $\{\epsilon_m\}_{m\in\mathbb{Z}_{\geq 0}}$ by a subsequence, we may assume that $T(J(\epsilon_m))$ and $\mathrm{rec}(J(\epsilon_m))$ are independent of m. Denote these by $\tilde{T}=T(J(\epsilon_m))$ and $\tilde{R}=\mathrm{rec}(J(\epsilon_m))$ respectively. Consider a sequence u_m

of elements in $\tau^\circ_{J(\epsilon_m)} \subset \tau^\circ_{\tilde{R}}$ such that $||u_m|| = 1$. After possibly replacing $\{\epsilon_m\}_{m\in\mathbb{Z}_{\geq 0}}$ by a subsequence, we may assume that $\lim_{m\to\infty} u_m = u \in \tau^\circ_J \subset \tau^\circ_{\mathrm{rec}(J)}$ exists, for some nonempty face J of P. Since $\tau_{\tilde{R}}$ is closed, $u \in \tau_{\tilde{R}}$, and hence $\tau_{\mathrm{rec}(J)} \subset \tau_{\tilde{R}}$ and $\tilde{R} \subset \mathrm{rec}(J)$. For any $t \in \tilde{T}$ and $t' \in T$,

$$\langle u, V_t\rangle = \lim_{m\to\infty}\langle u_m, V_t\rangle = \lim_{m\to\infty}\langle u_m, V_t(\epsilon_m)\rangle \leq \lim_{m\to\infty}\langle u_m, V_{t'}(\epsilon_m)\rangle = \lim_{m\to\infty}\langle u_m, V_{t'}\rangle = \langle u, V_{t'}\rangle, \tag{24}$$

and hence $V_t \in J$. We deduce that $\tilde{T} \subset T(J)$, a contradiction. □

Definition 5.8.5. *We say that* $\{P(\epsilon)\}_\epsilon$ *is* locally combinatorially constant *if for any* ϵ *sufficiently small, and for any nonempty face* $J(\epsilon)$ *of* $P(\epsilon)$*, there exists a (unique) nonempty face* J *of* P *such that* $T(J(\epsilon)) = T(J)$ *and* $\mathrm{rec}(J(\epsilon)) = rec(J)$*, and, moreover, every nonempty face* J *of* P *appears in this way.*

Lemma 5.8.6. *After possibly replacing* P *with* $P(\epsilon)$ *for some* $\epsilon \in \mathbb{Q}^\ell$*,* $\{P(\epsilon)\}_\epsilon$ *is locally combinatorially constant.*

Proof. Lemma 5.8.4 implies that we may order $\{P(\epsilon)\}_\epsilon$ by refinement. Since T is finite and σ has finitely many faces, there exists an $\epsilon \in \mathbb{R}^\ell$ such that $Q = P(\epsilon)$ is minimal under this ordering. Then $\{Q(\epsilon)\}_\epsilon$ is locally combinatorially constant. Consider $\epsilon' \in \mathbb{R}^\ell$ such that $\epsilon + \epsilon' \in \mathbb{Q}^\ell$, and let $Q' = Q(\epsilon')$. Then for ϵ' sufficiently small, $\{Q'(\epsilon)\}_\epsilon$ is locally combinatorially constant. □

Given a family of cones $\{C_k\}_{k=1}^\infty$, define $\limsup C_k$ to be the cone of points $u \in \mathbb{R}^n$ such that u is a limit point of a sequence of points $u_k \in C_k$, i.e., there exists a subsequence of u_k converging to u.

Lemma 5.8.7. *Assume that* $\{P(\epsilon)\}_\epsilon$ *is locally combinatorially constant. Fix a nonempty face* J *of* P*. For* ϵ *sufficiently small, let* $J(\epsilon)$ *be the nonempty face of* $P(\epsilon)$ *such that* $T(J(\epsilon)) = T(J)$ *and* $\mathrm{rec}(J(\epsilon)) = \mathrm{rec}(J)$*. Consider any* $\{\epsilon_k\}_{k\in\mathbb{Z}_{\geq 0}}$ *such that* $\lim_{m\to\infty}\epsilon_k = 0$*. Then* $\limsup_k \tau_{J(\epsilon_k)} \subset \tau_J$*, and, if we assume that* $\dim P = n$*, then* $\limsup_k \tau_{J(\epsilon_k)} = \tau_J$*.*

Proof. We first show that $\limsup_k \tau_{J(\epsilon_k)} \subset \tau_J$. Suppose that $u_k \in \tau_{J(\epsilon_k)}$, and that, after possibly replacing $\{\epsilon_k\}_{k\in\mathbb{Z}_{\geq 0}}$ with a subsequence, $\lim_{k\to\infty} u_k = u \in \mathbb{R}^n$ exists. Then $u \in \tau^\circ_{J'}$ for some nonempty face J' of P. Then the calculation in (24) implies that $T(J) = T(J(\epsilon_k)) \subset T(J')$, and hence $J \subset J'$ and $u \in \tau_{J'} \subset \tau_J$.

We need to prove the converse statement. Assume that $\dim P = n$. Then τ_J is generated by its rays $\{\gamma_m\}_{1\leq m\leq p}$. For any ϵ sufficiently small, let $\{\gamma_m(\epsilon)\}_{1\leq m\leq p}$ denote the corresponding rays in the dual fan to $P(\epsilon)$. Consider elements $\{u_{m,k} \in \gamma_m(\epsilon_k) : 1 \leq m \leq p, k \geq 0\}$ such that $||u_{m,k}|| = 1$. Then, after possibly replacing $\{\epsilon_k\}_{k\in\mathbb{Z}_{\geq 0}}$ with a subsequence, we may assume that $\lim_{k\to\infty} u_{m,k} = u_m$ exists, for $1 \leq m \leq p$. Then $u_m \in \limsup_k \gamma_m(\epsilon_k) \subset \gamma_m$ and $||u_m|| = 1$. Given an element $u \in \tau_J$, there exists $a_m \in \mathbb{R}^n_{\geq 0}$ such that $u = \sum_m a_m u_m$. Then $\lim_{k\to\infty}\sum_m a_m u_{m,k} = u \in \limsup_k \tau_{J(\epsilon_k)}$, as desired. □

With these lemmas in hand, we now return to our problem. Fix a restricted, weakly α-compatible pair $(\mathcal{Z}, \mathcal{F})$, where $\mathcal{Z} = (Z_s)_{s\in\mathcal{S}}$ and $\mathcal{F} = (F_s)_{s\in\mathcal{S}}$. We may apply Lemma 5.8.6 to both $P(\epsilon) = \mathrm{Newt}(f)_{\mathcal{Z}(\epsilon)}$ and $P(\epsilon) = Q_{\mathcal{Z}(\epsilon)}$. Hence, by Lemma 5.8.2, we may replace $\mathcal{Z}$ by $\mathcal{Z}(\epsilon)$ so that $\{Q_{\mathcal{Z}(\epsilon)}\}_\epsilon$ and $\{\mathrm{Newt}(f)_{\mathcal{Z}(\epsilon)}\}_\epsilon$ are locally combinatorially constant. Consider a nonempty face J of $Q_{\mathcal{Z}}$, dual to a cone τ_J in $\Sigma_{\mathcal{Z}}$. For ϵ sufficiently small, let $J(\epsilon)$ denote the corresponding nonempty face of $Q_{\mathcal{Z}(\epsilon)}$, dual to the cone $\tau_{J(\epsilon)}$ of $\Sigma_{\mathcal{Z}(\epsilon)}$. Similarly, given a cone C in $\Sigma \cap \Sigma_{\mathcal{Z}}$, we let $C(\epsilon)$ denote the corresponding cone in $\Sigma \cap \Sigma_{\mathcal{Z}(\epsilon)}$. If C is dual to a face of $\mathrm{Newt}(f)_{\mathcal{Z}}$ of the form $K' + J'$, for some $K' \in \Gamma$ and some nonempty face $J' \subset Q_{\mathcal{Z}(\epsilon)}$, then $C(\epsilon) = \sigma_{K'} \cap \tau_{J'(\epsilon)}$ is dual to the face $K' + J'(\epsilon)$ in $\mathrm{Newt}(f)_{\mathcal{Z}(\epsilon)}$.

Remark 5.8.8. Consider a nonempty face J of $Q_{\mathcal{Z}}$ such that $\sigma_M \cap \tau_J \neq \{0\}$. Since $\{Q_{\mathcal{Z}(\epsilon)}\}_\epsilon$ is locally combinatorially constant, for any $s \in \mathcal{S}$, $Z_s \in J$ if and only if $Z_s(\epsilon_s) \in J(\epsilon)$. In particular, if $\sigma_M \cap \tau_{J(\epsilon)} \neq \{0\}$, then $F_J = \max\{F_s : Z_s \in J\} = \max\{F_s : Z_s(\epsilon_s) \in J(\epsilon)\} = F_{J(\epsilon)}$.

The lemma below says that a pair (C, J) not being α-critical is an open condition.

Lemma 5.8.9. *Let J be a nonempty face of $Q_{\mathcal{Z}}$, and consider a cone $C \in \Sigma \cap \Sigma_{\mathcal{Z}}$. Suppose there exists a sequence $\{\epsilon_m\}_{m \in \mathbb{Z}_{\geq 0}}$ such that $\epsilon_m \in \mathbb{Q}^{\mathcal{S}}$, $\lim_{m \to \infty} \epsilon_m = 0$ and $(C(\epsilon_m), J(\epsilon_m))$ is α-critical for all m. Then (C, J) is α-critical.*

Proof. By Lemma 5.7.8 and since $\{\mathrm{Newt}(f)_{\mathcal{Z}(\epsilon)}\}_\epsilon$ is locally combinatorially constant, $C(\epsilon_m) \subset \tau_{J(\epsilon_m)}$ is dual to a face of the form $K + J'(\epsilon_m)$ of $\mathrm{Newt}(f)_{\mathcal{Z}(\epsilon_m)}$, where $K \in \Gamma$ and J' is a face of $Q_{\mathcal{Z}}$ such that $J(\epsilon_m) \subset J'(\epsilon_m)$, or, equivalently, $J \subset J'$. Then C is dual to $K + J'$. In particular, $C \subset \tau_{J'} \subset \tau_J$. By hypothesis, $C(\epsilon_m) \cap \sigma_M \neq \{0\}$. It follows from Bolzano–Weierstrass and Lemma 5.8.7 that $C \cap \sigma_M \neq \{0\}$. Then $\sigma_M \cap \tau_J \neq \{0\}$, and, by Remark 5.8.8, $F_J = F_{J(\epsilon_m)}$ for all m. The condition $C(\epsilon_m) \subset \sigma_{A_{F_J}}$ implies that $\sigma_K \subset \sigma_{A_{F_J}}$, and hence $C \subset \sigma_{A_{F_J}}$. By Lemma 5.8.7 and since $\mathrm{H}_{\alpha A_{F_J} + \mathbf{1}}$ is closed, $C = \limsup_m C(\epsilon_m) \subset \mathrm{H}_{\alpha A_{F_J} + \mathbf{1}}$, and hence C is critical with respect to (α, A_{F_J}). We conclude that (C, J) is α-critical. □

We say that ϵ can be chosen to be arbitrarily small if for any $\delta > 0$, there exists a choice of ϵ such that $|\epsilon_s| < \delta$ for all $s \in \mathcal{S}$.

Lemma 5.8.10. *Suppose that (C, J) is α-critical and $C \not\subset \sigma_M$. Then there exists an arbitrarily small choice of $\epsilon \in \mathbb{Q}^{\mathcal{S}}$ such that $(C(\epsilon), J(\epsilon))$ is not α-critical.*

Proof. By Lemma 5.7.8, C is dual to a face $K + J'$ of $\mathrm{Newt}(f)_{\mathcal{Z}}$, where $K \in \Gamma$ and J' is a face of $Q_{\mathcal{Z}}$ such that $K \subset F_{J'}$ and $J \subset J'$. Then $C(\epsilon) \subset \tau_{J(\epsilon)}$ is dual to the face $K + J'(\epsilon)$ of $\mathrm{Newt}(f)_{\mathcal{Z}(\epsilon)}$. Note that $C \subset \sigma_{A_{F_J}}$ implies that $\sigma_K \subset \sigma_{A_{F_J}}$, and hence $A_{F_J} \in K$. Since K, M are faces of Γ, $K \cap M$ is a (possibly empty) face of M, and $C_{K \cap M} = C_K \cap C_M$. Let $B_K = \mathrm{Gen}(C_{K \cap M}) \cup \mathcal{A}_M$.

Assume that $\mathbf{1} \in \mathrm{span}(B_K)$. We can write $\mathbf{1} = \sum_{V \in B_K} \lambda_V V$, for some coefficients λ_V, with $\lambda_V = 1$ for all $V \in \mathcal{A}_M$. Applying ψ_M to both sides gives $-\alpha = \sum_{V \in B_K \smallsetminus I_{K \cap M}} \lambda_V$. Hence we may equivalently write

$$\alpha A_{F_J} + \mathbf{1} = \sum_{V \in B_K \smallsetminus I_{K \cap M}} \lambda_V (V - A_{F_J}) + \sum_{V \in I_{K \cap M}} \lambda_V V.$$

Consider $u \in C^\circ \subset \sigma_K^\circ \cap \mathrm{H}_{\alpha A_{F_J} + \mathbf{1}}$. Consider $V \in \mathrm{Gen}(C_K)$. If $V \in I_K$, then $u \in \sigma_K^\circ$ implies that $\langle u, V \rangle = 0$. Since $A_{F_J} \in K$, if $V \in \mathrm{Vert}(K)$, then $u \in \sigma_K^\circ$ implies that $\langle u, V - A_{F_J} \rangle = 0$. We compute:

$$\begin{aligned} 0 = \langle u, \alpha A_{F_J} + \mathbf{1} \rangle &= \sum_{V \in B_K \smallsetminus I_{K \cap M}} \lambda_V \langle u, V - A_{F_J} \rangle + \sum_{V \in I_{K \cap M}} \lambda_V \langle u, V \rangle \\ &= \sum_{V \in B_K \smallsetminus C_K} \langle u, V - A_{F_J} \rangle. \end{aligned}$$

Since each term in the right-hand sum is positive, we deduce that the sum must be empty. It follows that $B_K \subset C_{K \cap M}$ and $\mathbf{1} \in \mathrm{span}(C_{K \cap M})$. Since M is minimal in $\mathrm{Contrib}(\alpha)$, we deduce that $K \cap M = M$. Then $C \subset \sigma_K \subset \sigma_M$, a contradiction.

We conclude that $\mathbf{1} \notin \mathrm{span}(B_K)$. Since $A_{F_J} \in B_K$, it follows that $\alpha A_{F_J} + \mathbf{1} \notin \mathrm{span}(B_K)$. Since $\alpha A_{F_J} + \mathbf{1} \in \mathrm{span}(M) \cap \mathbb{Q}^n$ and $\mathrm{Newt}(f)$ is α-simplicial, it follows that there exists $u' \in \mathbb{Q}^n$ such that

(1) $\langle u', \alpha A_{F_J} + \mathbf{1} \rangle = 1$,
(2) $\langle u', V \rangle = 0$ for all elements $V \in B_K$,
(3) $\langle u', V \rangle = 0$ for all $V \in \mathrm{Gen}(C_{F_{J'}} \smallsetminus C_M)$.

Consider an element $u \in C^\circ \cap \mathbb{Q}^n = \sigma_K^\circ \cap (\tau_{J'})^\circ \cap \mathbb{Q}^n \subset \mathrm{H}_{\alpha A_{F_J} + \mathbf{1}}$, and let $\hat{u}(\lambda) = u + \lambda u'$ for some choice of $\lambda \neq 0 \in \mathbb{R}$. Then property (1) implies that $\langle \hat{u}(\lambda), \alpha A_{F_J} + \mathbf{1} \rangle = \lambda \neq 0$, and hence $\hat{u}(\lambda) \notin \mathrm{H}_{\alpha A_{F_J} + \mathbf{1}}$. Properties (2) and (3) imply that for any V in C_K, $\langle \hat{u}(\lambda), V \rangle = \langle u, V \rangle + \lambda \langle u', V \rangle = \langle u, V \rangle$. It follows that $\hat{u}(\lambda) \in \sigma_K^\circ$ provided $|\lambda|$ is sufficiently small.

Recall that $\tau_{J'} = \cap_{Z_s \in J'} \tau_{Z_s}$. Consider $s \in \mathcal{S}$ such that $Z_s \in J'$. We claim that for a generic choice of $\lambda \in \mathbb{Q}$, we may choose $\epsilon_s \in \mathbb{Q}$ such that $\langle \hat{u}(\lambda), Z_s(\epsilon_s) \rangle = \langle u, Z_s \rangle$. Assume this claim holds. It follows that with this choice of $\epsilon = \{\epsilon_s\}_{s\in\mathcal{S}}$, $\hat{u}(\lambda) \in (\tau_{J'}(\epsilon))^\circ$ provided $|\lambda|$ is chosen sufficiently small. Then $\hat{u}(\lambda) \in \sigma_K^\circ \cap (\tau_{J'}(\epsilon))^\circ = C(\epsilon)^\circ$ and $\hat{u}(\lambda) \notin \mathrm{H}_{\alpha A_{F_J}+\mathbf{1}}$. Either $\sigma_M \cap \tau_{J(\epsilon)} = \{0\}$, or $\sigma_M \cap \tau_{J(\epsilon)} \neq \{0\}$ and, by Remark 5.8.8, $F_J = F_{J(\epsilon)}$ and $C(\epsilon) \not\subset \mathrm{H}_{\alpha A_{F_{J(\epsilon)}}+\mathbf{1}}$. In either case, $(C(\epsilon), J(\epsilon))$ is not α-critical.

It remains to verify the claim. We compute:

$$\begin{aligned}\langle \hat{u}(\lambda), Z_s(\epsilon_s)\rangle &= \langle \hat{u}(\lambda), Z_s + \epsilon_s V_M\rangle \\ &= \langle u, Z_s\rangle + \lambda\langle u', Z_s\rangle + \epsilon_s(\langle u, V_M\rangle + \lambda\langle u', V_M\rangle).\end{aligned}$$

Assume that $\langle u', V_M\rangle \neq 0$. Then for $\lambda \neq -\frac{\langle u,V_M\rangle}{\langle u',V_M\rangle}$, we may set $\epsilon_s = -\frac{\lambda\langle u',Z_s\rangle}{\langle u,V_M\rangle+\lambda\langle u',V_M\rangle}$, and the above calculation shows that $\langle \hat{u}(\lambda), Z_s(\epsilon_s)\rangle = \langle u, Z_s\rangle$. Assume that $\langle u', V_M\rangle = 0$, and let $\epsilon_s = 0$. Since $(\mathcal{Z}, \mathcal{F})$ is restricted, $Z_s \in \mathrm{span}(\{V_M\} \cup \mathrm{Gen}(C_{F_s} \smallsetminus C_M) \cup \mathcal{A}_M)$, for every $s \in \mathcal{S}$. Since $Z_s \in J'$, Definition 5.7.3 implies that $F_s \subset F_{J'}$. Then properties (2) and (3) imply that $\langle u', Z_s\rangle = 0$, and the above calculation shows that $\langle \hat{u}(\lambda), Z_s(\epsilon_s)\rangle = \langle u, Z_s\rangle$. □

Lemma 5.8.11. *Suppose there exists a restricted, weakly α-compatible pair $(\mathcal{Z}, \mathcal{F})$. Then there exists an α-compatible pair $(\mathcal{Z}, \mathcal{F})$.*

Proof. Consider the restricted, weakly α-compatible pair $(\mathcal{Z}, \mathcal{F})$ above. Suppose $(\mathcal{Z}, \mathcal{F})$ is not α-compatible. That is, suppose there exists a pair (C, J) that is α-critical and $C \not\subset \sigma_M$. Then Lemma 5.8.9 and Lemma 5.8.10 imply that we can deform $(\mathcal{Z}, \mathcal{F})$ and strictly increase the number of pairs (C, J) that do not have α-critical intersection. Since there are finitely many such pairs, by repeating this procedure we obtain an α-compatible pair. □

5.9. **Existence of restricted, weakly α-compatible sets.** In this section, we use the operative labeling assumption to explicitly construct a restricted, weakly α-compatible pair. Recall that $\mathrm{Gen}(C_F) = \mathrm{Vert}(F) \cup I_F$ is the set of distinguished vertices on the rays of C_F. Recall that because $\mathrm{Newt}(f)$ is α-simplicial, there is a bijection between $\{K \in \Gamma : M \subset K \subset F\}$ and subsets of $\mathrm{Gen}(C_F \smallsetminus C_M) = \mathrm{Gen}(C_F) \smallsetminus \mathrm{Gen}(C_M)$.

Consider an element $F \in \mathrm{Contrib}(\alpha)_M$. Given an element V in $\mathrm{Gen}(C_F)$, let $\zeta(V) \in F \subset \Gamma$ be defined by

$$\zeta(V) := \begin{cases} V & \text{if } V \in \mathrm{Vert}(F), \\ V + V_M & \text{if } V \in I_F. \end{cases}$$

Lemma 5.9.1. *Suppose that $F, F' \in \mathrm{Contrib}(\alpha)_M$ and $V \in \mathrm{Gen}(C_F)$. Then $\zeta(V) \in F'$ if and only if $V \in \mathrm{Gen}(C_{F'})$.*

Proof. First, suppose that $V \in \mathrm{Vert}(F)$. Then $\zeta(V) = V \in F'$ if and only if $V \in \mathrm{Vert}(F')$. Second, suppose that $V \in I_F$. Consider $u \in \sigma_{F'}^\circ$. Then $\langle u, V\rangle = \langle u, \zeta(V) - V_M\rangle$. Then $\zeta(V) \in F'$ if and only if $\langle u, \zeta(V) - V_M\rangle = 0$, if and only if $\langle u, V\rangle = 0$, if and only if $V \in I_{F'}$. The result follows. □

Let $\mathcal{S}$ be the set of saturated chains of faces in Γ starting at M. Let $s = F_\bullet$ be an element of $\mathcal{S}$. Let ℓ_s denote the length of $F_\bullet$ i.e., the number of elements in $F_\bullet$ minus 1. We let $F_{\bullet,i}$ denote the ith element of $F_\bullet$ for $0 \leq i \leq \ell_s$. For example, $F_{\bullet,0} = M$. We write $F \in F_\bullet$ if $F = F_{\bullet,i}$ for some $0 \leq i \leq \ell_s$.

Define $V_{s,0} = V_M$. Since $F_\bullet$ is saturated and $\mathrm{Newt}(f)$ is α-simplicial, we may define $V_{s,i}$ to be the unique element of $\mathrm{Gen}(C_{F_{\bullet,i}} \smallsetminus C_{F_{\bullet,i-1}})$, for $1 \leq i \leq \ell_s$.

Definition 5.9.2. *Let $\mathcal{S}$ be the set of saturated chains of faces in Γ starting at M. We define a pair $(\mathcal{Z}, \mathcal{F})$, where $\mathcal{Z} = (Z_s)_{s\in\mathcal{S}}$ and $\mathcal{F} = (F_s)_{s\in\mathcal{S}}$ are collections of elements of $\mathbb{Q}^n$ and $\mathrm{Contrib}(\alpha)_M$ respectively, as follows: for any element $s = F_\bullet$ of $\mathcal{S}$, let*

$$Z_s := \sum_{i=0}^{\ell_s} b_{i,\ell_s}\zeta(V_{s,i}),$$

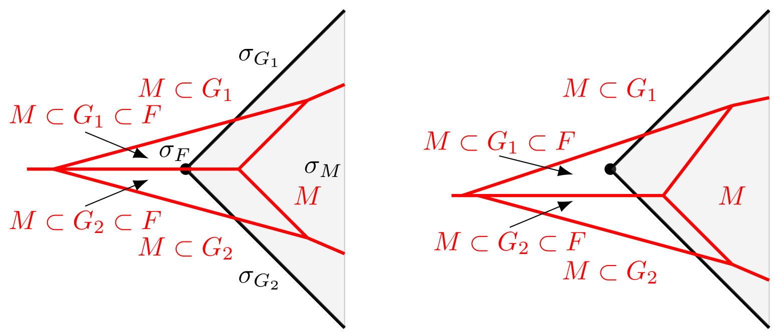

FIGURE 2. A complete fan with maximal cones indexed by saturated chains of faces starting at M and a corresponding deformation. We show the intersection of $\mathrm{span}(\sigma_M)$ with an affine hyperplane. The cone σ_M is shown in black and grey, while the codimension 1 cones of the complete fan appear in red, with their maximal cones labeled in red. For each such maximal cone, the maximal element of the corresponding chain is an element in $\mathrm{Contrib}(\alpha)$, and we associate a corresponding base direction.

where $\{b_{i,j}\}_{0\leq i,j\leq r}$*,* $r = n-1-\dim M$*, and*

$$b_{i,j} = b_{i,j}(\mu) = \begin{cases} 2^{-i} - 2i\mu, & \text{if } i = j \\ 2^{-(i+1)} - (i+j)\mu, & \text{if } i < j \\ 0, & \text{otherwise} \end{cases}$$

for some $\mu \in \mathbb{Q}$ *such that* $0 < \mu \ll 1$*. Let* $F_s := F_{\bullet,\ell_s}$ *be the maximal element of* $F_\bullet$*.*

For example, $b_{0,0} = 1$ and if $s = F_\bullet$, where $F_\bullet = \{M\}$, then $\ell_s = 0$, $Z_s = V_M$, and $F_s = M$. Note that we abuse notation above by not indicating the dependence of $(\mathcal{Z}, \mathcal{F})$ on the choice of μ. Below we fix a value of μ sufficiently small. Our goal is to show that we can construct a restricted, weakly α-compatible pair from $(\mathcal{Z}, \mathcal{F})$. Recall that we have fixed an operative labeling (A_F, e_F^*) of $\mathrm{Contrib}(\alpha)_M$.

Definition 5.9.3. *Let* $s = F_\bullet \in \mathcal{S}$*. Let* $\mathcal{A}_s = \{A_F : F \in F_\bullet\}$*. Given an element* A *in* $\mathcal{A}_s$*, we define a base direction* $e_{s,A}^*$ *as follows: if* $A = A_F$ *for some* $F \in F_\bullet$*, then* $e_{s,A}^* := e_F^*$*. We define a linear function* $\Phi_s \colon \mathbb{R}^n \to \mathbb{R}^n$ *by*

$$\Phi_s(X) = X - \sum_{A \in \mathcal{A}_s} \langle e_{s,A}^*, X\rangle (A - V_M).$$

The fact that $e_{s,A}^*$ is well-defined in Definition 5.9.3 is an immediate consequence of the operative labeling assumption. Explicitly, if $A = A_F = A_{F'}$ for some $F, F' \in F_\bullet$, then either $F \subset F'$ or $F' \subset F$, and the operative labeling assumption implies that $e_F^* = e_{F'}^*$.

Remark 5.9.4. For $s \in \mathcal{S}$, $u \in \sigma_M$ and $X \in \mathbb{R}^n$, $\langle u, \Phi_s(X)\rangle = \langle u, X\rangle$.

Lemma 5.9.5. *Let* $s = F_\bullet \in \mathcal{S}$*. Then* $\langle e_F^*, \Phi_s(X)\rangle = 0$ *for any* $F \in F_\bullet$ *and any* $X \in \mathbb{R}^n$*. If* $F_s \subset K$ *and* $\langle e_K^*, X\rangle = 0$ *for some* $X \in \mathbb{R}^n$*, then* $\langle e_K^*, \Phi_s(X)\rangle = 0$*.*

Proof. Recall from (15) that $\langle e_F^*, V_M\rangle = 0$ for all $F \supset M$. Suppose that $F \in F_\bullet$. Then $e_{s,A_F}^* = e_F^*$, and we compute $\langle e_F^*, \Phi_s(X)\rangle = \langle e_F^*, X - \langle e_{s,A_F}^*, X\rangle (A_F - V_M)\rangle = 0$. Suppose that $F_s \subset K$ and $\langle e_K^*, X\rangle = 0$. If $\langle e_K^*, A\rangle = 0$ for all $A \in \mathcal{A}_s$, then $\langle e_K^*, \Phi_s(X)\rangle = \langle e_K^*, X\rangle = 0$. Suppose that $\langle e_K^*, A_F\rangle \neq 0$ for some $F \in F_\bullet$.

Then $A_K = A_F$. Since $F \subset K$, the operative labeling assumption implies that $e^*_{s,A_F} = e^*_F = e^*_K$. As above, $\langle e^*_K, \Phi_s(X)\rangle = \langle e^*_K, X - \langle e^*_{s,A_F}, X\rangle(A_F - V_M)\rangle = 0$. □

Lemma 5.9.6. *With the notation of Definition 5.9.2, suppose that $(\mathcal{Z}, \mathcal{F})$ satisfies the the following property: Suppose that $\sigma_K^\circ \cap \tau_{Z_s} \cap \tau_{Z_{s'}} \neq \emptyset$, for some $K \in \mathrm{Contrib}(\alpha)_M$ and $s = F_\bullet, s' = F'_\bullet \in \mathcal{S}$. Then*

(1) $K \subset F_s$, and
(2) either $F_s \in F'_\bullet$ or $F_{s'} \in F_\bullet$.

Let $\Phi(\mathcal{Z}) := (\Phi_s(Z_s))_{s\in\mathcal{S}}$. Then $(\Phi(\mathcal{Z}), \mathcal{F})$ is restricted and weakly α-compatible.

Proof. It follows from Definition 5.9.2 and Definition 5.9.3 that $(\Phi(\mathcal{Z}), \mathcal{F})$ is restricted. Suppose that $\sigma_K^\circ \cap \tau_{\Phi_s(Z_s)} \cap \tau_{\Phi_{s'}(Z_{s'})} \neq \emptyset$, for some $K \in \mathrm{Contrib}(\alpha)_M$ and $s = F_\bullet, s' = F'_\bullet \in \mathcal{S}$. By Remark 5.9.4, the restriction of $\Sigma_{\mathcal{Z}}$ to σ_M equals the restriction of $\Sigma_{\Phi(\mathcal{Z})}$ to σ_M. Hence $\sigma_K^\circ \cap \tau_{Z_s} \cap \tau_{Z_{s'}} \neq \emptyset$. We deduce that $K \subset F_s$, and, either $F_s \in F'_\bullet$ or $F_{s'} \in F_\bullet$. The latter condition implies that either $F_s \subset F_{s'}$ or $F_{s'} \subset F_s$.

Applying Lemma 5.9.5 with $s = F_\bullet$ and $F = F_s$, gives $\langle e^*_{F_s}, \Phi_s(Z_s)\rangle = 0$. It remains to show that $\langle e^*_{F_s}, \Phi_{s'}(Z_{s'})\rangle = 0$. Suppose that $F_s \in F'_\bullet$. Applying Lemma 5.9.5 with $s' = F'_\bullet$ and $F = F_s$, gives $\langle e^*_{F_s}, \Phi_{s'}(Z_{s'})\rangle = 0$, as desired. Suppose that $F_{s'} \in F_\bullet$. Then $F_{s'} \subset F_s$. Applying Lemma 5.9.5 with $s' = F'_\bullet$ and $K = F_s$, gives $\langle e^*_{F_s}, \Phi_{s'}(Z_{s'})\rangle = 0$, provided $\langle e^*_{F_s}, Z_{s'}\rangle = 0$. By Definition 5.9.2, $Z_{s'} \in \mathrm{span}(\{V_M\} \cup \mathrm{Gen}(C_{F_{s'}} \smallsetminus C_M))$. Since $F_{s'} \subset F_s$, F_s is a B_1-face with base direction $e^*_{F_s}$, and $\langle e^*_{F_s}, V_M\rangle = 0$ by (15), it follows from Remark 5.5.6 that $\langle e^*_{F_s}, Z_{s'}\rangle = 0$, as desired. □

It remains to show that $(\mathcal{Z}, \mathcal{F})$ satisfies conditions (1) and (2) in Lemma 5.9.6. We will prove this through a series of lemmas.

Lemma 5.9.7. *There exists a constant $\lambda_M > 0$ such that for any $K, F, F' \in \mathrm{Contrib}(\alpha)_M$ that are not subfaces of a common face in $\mathrm{Contrib}(\alpha)_M$, and for any nonzero $u \in \sigma_K$, there exists an element $V \in \mathrm{Gen}(C_F \smallsetminus C_M) \cup \mathrm{Gen}(C_{F'} \smallsetminus C_M)$ such that $\langle u, \zeta(V)\rangle/N(u) \geq 1 + \lambda_M$.*

Proof. Fix $K, F, F' \in \mathrm{Contrib}(\alpha)_M$ that are not subfaces of a common face in $\mathrm{Contrib}(\alpha)_M$. Let $\mathcal{V} = \mathrm{Gen}(C_F \smallsetminus C_M) \cup \mathrm{Gen}(C_{F'} \smallsetminus C_M)$, and consider the continuous function $\phi\colon \sigma_K \smallsetminus \{0\} \to \mathbb{R}$ defined by

$$\phi(u) = \Big(\max_{V\in\mathcal{V}} \langle u, \zeta(V)\rangle/N(u)\Big) - 1.$$

We claim that image satisfies $\mathrm{im}(\phi) \subset \mathbb{R}_{>0}$. Indeed, suppose $\phi(u) = 0$. Let F_u be the face of $\partial\,\mathrm{Newt}(f)$ minimized by u. Then F_u contains K and $\{\zeta(V) : V \in \mathcal{V}\}$. By Lemma 5.9.1, C_{F_u} contains C_K, C_F and $C_{F'}$. Then K, F, F' are common subfaces of F_u, a contradiction.

Note that $\phi(\alpha u) = \phi(u)$ for all $\alpha \in \mathbb{R}_{>0}$ and $u \in \sigma_K \smallsetminus \{0\}$. Since $\sigma_K \cap S$ is compact, there exists $\lambda = \lambda(K, F, F') > 0$ such that $\phi(\sigma_K \cap S) \subset [\lambda, \infty)$. We let λ_M be the minimum value of $\lambda(K, F, F')$ over the finitely many choices of K, F, F'.

□

Below we fix $\lambda_M > 0$ satisfying Lemma 5.9.7.

Lemma 5.9.8. *Let $r = n - 1 - \dim M$. Assume that $\mu > 0$ is chosen sufficiently small. Then the coefficients $\{b_{i,j} = b_{i,j}(\mu)\}_{0\leq i,j\leq r}$ satisfy the following properties:*

(1) $b_{i,j} > 0$ for $i \leq j$,
(2) $b_{i,j} > b_{i,j+1}$ for $i \leq j < r$,
(3) $b_{i,j} > b_{i+1,j}$ for $i < j$
(4) $\sum_{i\geq k} b_{i,j} > \sum_{i\geq k} b_{i,j+1}$, for any $1 \leq k \leq j < r$,
(5) $\sum_{i\geq 0} b_{i,r} + b_{r,r}\lambda_M > 1$.

Proof. We check the conditions hold by direct computation, for μ sufficiently small. Condition (1) is clear. For condition (2), we compute, for $i = j < r$,

$$b_{i,i} = 2^{-i} - 2i\mu > b_{i,i+1} = 2^{-(i+1)} - (2i+1)\mu,$$

and, for $i < j < r$,

$$b_{i,j} = 2^{-(i+1)} - (i+j)\mu > b_{i,j+1} = 2^{-(i+1)} - (i+j+1)\mu.$$

For condition (3), we compute, for $i + 1 < j$,

$$b_{i,j} = 2^{-(i+1)} - (i+j)\mu > b_{i+1,j} = 2^{-(i+2)} - (i+j+1)\mu,$$

and, for $i + 1 = j$,

$$b_{i,i+1} = 2^{-(i+1)} - (2i+1)\mu > b_{i+1,i+1} = 2^{-(i+1)} - (2i+2)\mu.$$

For condition (4), define $c_{k,j} = \sum_{i\geq k} b_{i,j}$ for $k \leq j$. Then

$$\begin{aligned} c_{k,j} &= 2^{-k} - \sum_{i=k}^{j}(i+j)\mu \\ &= 2^{-k} - (j-k+1)(3j+k)\mu/2. \end{aligned}$$

For $j < r$, $c_{k,j} > c_{k,j+1}$, as desired. For condition (5), we compute

$$\begin{aligned} \sum_{i\geq 0} b_{i,r} + b_{r,r}\lambda_M - 1 &= c_{0,r} + b_{r,r}\lambda_M - 1 \\ &= -3r(r+1)\mu/2 + b_{r,r}\lambda_M \\ &= -3r(r+1)\mu/2 + (2^{-r} - 2r\mu)\lambda_M \\ &= 2^{-r}\lambda_M - \mu r(3(r+1)/2 + 2\lambda_M). \end{aligned}$$

The latter expression is positive for μ sufficiently small. □

Lemma 5.9.9. *Let $s = F_\bullet \in \mathcal{S}$ and suppose $u \in \sigma_M \cap \tau_{Z_s}$ is nonzero. Then $\langle u, \zeta(V_{s,i})\rangle \leq \langle u, \zeta(V_{s,j})\rangle$ for any $0 \leq i \leq j \leq \ell_s$. Moreover, if $F_s \subset F \in \Gamma$, then there exists a constant $0 \leq m < \lambda_M$ such that* $\mathrm{Gen}(C_{F_s} \smallsetminus C_M) = \{V \in \mathrm{Gen}(C_F \smallsetminus C_M) : \langle u, \zeta(V)\rangle/N(u) \leq 1 + m\}$.

Proof. Since $u \in \sigma_M$, $\zeta(V_{s,0}) = V_M \in M$, and $\zeta(V_{s,j}) \in \mathrm{Newt}(f)$, it follows that $\langle u, \zeta(V_{s,0})\rangle \leq \langle u, \zeta(V_{s,j})\rangle$ for any $0 \leq j \leq \ell_s$. Suppose that $\langle u, \zeta(V_{s,i})\rangle > \langle u, \zeta(V_{s,j})\rangle$ for some $0 < i < j \leq \ell_s$. Let $\pi = (i,j) \in \mathrm{Sym}_{\ell_s}$ be the permutation of $[\ell_s]$ switching i and j. Let $\pi(s)$ be the unique element in $\mathcal{S}$ such that $\ell_{\pi(s)} = \ell_s$ and $V_{\pi(s),i} = V_{s,\pi(i)}$ for $1 \leq i \leq \ell_s$. Using (3) in Lemma 5.9.8, we compute:

$$\begin{aligned} \langle u, Z_s - Z_{\pi(s)}\rangle &= b_{i,\ell_s}\langle u, \zeta(V_{s,i}) - \zeta(V_{\pi(s),i})\rangle + b_{j,\ell_s}\langle u, \zeta(V_{s,j}) - \zeta(V_{\pi(s),j})\rangle \\ &= (b_{i,\ell_s} - b_{j,\ell_s})\langle u, \zeta(V_{s,i}) - \zeta(V_{s,j})\rangle > 0. \end{aligned}$$

The latter contradicts the assumption that $u \in \tau_{Z_s}$. This completes the proof of the first statement.

Since M is interior and $u \in \sigma_M$, $N(u) > 0$. Let $m = (\langle u, \zeta(V_{s,\ell_s})\rangle / N(u)) - 1 \geq 0$. Assume that $m \geq \lambda_M$. Using all the statements of Lemma 5.9.8, we compute

$$\begin{aligned}\langle u, Z_s\rangle / N(u) &= \sum_{i=0}^{\ell_s} b_{i,\ell_s} \langle u, \zeta(V_{s,i})\rangle / N(u) \\ &\geq \sum_{i=0}^{\ell_s} b_{i,\ell_s} + b_{\ell_s,\ell_s}\lambda_M \\ &\geq \sum_{i=0}^{r} b_{i,r} + b_{r,r}\lambda_M > 1.\end{aligned}$$

On the other hand, if $\tilde{s}$ is the unique element in $\mathcal{S}$ with $\ell_{\tilde{s}} = 0$, then, since $u \in \sigma_M$,

$$\langle u, Z_{\tilde{s}}\rangle / N(u) = \langle u, V_M\rangle / N(u) = 1 < \langle u, Z_s\rangle / N(u).$$

This contradicts the assumption that $u \in \tau_{Z_s}$. We conclude that $m < \lambda_M$.

Since $\langle u, \zeta(V_{s,\ell_s})\rangle = \max_{0 \leq i \leq \ell_s} \langle u, \zeta(V_{s,i})\rangle$, we have

$$\mathrm{Gen}(C_{F_s} \smallsetminus C_M) \subset \{V \in \mathrm{Gen}(C_F \smallsetminus C_M) : \langle u, \zeta(V)\rangle / N(u) \leq 1 + m\}.$$

It remains to prove the reverse inclusion. Suppose that $V \in \mathrm{Gen}(C_F \smallsetminus C_M)$ and $\langle u, \zeta(V)\rangle / N(u) \leq 1 + m$. Equivalently, we assume that $\langle u, \zeta(V_{s,\ell_s}) - \zeta(V)\rangle \geq 0$. We argue by contradiction. Assume that $V \notin C_{F_s}$. Let s' be the unique element in $\mathcal{S}$ such that $\ell_{s'} = \ell_s + 1$, $V_{s',i} = V_{s,i}$ for $0 \leq i \leq \ell_s$, and $V_{s',\ell_{s'}} = V$. If we let $V_{s,-1} = V_{s',-1} = 0$, then we can write

$$\langle u, Z_s\rangle = \sum_{k=0}^{\ell_s} b_{k,\ell_s} \langle u, \zeta(V_{s,k})\rangle = \sum_{k=0}^{\ell_s} (\sum_{i=k}^{\ell_s} b_{i,\ell_s}) \langle u, \zeta(V_{s,k}) - \zeta(V_{s,k-1})\rangle,$$

$$\langle u, Z_{s'}\rangle = \sum_{k=0}^{\ell_s+1} b_{k,\ell_s+1} \langle u, \zeta(V_{s',k})\rangle = \sum_{k=0}^{\ell_s+1} (\sum_{i=k}^{\ell_s+1} b_{i,\ell_s+1}) \langle u, \zeta(V_{s',k}) - \zeta(V_{s',k-1})\rangle.$$

$$\langle u, Z_s - Z_{s'}\rangle = \sum_{k=0}^{\ell_s} (\sum_{i=k}^{\ell_s} b_{i,\ell_s} - \sum_{i=k}^{\ell_s+1} b_{i,\ell_s+1}) \langle u, \zeta(V_{s,k}) - \zeta(V_{s,k-1})\rangle + b_{\ell_s+1,\ell_s+1} \langle u, \zeta(V_{s,\ell}) - \zeta(V)\rangle.$$

Since $u \in \sigma_M$ and $V_{s,0} = V_M \in M$, we have $\langle u, \zeta(V_{s,0})\rangle = N(u) > 0$. By conditions (1) and (4) in Lemma 5.9.8, it follows that all terms above are nonnegative, and at least one term is positive. This contradicts the assumption that $u \in \tau_{Z_s}$. □

The following lemma completes our proof.

Lemma 5.9.10. *With the notation of Definition 5.9.2, $(\mathcal{Z}, \mathcal{F})$ satisfies the the following property: Suppose that $\sigma_K^\circ \cap \tau_{Z_s} \cap \tau_{Z_{s'}} \neq \emptyset$, for some $K \in \mathrm{Contrib}(\alpha)_M$ and $s = F_\bullet, s' = F'_\bullet \in \mathcal{S}$. Then*

(1) $K \subset F_s$, and

(2) either $F_s \in F'_\bullet$ or $F_{s'} \in F_\bullet$.

Proof. Fix $u \in \sigma_K^\circ \cap \tau_{Z_s} \cap \tau_{Z_{s'}}$. By Lemma 5.9.9, $\langle u, \zeta(V)\rangle / N(u) < 1 + \lambda_M$ for all $V \in \mathrm{Gen}(C_{F_s} \smallsetminus C_M) \cup \mathrm{Gen}(C_{F_{s'}} \smallsetminus C_M)$. By Lemma 5.9.7, there exists a face $F \in \mathrm{Contrib}(\alpha)$ such that $K, F_s, F_{s'} \subset F$. By Lemma 5.9.9, there exists $m, m' \geq 0$ such that

$$\mathrm{Gen}(C_{F_s} \smallsetminus C_M) = \{V \in \mathrm{Gen}(C_F \smallsetminus C_M) : \langle u, \zeta(V)\rangle / N(u) \leq 1 + m\},$$
$$\mathrm{Gen}(C_{F_{s'}} \smallsetminus C_M) = \{V \in \mathrm{Gen}(C_F \smallsetminus C_M) : \langle u, \zeta(V)\rangle / N(u) \leq 1 + m'\}.$$

Since $u \in \sigma_K^\circ$, it follows from Lemma 5.9.1 that $\mathrm{Gen}(C_K \smallsetminus C_M) = \{V \in \mathrm{Gen}(C_F \smallsetminus C_M) : \langle u, \zeta(V)\rangle/N(u) = 1\}$, and hence $K \subset F_s$. It remains to establish (2). Without loss of generality, we may assume that $m \leq m'$. Then $\mathrm{Gen}(C_{F_s} \smallsetminus C_M) = \{V \in \mathrm{Gen}(C_{F_{s'}} \smallsetminus C_M) : \langle u, \zeta(V)\rangle/N(u) \leq 1 + m\}$. By Lemma 5.9.9, $\langle u, \zeta(V_{s',i})\rangle \leq \langle u, \zeta(V_{s',j})\rangle$ for $0 \leq i \leq j \leq \ell_{s'}$. We deduce that $F_s \in F'_\bullet$. □

A corollary of the proof of Lemma 5.9.10 is that $K \in F_\bullet$.

Proof of Theorem 5.7.12. Let $\alpha \notin \mathbb{Z}_{<0}$, and assume that all faces of $\mathrm{Contrib}(\alpha)$ are UB_1 and $\mathrm{Newt}(f)$ is α-simplicial. Let M be a minimal face of $\mathrm{Contrib}(\alpha)$. Then Lemma 5.9.6 and Lemma 5.9.10 imply that there is a restricted weakly α-compatible pair. Lemma 5.8.11 then implies that there is an α-compatible pair. □

References

[A'C75] N. A'Campo, *La fonction zêta d'une monodromie*, Comment. Math. Helv. **50** (1975), 233–248.

[Ath12a] C. Athanasiadis, *Cubical subdivisions and local h-vectors*, Ann. Comb. **16** (2012), no. 3, 421–448.

[Ath12b] ———, *Flag subdivisions and γ-vectors*, Pacific J. Math. **259** (2012), no. 2, 257–278.

[BN20] E. Bultot and J. Nicaise, *Computing motivic zeta functions on log smooth models*, Math. Z. **295** (2020), no. 1-2, 427–462.

[BO16] S. Brzostowski and G. Oleksik, *On combinatorial criteria for non-degenerate singularities*, Kodai Math. J. **39** (2016), no. 2, 455–468. MR 3520525

[BS66] S. Borewicz and I. Shafarevich, *Zahlentheorie*, Lehrbücher und Monographien aus dem Gebiete der Exakten Wissenschaften, Mathematische Reihe, Band 32, Birkhäuser Verlag, Basel-Stuttgart, 1966, Aus dem Russischen übersetzt von Helmut Koch.

[BV16] B. Bories and W. Veys, *Igusa's p-adic local zeta function and the monodromy conjecture for non-degenerate surface singularities*, Mem. Amer. Math. Soc. **242** (2016), no. 1145, vii+131.

[dCMM18] M. de Cataldo, L. Migliorini, and M. Mustaţă, *Combinatorics and topology of proper toric maps*, J. Reine Angew. Math. **744** (2018), 133–163. MR 3871442

[Den84] J. Denef, *The rationality of the Poincaré series associated to the p-adic points on a variety*, Invent. Math. **77** (1984), no. 1, 1–23.

[Den85] ———, *Poles of p-adic complex powers and Newton polyhedra*, Study group on ultrametric analysis, 12th year, 1984/85, No. 1, Secrétariat Math., Paris, 1985, pp. Exp. No. 17, 3.

[Den87] ———, *On the degree of Igusa's local zeta function*, Amer. J. Math. **109** (1987), no. 6, 991–1008.

[Den91a] ———, *Local zeta functions and Euler characteristics*, Duke Math. J. **63** (1991), no. 3, 713–721.

[Den91b] ———, *Report on Igusa's local zeta function*, no. 201-203, 1991, Séminaire Bourbaki, Vol. 1990/91, pp. Exp. No. 741, 359–386 (1992).

[DH01] J. Denef and K. Hoornaert, *Newton polyhedra and Igusa's local zeta function*, J. Number Theory **89** (2001), no. 1, 31–64.

[DL92] J. Denef and F. Loeser, *Caractéristiques d'Euler-Poincaré, fonctions zêta locales et modifications analytiques*, J. Amer. Math. Soc. **5** (1992), no. 4, 705–720.

[DL98] ———, *Motivic Igusa zeta functions*, J. Algebraic Geom. **7** (1998), no. 3, 505–537.

[dMGP+20] A. de Moura, E. Gunther, S. Payne, J. Schuchardt, and A. Stapledon, *Triangulations of simplices with vanishing local h-polynomial*, Algebr. Comb. **3** (2020), no. 6, 1417–1430. MR 4184051

[ELT22] A. Esterov, A Lemahieu, and K. Takeuchi, *On the monodromy conjecture for non-degenerate hypersurfaces*, Jour. Eur. Math. Soc. (2022), preprint arXiv:1309.0630v9.

[Est21] A. Esterov, *The ring of local tropical fans and tropical nearby monodromy eigenvalues*, preprint arXiv:1807.00609 v3, 2021.

[GJ00] E. Gawrilow and M. Joswig, `polymake`*: a framework for analyzing convex polytopes*, Polytopes—combinatorics and computation (Oberwolfach, 1997), DMV Sem., vol. 29, Birkhäuser, Basel, 2000, pp. 43–73. MR 1785292

[Igu75] J.-i. Igusa, *Complex powers and asymptotic expansions. II. Asymptotic expansions*, J. Reine Angew. Math. **278/279** (1975), 307–321.

[Igu78] ———, *Forms of higher degree*, Tata Institute of Fundamental Research Lectures on Mathematics and Physics, vol. 59, Tata Institute of Fundamental Research, Bombay; by the Narosa Publishing House, New Delhi, 1978.

[Igu88] ———, *b-functions and p-adic integrals*, Algebraic analysis, Vol. I, Academic Press, Boston, MA, 1988, pp. 231–241.

[JKYS19] S.-J. Jung, I.-K. Kim, Y. Yoon, and M. Saito, *Spectrum of nondegenerate functions with simplicial Newton polyhedra*, preprint arXiv:1911.09465, 2019.

[Kou76] A. Kouchnirenko, *Polyèdres de Newton et nombres de Milnor*, Invent. Math. **32** (1976), no. 1, 1–31.

[KS16] E. Katz and A. Stapledon, *Local h-polynomials, invariants of subdivisions, and mixed Ehrhart theory*, Adv. Math. **286** (2016), 181–239.

[Loe88] F. Loeser, *Fonctions d'Igusa p-adiques et polynômes de Bernstein*, Amer. J. Math. **110** (1988), no. 1, 1–21.

[Loe90] ———, *Fonctions d'Igusa p-adiques, polynômes de Bernstein, et polyèdres de Newton*, J. Reine Angew. Math. **412** (1990), 75–96.

[LPS22] M. Larson, S. Payne, and A. Stapledon, *Resolutions of local face modules, functoriality, and nonvanishing of local h-vectors*, 2022.

[LVP11] A. Lemahieu and L. Van Proeyen, *Monodromy conjecture for nondegenerate surface singularities*, Trans. Amer. Math. Soc. **363** (2011), no. 9, 4801–4829.

[Mal74] B. Malgrange, *Intégrales asymptotiques et monodromie*, Ann. Sci. École Norm. Sup. (4) **7** (1974), 405–430 (1975).

[Nic10] J. Nicaise, *An introduction to p-adic and motivic zeta functions and the monodromy conjecture*, Algebraic and analytic aspects of zeta functions and L-functions, MSJ Mem., vol. 21, Math. Soc. Japan, Tokyo, 2010, pp. 141–166.

[Nil12] B. Nill, *Combinatorial questions related to stringy E-polynomials of Gorenstein polytopes*, in Toric Geometry (K. Altmann et. al., eds.), 2012, pp. 62–64.

[Poo02] B. Poonen, *The Grothendieck ring of varieties is not a domain*, Math. Res. Lett. **9** (2002), no. 4, 493–497.

[Que22] M. H. Quek, *Around the motivic monodromy conjecture for non-degenerate hypersurfaces*, preprint arXiv:2208.02566, 2022.

[RV03] B. Rodrigues and W. Veys, *Poles of zeta functions on normal surfaces*, Proc. London Math. Soc. (3) **87** (2003), no. 1, 164–196.

[Sta92] R. Stanley, *Subdivisions and local h-vectors*, J. Amer. Math. Soc. **5** (1992), no. 4, 805–851.

[Sta17] A. Stapledon, *Formulas for monodromy*, Res. Math. Sci. **4** (2017), Paper No. 8, 42 pp.

[Var76] A. Varchenko, *Zeta-function of monodromy and Newton's diagram*, Invent. Math. **37** (1976), no. 3, 253–262.

[Vey93] W. Veys, *Poles of Igusa's local zeta function and monodromy*, Bull. Soc. Math. France **121** (1993), no. 4, 545–598.

[Vey07] ———, *Monodromy eigenvalues and zeta functions with differential forms*, Adv. Math. **213** (2007), no. 1, 341–357. MR 2331246

[VS12] J. Viu-Sos, *Computing the Igusa and Topological zeta functions of a Newton non-degenerated polynomial*, `https://github.com/jviusos/ZetaFunctionsNewtonND-Sagemath`, 2012.

[VS22] ———, *An introduction to p-adic and motivic integration, zeta functions and invariants of singularities*, p-adic analysis, arithmetic and singularities, Contemp. Math., vol. 778, Amer. Math. Soc., [Providence], RI, [2022] ©2022, pp. 103–176. MR 4419244

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