

KAPRANOV DEGREES

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ABSTRACT. The moduli space of stable rational curves with marked points has two distinguished families of maps: the forgetful maps, given by forgetting some of the markings, and the Kapranov maps, given by complete linear series of ψ -classes. The collection of all these maps embeds the moduli space into a product of projective spaces. We call the multidegrees of this embedding “Kapranov degrees,” which include as special cases the work of Witten, Silversmith, Gallet–Grassegger–Schicho, Castravet–Tevelev, Postnikov, Cavalieri–Gillespie–Monin, and Gillespie–Griffins–Levinson. We establish, in terms of a combinatorial matching condition, upper bounds for Kapranov degrees and a characterization of their positivity. The positivity characterization answers a question of Silversmith and gives a new proof of Laman’s theorem characterizing generically rigid graphs in the plane. We achieve this by proving a recursive formula for Kapranov degrees and by using tools from the theory of error correcting codes.

1. INTRODUCTION

For a finite set S with cardinality $|S| \geq 3$, let $\overline{M}_{0,S}$ be the Deligne–Mumford–Knudsen moduli space of stable rational curves with marked points labelled by S . We write $\overline{M}_{0,n}$ for $\overline{M}_{0,[n]}$, where $[n] := \{1, \dots, n\}$ for an integer $n \geq 3$. For a pair (S, i) consisting of a subset $S \subseteq [n]$ with $|S| \geq 3$ and an element $i \in S$, let

$f_S: \overline{M}_{0,n} \rightarrow \overline{M}_{0,S}$ be the forgetful map induced by forgetting the markings not in S , and $|\psi_i|: \overline{M}_{0,S} \rightarrow \mathbb{P}^{|S|-3}$ be the *Kapranov map* induced by the complete linear series of the *psi-class* ψ_i ,

where ψ_i is the divisor class corresponding to the base-point-free line bundle whose fiber over a stable marked curve $C \in \overline{M}_{0,S}$ is the cotangent line at the i -th marked point of C [Kap93]. See [GGL23a, Section 2] for a convenient summary of relevant facts about these maps. For such a pair (S, i) , we define $X_{S,i} := f_S^* \psi_i$. These divisor classes were first considered in [Sin04, EHKR10], where they were used to give a basis for the cohomology ring of $\overline{M}_{0,n}$.

Definition 1.1. For a collection $(S_1, i_1), \dots, (S_{n-3}, i_{n-3})$ of $n - 3$ such pairs, we define its *Kapranov degree* to be the intersection number

$$\int_{\overline{M}_{0,n}} X_{S_1, i_1} \cdots X_{S_{n-3}, i_{n-3}},$$

where $\int_{\overline{M}_{0,n}}: H^*(\overline{M}_{0,n}) \rightarrow \mathbb{Z}$ is the degree map on the cohomology ring of $\overline{M}_{0,n}$.

In other words, Kapranov degrees are exactly the coefficients of the multidegree of the embedding

$$\overline{M}_{0,n} \hookrightarrow \prod_{\substack{(S,i) \text{ such that} \\ i \in S \subseteq [n] \text{ and } |S| \geq 3}} \mathbb{P}^{|S|-3}$$

of $\overline{M}_{0,n}$ into a product of projective spaces obtained by collecting together all compositions of forgetful maps and Kapranov maps. Much of the enormous amount of work on intersection theory on $\overline{M}_{0,n}$ can be viewed as special cases of Kapranov degrees:

- (i) When $S_j = [n]$ for all $j = 1, \dots, n-3$, Kapranov degrees in this case are intersection numbers of ψ -classes, for which Witten [Wit91] showed that $\int_{\overline{M}_{0,n}} \psi_1^{a_1} \cdots \psi_n^{a_n} = \binom{n-3}{a_1, \dots, a_n}$. This is the first case of Witten's conjecture on the intersection numbers of ψ -classes on $\overline{M}_{g,n}$.
- (ii) When $|S_j| = 4$ for all j , Kapranov degrees in this case were studied by Silversmith [Sil22] under the name of *cross-ratio degrees*, because the divisor class $X_{S,i}$ when $|S| = 4$ is the pullback of the hyperplane class under the map $\overline{M}_{0,n} \rightarrow \mathbb{P}^1$ given by taking the cross-ratio of the four points marked by S . That is, the Kapranov degree in this case can be interpreted as the number of ways to choose n marked points on \mathbb{P}^1 with $n-3$ prescribed cross-ratios.
- (iii) When further $n = 2v$, and the size 4 subsets S_j are obtained from a graph G with vertex set $[v]$ and $2v-3$ edges by assigning to each edge (a,b) the subset $\{a, b, a+v, b+v\}$, the cross-ratio degree counts the number of (complex) realizations of G on the 2-dimensional sphere up to isometries for any fixed generic edge lengths of G [GG20].
- (iv) When $\{n-1, n\} \subset S_j$ and $i_j \in \{n-1, n\}$ for all j , Kapranov degrees are certain mixed volumes of simplices, for which combinatorial formulas were given in special cases in [Pos09] and [EFLS22] in terms of matching conditions. See Remark 5.3 for a detailed discussion.
- (v) When $S = [i]$, the divisor class $X_{[i],i}$ is also called an ω -class, and the intersection numbers of ω -classes and ψ -classes were studied in [CGM21, GGL23b, GGL23a] in terms of combinatorial structures such as parking functions, tournaments, and slide rules.
- (vi) When $i_j = n$ for all j , the Kapranov degrees were computed in [CT13, Theorem 3.2]. In this case the Kapranov degree is either 0 or 1.

The more general framework that we consider here allows access to additional tools to study these special cases. Let us begin by noting some immediate properties of Kapranov degrees.

Proposition 1.2. Kapranov degrees satisfy the following.

- (1) (Permutation invariance) $\int_{\overline{M}_{0,n}} X_{S_1, i_1} \cdots X_{S_{n-3}, i_{n-3}} = \int_{\overline{M}_{0,n}} X_{\sigma(S_1), \sigma(i_1)} \cdots X_{\sigma(S_{n-3}), \sigma(i_{n-3})}$ for any permutation σ of $[n]$.
- (2) (Nonnegativity) $\int_{\overline{M}_{0,n}} X_{S_1, i_1} \cdots X_{S_{n-3}, i_{n-3}} \geq 0$.
- (3) (Monotonicity) $\int_{\overline{M}_{0,n}} X_{S_1, i_1} \cdots X_{S_{n-3}, i_{n-3}} \leq \int_{\overline{M}_{0,n}} X_{S_1 \cup T_1, i_1} \cdots X_{S_{n-3} \cup T_{n-3}, i_{n-3}}$ for any subsets $T_1, \dots, T_{n-3} \subseteq [n]$.

Proof. Permutation invariance is clear. Nonnegativity follows from the fact that the divisor classes $X_{S,i}$ are nef, as they are pullbacks of ψ -classes, which are base-point-free. Monotonicity follows by further noting that $X_{S \cup T, i} - X_{S, i}$ is an effective divisor class (see Lemma 2.1). \square

Our first main result gives upper bounds for Kapranov degrees and a characterization of their positivity.

Theorem A. Kapranov degrees satisfy the following.

- (a) For any fixed size 3 subset $\{p, q, r\} \subseteq [n]$, we define a *matching* for S_1, \dots, S_{n-3} to be a bijection $m: [n-3] \rightarrow [n] \setminus \{p, q, r\}$ such that $m(j) \in S_j$ for all $j \in [n-3]$. We define the *weight* of a matching m to be $w(m) = \prod_j (|S_j| - 3)$, where the product is taken over $j \in [n-3]$ such that $m(j) = i_j$. The empty product is set to be 1. Then, we have that

$$\int_{\overline{M}_{0,n}} X_{S_1, i_1} \cdots X_{S_{n-3}, i_{n-3}} \leq \sum_{m \text{ a matching}} w(m).$$

- (b) Let us say that S_1, \dots, S_{n-3} satisfies the *Cerberus condition* if $|\bigcup_{j \in J} S_j| \geq |J| + 3$ for all nonempty $J \subseteq [n-3]$. The Kapranov degree $\int_{\overline{M}_{0,n}} X_{S_1, i_1} \cdots X_{S_{n-3}, i_{n-3}}$ is positive if and only if S_1, \dots, S_{n-3} satisfies the Cerberus condition.

In the case of cross-ratio degrees, the upper bound in Theorem A(a) was proven in [Sil22] using Gromov–Witten theory. We prove Theorem A(a) in Section 3 by comparing the Kapranov degree with the intersection number of the pushforwards of the $X_{S, i}$'s under the map $f: \overline{M}_{0,n} \rightarrow (\mathbb{P}^1)^{n-3}$ given by taking the cross-ratio of the points marked by $\{j, p, q, r\}$ for $j \in [n] \setminus \{p, q, r\}$.

As a step in the proof of Theorem A(a), we establish a recursive formula for Kapranov degrees, described as follows. Let $\{\star\}$ be a singleton disjoint from $[n]$. Given a partition $[n] = P \sqcup Q$ with $|P| \geq 2$ and $|Q| \geq 2$, and a pair (S, i) with $S \subseteq [n]$, $|S| \geq 3$, and $i \in S$, we define an element $\text{res}_P(S, i)$ in the cohomology of $\overline{M}_{0, P \cup \star}$ by

$$\text{res}_P(S, i) = \begin{cases} X_{S, i} & \text{if } S \subseteq P \\ X_{(S \setminus i) \cup \star, \star} & \text{if } S \cap P = S \setminus i \\ X_{(S \cap P) \cup \star, i} & \text{if } S \cap P \supseteq \{i, j\} \text{ for some } i \neq j \in S \\ 1 & \text{otherwise, i.e., if } S \cap P = \{i\} \text{ or } S \cap P \subseteq S \setminus \{i, j\} \text{ for some } i \neq j \in S. \end{cases}$$

One may find an illustration of these cases in Figure 1 helpful for visualizing the above conditions. We similarly define $\text{res}_Q(S, i)$ by replacing P with Q in the above. It follows from the definition that $\text{res}_P(S, i) = 1$ if and only if $\text{res}_Q(S, i) \neq 1$ and vice versa.

Theorem B. Kapranov degrees satisfy the following recursion: for an element $a \in S_1 \setminus i_1$,

$$\begin{aligned} \int_{\overline{M}_{0,n}} X_{S_1, i_1} \cdots X_{S_{n-3}, i_{n-3}} &= \int_{\overline{M}_{0,n}} X_{S_1 \setminus a, i_1} X_{S_2, i_2} \cdots X_{S_{n-3}, i_{n-3}} \\ &+ \sum_{\substack{[n] = P \sqcup Q \\ \{a, i_1\} \subseteq P \\ S_1 \setminus \{a, i_1\} \subseteq Q}} \left(\int_{\overline{M}_{0, P \cup \star}} \prod_{j=2}^{n-3} \text{res}_P(S_j, i_j) \right) \cdot \left(\int_{\overline{M}_{0, Q \cup \star}} \prod_{j=2}^{n-3} \text{res}_Q(S_j, i_j) \right). \end{aligned}$$

When $|S_j| = 4$ for all $j \in [n-3]$, i.e., the case of cross-ratio degrees, Theorem B was proven in [GGS20]. It was also proven in [Gol21, Lemma 3.11] using tropical geometry (see [Sil22]). We prove Theorem B in Section 2 by writing $X_{S, i} - X_{S \setminus a, i}$ as a sum of boundary divisors and using that the boundary divisors are products of smaller $\overline{M}_{0, n'}$'s. Theorem B gives a useful algorithm for computing

Kapranov degrees.¹ However, we do not know how to deduce Theorem A(b) using this recursive formula; see Remark 4.8 for a discussion.

Theorem A(b) affirmatively answers [Sil22, Question 4.2]. When combined with [GGS20], Theorem A(b) gives a new proof of Laman’s theorem [PG27, Lam70] concerning minimally rigid graphs in the plane; see Corollary 4.9. When combined with Proposition 1.2(3) and the observation that $\int_{\overline{M}_{0,n}} \psi_n^{n-3} = 1$, Theorem A(b) also recovers [CT13, Theorem 3.2].

The “only if” part of Theorem A(b) is straightforward: if the Cerberus condition is violated, then $|\bigcup_{j \in J} S_j| < |J| + 3$ for some nonempty $J \subseteq [n-3]$, and the product $\prod_{j \in J} X_{S_j, i_j}$ is zero because it is pulled back from $\overline{M}_{0,S}$ where $S = \bigcup_{j \in J} S_j$, which has dimension strictly less than $|J|$. We prove the “if” part in Section 4 by reducing to the case of cross-ratio degrees, in which case the positivity is equivalent to the algebraic independence of the rational functions corresponding to the cross-ratios. We check the linear independence of the differentials of these rational functions using the GM-MDS theorem [Lov21, YH19] from the theory of error correcting codes.

Lastly, we give an explicit formula for the special case of Kapranov degrees where all the S_j share a subset of size 3, say $\{n-2, n-1, n\}$, and each i_j lies in $\{n-2, n-1, n\}$. For a pair (T, j) consisting of $T \subseteq [n-3]$ and $j \in \{n-2, n-1, n\}$, let $Y_{T,j} = X_{T \cup \{n-2, n-1, n\}, j}$. For $n-3$ such pairs $(T_1, j_1), \dots, (T_{n-3}, j_{n-3})$, we define a 3-transversal to be a map $t: [n-3] \rightarrow \{n-2, n-1, n\}$ such that there is a bijection $m: [n-3] \rightarrow [n-3]$ satisfying $m(i) \in T_i$ and $t(m(i)) = j_i$ for all $i \in [n-3]$.

Theorem C. The Kapranov degree $\int_{\overline{M}_{0,n}} Y_{T_1, j_1} \cdots Y_{T_{n-3}, j_{n-3}}$ is equal to the number of 3-transversals of $(T_1, j_1), \dots, (T_{n-3}, j_{n-3})$.

When further $j_i \in \{n-1, n\}$ for all $i \in [n-3]$, this follows from [EFLS22, Theorem B]; see Remark 5.3. We prove Theorem C by repeatedly applying Theorem B.

Open questions. We conclude with some open problems:

- When are Kapranov degrees 1? We don’t have a conjecture even in the case of cross-ratio degrees. A special case of this problem arose in [AHHLT23].
- It would be interesting to study the K -theoretic analogue of Kapranov degrees. A generalization to K -theory of the result in [Wit91] was given in [Pan97, Lee97], and a generalization of [CT13, Theorem 3.2] was given in [LLPP22, Section 9].
- To what extent can these results be generalized to positive genus? If $g = 1$, the analogue of Theorem A(b) does not hold: $\int_{\overline{M}_{1,2}} f_1^* \psi_1 f_2^* \psi_2 = 0$.

A note on ground field characteristic. We assume the ground field has characteristic zero in the proof of Theorem A(a) when making Observation 3.1, but our results are valid for $\overline{M}_{0,n}$ in arbitrary characteristic because the Chow ring of $\overline{M}_{0,n}$ is independent of the ground field. See also Remark 4.7.

¹An implementation in SageMath is available at <https://github.com/MattLarson2399/KapranovDegrees>.

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2. RECURSION

In this section, we prove Theorem B and deduce from it an interpretation of cross-ratio degrees, i.e., Kapranov degrees when $|S_j| = 4$ for all j , in terms of marked trivalent trees (Corollary 2.3).

We prepare with two lemmas about $X_{S,i}$ and boundary divisors on $\overline{M}_{0,n}$. For a partition $[n] = P \sqcup Q$ with $|P| \geq 2$ and $|Q| \geq 2$, let $D_{P|Q}$ be the corresponding boundary divisor, i.e.,

$$D_{P|Q} = \text{the closure of } \left\{ C \in \overline{M}_{0,n} : \begin{array}{l} C \text{ has two } \mathbb{P}^1\text{-components, each with distinct marked points} \\ \text{labelled by } P \text{ or } Q \text{ respectively} \end{array} \right\}.$$

With $\{\star\}$ a singleton disjoint from $[n]$, the gluing map $\text{gl}_{P|Q}: \overline{M}_{0,P \cup \star} \times \overline{M}_{0,Q \cup \star} \rightarrow \overline{M}_{0,n}$, which glues two marked curves $C_1 \in \overline{M}_{0,P \cup \star}$ and $C_2 \in \overline{M}_{0,Q \cup \star}$ by identifying the points marked by \star , is an isomorphism onto the boundary divisor $D_{P|Q}$. Let $\pi_P: \overline{M}_{0,P \cup \star} \times \overline{M}_{0,Q \cup \star} \rightarrow \overline{M}_{0,P \cup \star}$ be the projection onto the first factor, and similarly for π_Q .

Lemma 2.1. Let (S, i) be a pair with $S \subseteq [n]$, $|S| \geq 4$, and $i \in S$. If $a \in S$ is an element distinct from i , then we have that

$$X_{S,i} = X_{S \setminus a, i} + \sum_{P, Q} D_{P|Q},$$

where the sum ranges over all bipartitions $[n] = P \sqcup Q$ such that $\{a, i\} \subseteq P$ and $S \setminus \{a, i\} \subseteq Q$.

Proof. Consider the forgetful map $f_{S \setminus a}: \overline{M}_{0,S} \rightarrow \overline{M}_{0,S \setminus a}$. By [AC98, Lemma 3.1], we have that $\psi_i = X_{S \setminus a, i} + D_{\{a, i\} | (S \setminus \{a, i\})}$ on $\overline{M}_{0,S}$. The pullback of any $D_{P'|Q'}$ along $f_S: \overline{M}_{0,n} \rightarrow \overline{M}_{0,S}$ is the sum of the boundary divisors on $\overline{M}_{0,n}$ corresponding to the partitions obtained by adding elements of $[n] \setminus S$ to either P' or Q' . The result then follows. \square

Lemma 2.2. Let (S, i) be a pair with $S \subseteq [n]$, $|S| \geq 4$, and $i \in S$. For a partition $[n] = P \sqcup Q$ with $|P| \geq 2$ and $|Q| \geq 2$, we have

$$\text{gl}_{P|Q}^* X_{S,i} = \begin{cases} \pi_P^* X_{S,i} & \text{if } S \subseteq P \\ \pi_P^* X_{(S \setminus i) \cup \star, \star} & \text{if } S \cap P = S \setminus i \\ \pi_P^* X_{(S \cap P) \cup \star, i} & \text{if } S \cap P \supseteq \{i, j\} \text{ for some } i \neq j \in S \end{cases},$$

and by symmetry, one obtains the remaining three cases by replacing P with Q .

Proof. For a curve $C \in \overline{M}_{0,n}$, the curve $f_S(C) \in \overline{M}_{0,S}$ is obtained by forgetting the points not in S and contracting unstable components. The lemma follows from the fact that the divisor $X_{S,i}$ corresponds to the line bundle whose fiber at C is the cotangent line at the i -th marked point on $f_S(C)$. \square

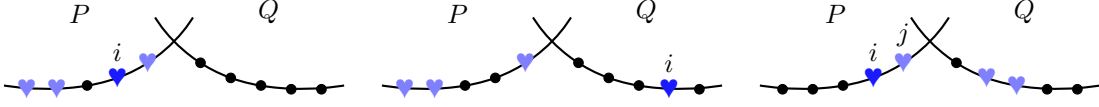


FIGURE 1. Illustrations of rational curves in the three cases of Lemma 2.2. The left rational component supports marked points indexed by P , the black round marked points are not indexed by S , and the blue hearts are marked points indexed by S .

Proof of Theorem B. By Lemma 2.1, we have that

$$\begin{aligned} \int_{\overline{M}_{0,n}} X_{S_1, i_1} \cdots X_{S_{n-3}, i_{n-3}} &= \int_{\overline{M}_{0,n}} X_{S_1 \setminus \{a, i_1\}} X_{S_2, i_2} \cdots X_{S_{n-3}, i_{n-3}} \\ &\quad + \sum_{\substack{[n] = P \sqcup Q \\ \{a, i_1\} \subseteq P \\ S \setminus \{a, i_1\} \subseteq Q}} \int_{\overline{M}_{0,n}} D_{P|Q} X_{S_2, i_2} \cdots X_{S_{n-3}, i_{n-3}}. \end{aligned}$$

By the projection formula,

$$\int_{\overline{M}_{0,n}} D_{P|Q} X_{S_2, i_2} \cdots X_{S_{n-3}, i_{n-3}} = \int_{\overline{M}_{0, P \cup \star} \times \overline{M}_{0, Q \cup \star}} \text{gl}_{P|Q}^* X_{S_2, i_2} \cdots \text{gl}_{P|Q}^* X_{S_{n-3}, i_{n-3}}.$$

By Lemma 2.2, each factor $\text{gl}_{P|Q}^* X_{S_j, i_j}$ on the right-hand-side is pulled back via either π_P or π_Q . By the Künneth formula, the intersection number is a product of intersection numbers on $\overline{M}_{0, P \cup \star}$ and $\overline{M}_{0, Q \cup \star}$. Then Lemma 2.2 gives the result. \square

When $|S_j| = 4$ for all j , i.e., the case of cross-ratio degrees, a repeated application of Theorem B yields a combinatorial interpretation of Kapranov degrees, stated in Corollary 2.3 below. However, this combinatorial interpretation depends on many auxiliary choices, and we do not know whether one can use it to prove Theorem A(b).

For $T \subseteq [n]$ with $|T| = 4$, we write $X_T = X_{T, i}$ for any $i \in T$ because the class does not depend on the choice of i . We say that two vertex-disjoint paths on a tree are *separated by an edge* if there is a (necessarily unique) edge on the tree incident to a vertex from each path.

Corollary 2.3. Let S_1, \dots, S_{n-3} be subsets of $[n]$ of size 4, and for each $j \in [n-3]$ choose a partition $S_j = \{a_j, b_j\} \sqcup \{c_j, d_j\}$. Then, the cross-ratio degree $\int_{\overline{M}_{0,n}} X_{S_1} \cdots X_{S_{n-3}}$ is the number of trivalent trees with n leaves marked by $[n]$ that satisfy the following condition: The unique path from the leaf marked a_1 to the leaf marked b_1 is separated by an edge from the unique path between the leaves marked c_1 and d_1 . After contracting this edge, the unique path from a_2 to b_2 is separated by an edge from the unique path from c_2 to d_2 . After contracting this edge, similarly for S_3 and so forth.

3. UPPER BOUND

In this section, we prove Theorem A(a) in the following way. We may assume that $\{p, q, r\} = \{n-2, n-1, n\}$. For $T \subseteq [n]$ with $|T| = 4$, the forgetful map $f_T: \overline{M}_{0,n} \rightarrow \overline{M}_{0,T} \simeq \mathbb{P}^1$ is given by



FIGURE 2. The trivalent trees which, under Corollary 2.3, correspond to the computation $\int_{\overline{M}_{0,7}} X_{1234} X_{1235} X_{4567} X_{1267} = 2$.

taking the cross-ratio of the points marked by T . Consider the birational map

$$f: \overline{M}_{0,n} \rightarrow \prod_{j=1}^{n-3} \overline{M}_{0,\{j,n-2,n-1,n\}} \simeq (\mathbb{P}^1)^{n-3}.$$

Using Poincaré duality on $(\mathbb{P}^1)^{n-3}$ and Theorem B, we will first compute the pushforward $f_* X_{S,i}$, and then deduce Theorem A(b) by comparing the Kapranov degree with the intersection number of the pushforwards. We start with an observation that we repeatedly use.

Observation 3.1. We may choose divisors D_1, \dots, D_{n-3} in the linear series of $X_{S_1, i_1}, \dots, X_{S_{n-3}, i_{n-3}}$, respectively, such that they intersect transversely in finitely many points, all of which are contained in the open moduli subspace $M_{0,n}$ of irreducible marked curves. This is possible because the line bundles corresponding to the $X_{S,i}$ are base-point-free.

We now prepare the computation of the pushforward $f_* X_{S,i}$ with a lemma. As before, when $|T| = 4$ we write $X_T = X_{T,i}$ for any $i \in T$, which is the pullback of the hyperplane class from $\overline{M}_{0,n} \rightarrow \overline{M}_{0,T} \simeq \mathbb{P}^1$.

Lemma 3.2. Let $S_1, \dots, S_{n-3} \subset [n]$ be subsets of size 4 such that, after reordering, we have that $|\bigcup_{j=1}^{\ell} S_j| = \ell + 3$ for all $\ell = 1, \dots, n-3$. Then

$$\int_{\overline{M}_{0,n}} X_{S_1} \cdots X_{S_{n-3}} = 1.$$

Proof. By Observation 3.1, for generic complex numbers $\lambda_1, \dots, \lambda_{n-3}$, the intersection number is the number of points in $M_{0,n}$ such that the cross-ratio of the points in S_j is λ_j for all $j = 1, \dots, n-3$. There is a unique such point: we set three of the marked points in S_1 to be 0, 1, and ∞ , and imposing the cross-ratio condition given by each S_j determines an additional point. \square

Let $T_j = \{j, n-2, n-1, n\}$ for $j \in \{1, \dots, n-3\}$. The following proposition will amount to computing the pushforward $f_* X_{S,i}$.

Proposition 3.3. For $S \subseteq [n]$ with $|S| \geq 4$ and $i \in S$, we have

$$\int_{\overline{M}_{0,n}} X_{S,i} X_{T_2} X_{T_3} \cdots X_{T_{n-3}} = \begin{cases} |S| - 3 & \text{if } i = 1 \\ 1 & \text{if } 1 \in S \setminus i \\ 0 & \text{otherwise, i.e., if } 1 \notin S. \end{cases}$$

Proof. Let us first recall the general observation that a Kapranov degree $\int_{\overline{M}_{0,n}} X_{S_1, i_1} \cdots X_{S_{n-3}, i_{n-3}}$ is zero when $\bigcup_{i=1}^{n-3} S_i \neq [n]$ because in that case $X_{S_1, i_1} \cdots X_{S_{n-3}, i_{n-3}}$ is the pullback of a product of $n-3$ divisors on $\overline{M}_{0, \bigcup_i S_i}$, which has dimension at most $n-4$.

If $1 \notin S$, this immediately implies that the Kapranov degree in our proposition is zero. For the remaining cases, we first note that, when we apply Theorem B with $a \in S \setminus i$, the term

$$\left(\int_{\overline{M}_{0, P \sqcup \star}} \prod_{j=2}^{n-3} \text{res}_P(T_j, j) \right) \cdot \left(\int_{\overline{M}_{0, Q \sqcup \star}} \prod_{j=2}^{n-3} \text{res}_Q(T_j, j) \right)$$

in the summation, for a partition $[n] = P \sqcup Q$ with $P \ni 1$, is nonzero only if $P = \{1, b\}$ for some $b \neq 1$ since $1 \notin T_j$ for all $j > 1$. When $P = \{1, b\}$ for some $b \neq 1$, we find that the term evaluates to 1 as follows: $\int_{\overline{M}_{0, P \sqcup \star}} \prod_{j=2}^{n-3} \text{res}_P(T_j, j) = 1$ because $\text{res}_P(T_j, j) = 1$ for all $j > 1$ and $\overline{M}_{0, P \sqcup \star}$ is a point, while $\int_{\overline{M}_{0, Q \sqcup \star}} \prod_{j=2}^{n-3} \text{res}_Q(T_j, j) = 1$ by Lemma 3.2 because $\text{res}_Q(T_j, j) = X_{\{j, \star, n-1, n\}}$ if $b = n-2$ (and likewise if $b = n-1$ or n), and otherwise $\text{res}_Q(T_j, j) = X_{\{j, n-2, n-1, n\}}$ for $j \neq b$ and $X_{\{\star, n-2, n-1, n\}}$ for $j = b$.

Now, if $1 \in S \setminus i$, then applying Theorem B with $a = 1$, we find $\int_{\overline{M}_{0,n}} X_{S \setminus 1, i} X_{T_2} \cdots X_{T_{n-3}} = 0$ by the general observation, and all the terms in the summation over the appropriate partitions $[n] = P \sqcup Q$ vanish except when $P = \{1, i\}$, which contributes 1.

If $i = 1$, we induct on $|S|$. When $|S| = 4$, the result follows from Lemma 3.2. If $|S| > 4$, when we apply Theorem B with $a \in S \setminus \{1, n-2, n-1, n\}$, the only nonzero term in the summation over partitions is when $P = \{1, a\}$, which contributes 1. Now the desired result follows from the induction hypothesis on the remaining term $\int_{\overline{M}_{0,n}} X_{S \setminus a, 1} X_{T_2} \cdots X_{T_{n-3}}$. \square

Proof of Theorem A(a). Without loss of generality, let $\{p, q, r\} = \{n-2, n-1, n\}$. Let $f: \overline{M}_{0,n} \rightarrow (\mathbb{P}^1)^{n-3}$ be the birational map given by taking the cross-ratio of the points $\{j, n-2, n-1, n\}$ for $j \in \{1, \dots, n-3\}$. Let y_1, \dots, y_{n-3} be the basis for $H^2((\mathbb{P}^1)^{n-3})$ such that $f^* y_j = X_{T_j}$. Note that under the Poincaré pairing on the cohomology ring of $(\mathbb{P}^1)^{n-3}$, the monomials $\frac{y_1 \cdots y_{n-3}}{y_1}, \dots, \frac{y_1 \cdots y_{n-3}}{y_{n-3}}$ form the dual basis. Then, by Poincaré duality on $(\mathbb{P}^1)^{n-3}$, the projection formula, and Proposition 3.3, we have that

$$f_* X_{S,i} = \delta_{S,i} + \sum_{\substack{a \in S \setminus i \\ a \in [n-3]}} y_a \quad \text{where} \quad \delta_{S,i} = \begin{cases} (|S| - 3) y_i & \text{if } i \in [n-3] \\ 0 & \text{if } i \in \{n-2, n-1, n\}. \end{cases}$$

Now, let D_1, \dots, D_{n-3} be a choice of divisors as in Observation 3.1. As f is an isomorphism on $M_{0,n}$, we let D'_j be the closure of $D_j \cap M_{0,n}$ in $(\mathbb{P}^1)^{n-3}$, which as a divisor class is equivalent to $f_* X_{S_j, i_j}$. The intersection of the D'_j contains the intersection of the D_j , but it may contain extra

points in the boundary. By [Fu198, Example 12.2.7(a)], we have that

$$\begin{aligned}
\int_{\overline{M}_{0,n}} X_{S_1, i_1} \cdots X_{S_{n-3}, i_{n-3}} &\leq \int_{(\mathbb{P}^1)^{n-3}} D'_1 \cdots D'_{n-3} \\
&= \int_{(\mathbb{P}^1)^{n-3}} \prod_{j=1}^{n-3} \left(\delta_{S_j, i_j} + \sum_{\substack{a \in S_j \setminus i_j \\ a \in [n-3]}} y_a \right) \\
&= \sum_{m \text{ a matching}} w(m). \quad \square
\end{aligned}$$

The upper bound in Theorem A(a) and the upper bound obtained by using Proposition 1.2(3) to reduce to intersections of ψ -classes are not comparable. For instance, in the example featured in Figure 2, the upper bound in Theorem A(a) is tight. The upper bound in Theorem A(a) may not be tight for any choice of $\{p, q, r\}$ even in the case of cross-ratio degrees [Sil22, Section 4]. By Theorem A(b) and Hall's marriage theorem, the upper bound is tight for some choice of $\{p, q, r\}$ whenever the Kapranov degree vanishes.

4. POSITIVITY

In this section, we prove Theorem A(b). Recall that a collection of subsets S_1, \dots, S_{n-3} of $[n]$ satisfies the *Cerberus condition* if

$$(\text{Cerberus}) \quad \left| \bigcup_{j \in J} S_j \right| \geq |J| + 3 \quad \text{for all nonempty } J \subseteq [n].$$

We noted in the introduction that the Kapranov degree is zero if the Cerberus condition is not satisfied. For the converse, i.e., the "if" part of Theorem A(b), we reduce to the case of cross-ratio degrees by combining Proposition 1.2(3) with a combinatorial argument (Proposition 4.1). We then analyze the cross-ratio degree case using the GM-MDS theorem (Theorem 4.5) from the theory of error correcting codes. We conclude with an application to graph rigidity (Corollary 4.9).

4.1. Reduction to the case of cross-ratio degrees.

Proposition 4.1. Let $S_1, \dots, S_{n-3} \subset [n-3]$, and choose $i_j \in S_j$. If S_1, \dots, S_{n-3} satisfies the Cerberus condition, then we can find $S'_j \subset S_j$ with $i_j \in S'_j$ and $|S'_j| = 4$ such that S'_1, \dots, S'_{n-3} satisfies the Cerberus condition.

Proposition 4.1 is closely related to the "generalized Hall's marriage theorem" results in [DSY14, BGM23]. We prove Proposition 4.1 using matroid theory. See [Oxl11] for undefined matroid terminology. A reader uninterested in the proof of Proposition 4.1 may skip to Section 4.2.

We begin by recalling the Hall-Rado theorem, as stated in [Oxl11, Theorem 11.2.2].

Proposition 4.2. Let M be a matroid with rank function rk_M , and let A_1, \dots, A_ℓ be subsets of the ground set of M . Then, there is an independent set $\{a_1, \dots, a_\ell\}$ of M satisfying $a_j \in A_j$ for all $j \in [\ell]$

if and only if

$$\text{rk}_M \left(\bigcup_{j \in J} A_j \right) \geq |J| \quad \text{for all } J \subseteq [\ell].$$

In particular, such a subset $\{a_1, \dots, a_\ell\}$ exists if and only if it exists after replacing each A_j with its closure in M .

We will apply the Hall–Rado theorem to the matroid known as the *third Dilworth truncation* of the boolean matroid, denoted $D_3(U_{n,n})$. See, for example, [Whi90]. Its ground set is the collection $\binom{[n]}{4}$ of all subsets of $[n]$ of size 4, and its set of bases consists of collections of $n - 3$ subsets of size 4 that satisfy the Cerberus condition. In [Ath99, Theorem 3.2], the flats of $D_3(U_{n,n})$ are calculated to be as follows. They are in bijection with anti-chains $\{T_1, \dots, T_k\}$ in the poset of subsets of $[n]$ that satisfy the conditions

$$|T_i| \geq 4 \quad \text{and} \quad \left| \bigcup_{i \in I} T_i \right| > 3 + \sum_{i \in I} (|T_i| - 3) \quad \text{for all } I \subseteq [k] \text{ with } |I| \geq 2.$$

The flat of $D_3(U_{n,n})$ corresponding to $\{T_1, \dots, T_k\}$ is the set

$$F_{T_1, \dots, T_k} = \left\{ S \in \binom{[n]}{4} : S \subseteq T_i \text{ for some } i \in [k] \right\},$$

and its rank in $D_3(U_{n,n})$ is $\sum_{i=1}^k (|T_i| - 3)$. The flat F_{T_1, \dots, T_k} is connected if and only if $k = 1$. The poset relation on the lattice of flats of $D_3(U_{n,n})$ is by refinement: $F_{T_1, \dots, T_k} \leq F_{T'_1, \dots, T'_k}$ if each T_i is contained in some T'_i .

Proof of Proposition 4.1. For a pair (S, i) , let $T_{S,i}$ be the set $\{S' \in \binom{[n]}{4} : i \in S' \subseteq S\}$. We claim that because S_1, \dots, S_{n-3} satisfies Cerberus, the collection $T_{S_1, i_1}, \dots, T_{S_{n-3}, i_{n-3}}$ of subsets of $\binom{[n]}{4}$ satisfies (Hall–Rado) with $M = D_3(U_{n,n})$. The result then follows by applying Proposition 4.2 with $A_j = T_{S_j, i_j}$.

For the claim, we first note that the closure of $T_{S,i}$ in $D_3(U_{n,n})$ is the flat F_S corresponding to $\{S\}$ because F_S contains $T_{S,i}$ and they both have rank $|S| - 3$. We now need show that the collection of flats $F_{S_1}, \dots, F_{S_{n-3}}$ of $D_3(U_{n,n})$ satisfies (Hall–Rado). Suppose on the contrary that it fails (Hall–Rado), and take a minimal $J \subseteq [n - 3]$ witnessing the failure. For each $j \in J$, because the flat F_{S_j} is connected, it is contained in a connected component of the join $\bigvee_{j \in J} F_{S_j}$. Since ranks add over connected components, the minimality of J implies that $\bigvee_{j \in J} F_{S_j}$ is a connected flat of rank less than $|J|$ in $D_3(U_{n,n})$. But the smallest connected flat containing F_{S_j} for all $j \in J$ is the flat $F_{\bigcup_{j \in J} S_j}$, which has rank $|\bigcup_{j \in J} S_j| - 3 \geq |J|$ because of the Cerberus condition. \square

4.2. Positivity for cross-ratio degrees.

Proof of Theorem A(b). By Proposition 4.1 and the monotonicity property Proposition 1.2(3), it suffices to show the theorem in the case of cross-ratio degrees, which is Proposition 4.3 established below. \square

Proposition 4.3. Let $S_1, \dots, S_{n-3} \subseteq [n]$, each with $|S_j| = 4$, satisfying the Cerberus condition. Then, we have

$$\int_{\overline{M}_{0,n}} X_{S_1} \cdots X_{S_{n-3}} > 0.$$

Note that $\int_{\overline{M}_{0,n}} X_{S_1} \cdots X_{S_{n-3}}$ is the degree of the map $\overline{M}_{0,n} \rightarrow (\mathbb{P}^1)^{n-3}$ induced by taking the cross-ratio of the points marked by S_1, \dots, S_{n-3} , which is positive exactly when the map is dominant. In other words, denoting by $\text{cr}_{S_i} \in K(\overline{M}_{0,n})$ the rational function corresponding to the cross-ratio of the points marked by S_i , we have

$$\int_{\overline{M}_{0,n}} X_{S_1} \cdots X_{S_{n-3}} > 0 \quad \text{if and only if} \quad \text{cr}_{S_1}, \dots, \text{cr}_{S_{n-3}} \text{ are algebraically independent.}$$

In order to check this, it suffices to show that

$$d \text{cr}_{S_1}, \dots, d \text{cr}_{S_{n-3}} \text{ are linearly independent in } \Omega_{K(\overline{M}_{0,n})/K},$$

where $\Omega_{K(\overline{M}_{0,n})/K}$ is the module of differentials of the function field $K(\overline{M}_{0,n})$ over the ground field K . The quotient map $(\mathbb{P}^1)^n \dashrightarrow \overline{M}_{0,n}$ induces an injection

$$\Omega_{K(\overline{M}_{0,n})/K} \otimes_K K(x_1, \dots, x_n) \rightarrow \Omega_{K(x_1, \dots, x_n)/K} = \bigoplus dx_i \cdot K(x_1, \dots, x_n).$$

In order to prove Proposition 4.3, it therefore suffices to prove the following result.

Proposition 4.4. Let $S_1, \dots, S_{n-3} \subset [n]$ with each S_j of size 4. If S_1, \dots, S_{n-3} satisfies the Cerberus condition, then $d \text{cr}_{S_1}, \dots, d \text{cr}_{S_{n-3}}$ are linearly independent in $\Omega_{K(x_1, \dots, x_n)/K}$.

To prove Proposition 4.4, we use a result from the theory of error correcting codes called the GM-MDS theorem.² Although the theorem has several variants, we present the result in the manner which is most useful for us.

Theorem 4.5 (GM-MDS Theorem, [DSY14], [YH19, Theorem 1 and Eq. (2)], [Lov21]). Let $n \geq m \geq 1$ be positive integers, and let $T_1, \dots, T_m \subset [n]$ be sets such that for all $I \subseteq [m]$ nonempty, we have that $|\bigcup_{i \in I} T_i| \geq n - m + |I|$. Let x_1, \dots, x_n be indeterminates, and let M be the $m \times n$ matrix with $M_{ij} = x_j^{i-1}$. Then there is $G \in \text{GL}_m$ such that $(GM)_{ij} = 0$ unless $j \in T_i$.

We also note the following observation of [DSY14], which gives additional structure for G .

Proposition 4.6 ([DSY14, Eq.(18-21)]). In the context of Theorem 4.5, if $|T_i| = n - m + 1$ for all $i \in [m]$, then for all $i, \ell \in [m]$, we can take $G_{i,\ell}$ to be the coefficient of t^ℓ in $\prod_{j \in [n] \setminus T_i} (t - x_j)$. Furthermore, in that case $(GM)_{ij} = \prod_{k \in [n] \setminus T_i} (x_j - x_k)$.

We can now prove Proposition 4.4.

Proof of Proposition 4.4. Recall that the cross-ratio of points x_1, x_2, x_3, x_4 in \mathbb{P}^1 is

$$\text{cr}_{\{1,2,3,4\}} = \frac{(x_1 - x_3)(x_2 - x_4)}{(x_1 - x_4)(x_2 - x_3)}.$$

²For some additional context, a linear code C is a k -dimensional subspace of \mathbb{F}_q^n for some q . A code is maximum distance separable (MDS) if all Plücker coordinates of this subspace are nonzero. A *generator matrix* is a matrix whose rows form a basis for C . A code C has many possible generator matrices. The GM-MDS conjecture of [DSY14] gave a combinatorial description of which entries of a generator matrix can be chosen to be zero for some MDS code C over \mathbb{F}_q . In the course of proving the GM-MDS conjecture, it was found that it is sufficient for C to be a particular kind of code known as a Reed–Solomon code. This version of the result is how we formulate Theorem 4.5.

Using the quotient rule and the product rule, we compute

$$d \operatorname{cr}_{\{1,2,3,4\}} = \left[(x_2 - x_3)(x_2 - x_4)(x_3 - x_4)dx_1 - (x_1 - x_3)(x_1 - x_4)(x_3 - x_4)dx_2 \right. \\ \left. + (x_1 - x_2)(x_1 - x_4)(x_2 - x_4)dx_3 - (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)dx_4 \right] (x_1 - x_4)^{-2}(x_2 - x_3)^{-2}.$$

It suffices to show that the $(n-3) \times n$ matrix whose rows are given by the $d \operatorname{cr}_{S_i}$ has full rank; we may check this after scaling each row to clear denominators. By the computation above, if $S_j = \{j_1, j_2, j_3, j_4\}$, then i -th row has $(-1)^{\ell+1} \prod_{a < b \in S_i, a, b \neq j_\ell} (x_a - x_b)$ in the j_ℓ -th entry and is zero otherwise.

Let M be the $(n-3) \times n$ matrix whose i -th column is given by $(1, x_i, \dots, x_i^{n-4})$. We apply Theorem 4.5 with $m = n-3$ and each $T_i = S_i$. Because S_1, \dots, S_{n-3} satisfies the Cerberus condition, there is an invertible $(n-3) \times (n-3)$ matrix G such $(GM)_{ij} = 0$ unless $j \in S_i$. By Proposition 4.6, we may pick G such that $(GM)_{ij} = \prod_{k \in [n] \setminus S_i} (x_j - x_k)$.

If we multiply the i -th row of GM by $\prod_{a < b \in S_i} (x_a - x_b)$ and divide the j -th column of GM by $\prod_{k \in [n] \setminus j} (x_j - x_k)$, then the (i, j) -th entry is $\frac{\prod_{a < b \in S_i} (x_a - x_b)}{\prod_{k \in S_i, k \neq j} (x_j - x_k)}$. This is exactly the matrix obtained from taking the differentials of the cross-ratios and clearing denominators. Because G is invertible and M has full-rank, we see that GM has full rank. \square

Remark 4.7. Because the GM-MDS theorem holds over a field of any characteristic, Proposition 4.4 implies that, in any characteristic, the rational functions corresponding to cross-ratios form a separating transcendence basis when the Cerberus condition is satisfied. This implies that the map $\bar{M}_{0,n} \rightarrow (\mathbb{P}^1)^{n-3}$ given by $n-3$ cross-ratios is generically étale when it is dominant.

Remark 4.8. It is tempting to try to prove Theorem A(b) using the non-negative recursion given by Theorem B. The following combinatorial statement is equivalent to Proposition 4.3: Let $S_1, \dots, S_{n-3} \subset [n]$ be sets satisfying the Cerberus condition with each S_j of size 4, and suppose $S_1 = \{1, 2, 3, 4\}$. Then we can partition $[n]$ into two parts P and Q with $\{1, 2\} \subset P$ and $\{3, 4\} \subset Q$ such that

- (1) For all $j > 1$, $|S_j \cap P| \neq 2$.
- (2) Say S_2, \dots, S_k have $|S_j \cap P| > 2$, and S_{k+1}, \dots, S_{n-3} have $|S_j \cap Q| > 2$. Form a set system T_2, \dots, T_k in $P \cup \star$ by setting $T_j = S_j$ if $|S_j \cap P| = 4$, and $T_j = (S_j \cap P) \cup \star$ if $|S_j \cap P| = 3$ for $j = 2, \dots, k$, and similarly form a set system in $Q \cup \star$ using S_{k+1}, \dots, S_{n-3} . Then both of these set systems satisfy the Cerberus condition.

We have been unable to prove this statement directly. In a communication with Hunter Spink we were able to prove it in the case when $1 \in S_j$ for all j , when there is a unique choice of bipartition that works.

Let I be the group of Euclidean isometries of the plane \mathbb{R}^2 . For a graph G with vertex set $[v]$ and edges E , consider the map $\Psi_G: (\mathbb{R}^2)^{[v]}/I \rightarrow \mathbb{R}^E$ given by $[p: [v] \rightarrow \mathbb{R}^2] \mapsto (\|p(a) - p(b)\|^2)_{(a,b) \in E}$. The graph G is said to be *generically rigid* if Ψ_G has finite fibers over general points of $\operatorname{Image}(\Psi_G)$. Note that $\dim \operatorname{Image}(\Psi_G) = 2v - 3$ if G is generically rigid, and thus, if $|E| > 2v - 3$, there is some $E' \subsetneq E$ such that $G' = ([v], E')$ is generically rigid. Hence, a generically rigid graph G is said to be

minimally rigid if Ψ_G is further dominant. By combining Theorem A(b) with [GG20], we obtain a new proof of Laman's theorem [PG27, Lam70], stated below.

Corollary 4.9. A graph $G = ([v], E)$ is minimally rigid if and only if it has $2v - 3$ edges and every induced subgraph on $v' > 1$ vertices has at most $2v' - 3$ edges.

Proof. Let $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ be the 2-dimensional sphere, and consider the map $\tilde{\Psi}_G : S^{[v]}/SO_3(\mathbb{R}) \rightarrow \mathbb{R}^E$ given by $[p: [v] \rightarrow S] \mapsto \left(\frac{1-p(a) \bullet p(b)}{2}\right)_{(a,b) \in E}$, where \bullet denotes the standard dot product, which is essentially the arc length between a and b . A direct argument [Pog73] shows that a graph is minimally rigid (in the plane) if and only if it is *minimally rigid in the sphere*, i.e., the map $\tilde{\Psi}_G$ is generically finite. By dimensional considerations, this happens only if $|E| = 2v - 3$, and in this case we need check when $\tilde{\Psi}_G$ is dominant, which can be done after base changing to \mathbb{C} . [GG20] showed that the degree of the complexified map $\tilde{\Psi}_{G, \mathbb{C}}$ is equal to a cross-ratio degree on $\overline{M}_{0, 2v}$, as described in (iii). Translating the Cerberus condition on subsets of $[2v]$ of the form $\{a, b, a+v, b+v\}$ for $(a, b) \in E$ to a condition on induced subgraphs of G and applying Theorem A(b) yields the desired result. \square

5. 3-TRANSVERSALS

In this section, we prove Theorem C. Our strategy is to apply Theorem B repeatedly along with Lemma 5.2, which involves the following close cousin of the Cerberus condition: For subsets S_1, \dots, S_ℓ of a finite set T , we say that they satisfy the *dragon marriage condition* if

(dragon marriage) $\ell = |T| - 1$ and $|\bigcup_{i \in U} S_i| \geq |U| + 1$ for all nonempty subsets $U \subseteq T$.

Note that subsets S_1, \dots, S_{n-3} of $[n-2]$ satisfy the dragon marriage condition if and only if $S_1 \cup \{n-1, n\}, \dots, S_{n-3} \cup \{n-1, n\}$ satisfies the Cerberus condition.

Proposition 5.1.

- (1) Subsets S_1, \dots, S_ℓ of T satisfy the dragon marriage condition if and only if, for any choice of $i \in T$, there is a bijection $m: [|T| - 1] \rightarrow T \setminus i$ such that $m(j) \in S_j$ for all $j \in [|T| - 1]$.
- (2) For subsets S_1, \dots, S_{n-3} of $[n-2]$, the Kapranov degree

$$\int_{\overline{M}_{0, n}} X_{S_1 \cup \{n-1, n\}, n} \cdots X_{S_{n-3} \cup \{n-1, n\}, n}$$

is 1 if they satisfy the dragon marriage condition and 0 otherwise.

Proof. Statement (1) is [Pos09, Proposition 5.4], which can be deduced from the Hall-Rado theorem (Proposition 4.2). For statement (2), Proposition 1.2(3) implies that the Kapranov degree is at most $\int_{\overline{M}_{0, n}} X_{[n], n}^{n-3} = \int_{\overline{M}_{0, n}} \psi_n^{n-3} = 1$, so Theorem A(b) implies the statement. Alternatively, the statement (2) can be interpreted as [Pos09, Theorem 5.1]; see Remark 5.3. \square

We now proceed to prove Theorem C. Recall that for a pair (T, j) consisting of $T \subseteq [n-3]$ and $j \in \{n-2, n-1, n\}$, we denoted $Y_{T, j} = X_{T \cup \{n-2, n-1, n\}, j}$. Let $\mathcal{T} = \{(T_1, j_1), \dots, (T_{n-3}, j_{n-3})\}$ be a collection of such pairs.

Lemma 5.2. For a partition $[n] = P \sqcup Q$ with $|P| \geq 2$, $n-2 \in P$, and $\{n-1, n\} \subseteq Q$, define a collection $\mathcal{T}|_P$ of subsets of $P \setminus (n-2)$, and a collection $\mathcal{T}|_Q$ of pairs (T, j) where $T \subseteq Q \setminus \{n-1, n\}$ and $j \in \{n-2, n-1, n\}$, by

$$\begin{aligned} \mathcal{T}|_P &= \{T_i \cap P : j_i = n-2 \text{ and } T_i \cap P \neq \emptyset\} \quad \text{and} \\ \mathcal{T}|_Q &= \{(T_i \cap Q, j_i) : j_i = n-2 \text{ and } T_i \cap P = \emptyset\} \cup \{(T_i \cap Q, j_i) : j_i \in \{n-1, n\}\}. \end{aligned}$$

Then, we have

$$\int_{D_{P|Q}} \text{gl}_{P|Q}^* Y_{T_1, j_1} \cdots \text{gl}_{P|Q}^* Y_{T_{n-3}, j_{n-3}} = \int_{\overline{M}_{0, Q \cup \{n-2\}}} \prod_{(T, j) \in \mathcal{T}|_Q} Y_{T, j}$$

if the collection $\mathcal{T}|_P$ of subsets of $P \setminus (n-2)$ satisfies the dragon marriage condition, and 0 otherwise.

Proof. By Lemma 2.2, we have that

$$\text{gl}_{P|Q}^* Y_{T_i, j_i} = \begin{cases} \pi_P^* X_{T_i \cap P \cup \{n-2, \star\}, n-2} & \text{if } j_i = n-2 \text{ and } T_i \cap P \neq \emptyset \\ \pi_Q^* X_{T_i \cup \{n-1, n, \star\}, \star} & \text{if } j_i = n-2 \text{ and } T_i \cap P = \emptyset \\ \pi_Q^* X_{T_i \cap Q \cup \{n-1, n, \star\}, j_i} & \text{otherwise, i.e. } j_i \in \{n-1, n\}, \end{cases}$$

so that the intersection number is the product

$$\left(\int_{\overline{M}_{0, P \cup \star}} \prod_{T \in \mathcal{T}|_P} X_{T \cup \{n-2, \star\}, n-2} \right) \cdot \left(\int_{\overline{M}_{0, Q \cup \{n-2\}}} \prod_{(T, j) \in \mathcal{T}|_Q} Y_{T, j} \right),$$

where for the second factor about $\overline{M}_{0, Q \cup \{n-2\}}$, we relabeled \star by $n-2$. The lemma now follows from Proposition 5.1(2). \square

Proof of Theorem C. First, we note that the theorem holds when $T_i = [n-3]$ for all $i \in [n-3]$. In that case, it is straightforward to verify that the number of 3-transversals is the multinomial $\binom{n-3}{d_{n-2}, d_{n-1}, d_n}$, where d_k denotes the number of $i \in [n-3]$ such that $j_i = k$. On the other hand, by [Wit91] (as stated in (i)), we have $\int_{\overline{M}_{0, n}} \psi_{n-2}^{d_{n-2}} \psi_{n-1}^{d_{n-1}} \psi_n^{d_n} = \binom{n-3}{d_{n-2}, d_{n-1}, d_n}$.

We may assume that $j_1 = n-2$. Inducting on n , we show that

$$\int_{\overline{M}_{0, n}} Y_{T_1 \cup a, n-2} \cdots Y_{T_{n-3}, j_{n-3}} - \int_{\overline{M}_{0, n}} Y_{T_1, n-2} \cdots Y_{T_{n-3}, j_{n-3}} \leq |\{\text{new 3-transversals}\}|,$$

where a 3-transversal t of $(T_1 \cup a, n-2), \dots, (T_{n-3}, j_{n-3})$ is “new” if it is not a 3-transversal of $(T_1, n-2), \dots, (T_{n-3}, j_{n-3})$. This implies the theorem, because if this inequality were ever strict then we would have

$$\int_{\overline{M}_{0, n}} Y_{U_1, n-2} \cdots Y_{U_{n-3}, j_{n-3}} < |\{\text{3-transversals of } (U_1, n-2), \dots, (U_{n-3}, j_{n-3})\}|$$

for all $U_1 \supseteq T_1 \cup a$, $U_2 \supseteq T_2, \dots, U_{n-3} \supseteq T_{n-3}$. This contradicts the fact we have equality when $U_i = [n-3]$ for all i .

Applying Theorem B and then Lemma 5.2 with $\mathcal{T} = \{(T_2, j_2), \dots, (T_{n-3}, j_{n-3})\}$, the induction hypothesis implies that

$$\int_{\overline{M}_{0,n}} Y_{T_1 \cup a, n-2} \cdots Y_{T_{n-3}, j_{n-3}} - \int_{\overline{M}_{0,n}} Y_{T_1, n-2} \cdots Y_{T_{n-3}, j_{n-3}} = \sum_{P|Q} |\{3\text{-transversals of } \mathcal{T}|_Q\}|$$

where the sum is over all partitions $[n] = P \sqcup Q$ with $\{a, n-2\} \subseteq P$ and $T_1 \cup \{n-1, n\} \subseteq Q$ such that the subsets of $P \setminus (n-2)$ in $\mathcal{T}|_P$ satisfy the dragon marriage condition. Given such a partition $[n] = P \sqcup Q$ and a 3-transversal t of $\mathcal{T}|_Q$, extend t to a 3-transversal \tilde{t} of $(T_1 \cup a, n-2), \dots, (T_{n-3}, j_{n-3})$ by setting $\tilde{t}(i) = n-2$ if $i \in P \setminus (n-2)$, and $\tilde{t}(i) = t(i)$ if $i \in Q \setminus \{n-1, n\}$. This is indeed a 3-transversal: if m is a bijection $\{i : (T_i, j_i) \in \mathcal{T}|_Q\} \rightarrow Q \setminus \{n-1, n\}$ that witnesses the 3-transversal t , then we can extend m to $\tilde{m} : [n-3] \rightarrow [n-3]$ by setting $\tilde{m}(1) = a$ and using Proposition 5.1(1) to extend \tilde{m} to the remaining indices $\{i : T_i \in \mathcal{T}|_P\}$.

In order to prove the claimed bound, we need to check that the 3-transversal obtained is new, and that no such new 3-transversal can be obtained from two different partitions $[n] = P \sqcup Q$. For verifying this, we note the following observation.

Observation: Any bijection $m : \{i : (T_i, j_i) \in \mathcal{T}|_Q\} \rightarrow Q \setminus \{n-1, n\}$ witnessing the 3-transversal t of $\mathcal{T}|_Q$ must restrict to a bijection $\{i : j_i = n-2 \text{ and } T_i \cap P = \emptyset\} \rightarrow \{b \in Q \setminus \{n-1, n\} : t(b) = n-2\}$, since the other indices $i' \in \{i : (T_i, j_i) \in \mathcal{T}|_Q\} \setminus \{i : j_i = n-2 \text{ and } T_i \cap P = \emptyset\}$ have $j_{i'} \in \{n-1, n\}$. Here, note that $i \geq 2$ throughout.

Suppose now that \tilde{t} was not a new 3-transversal, which would imply that there is a bijection \tilde{m}' realizing \tilde{t} with $\tilde{m}'(1) \neq a$. Then, we have $\tilde{m}'(1) \in T_1 \subseteq Q \setminus \{n-1, n\}$ and $\tilde{t}(\tilde{m}'(1)) = n-2$. But this contradicts the observation that \tilde{m}' must restrict to a bijection $\{i : j_i = n-2 \text{ and } T_i \cap P = \emptyset\} \rightarrow \{b \in Q \setminus \{n-1, n\} : t(b) = n-2\}$.

Lastly, suppose now that there is some other partition $[n] = P' \sqcup Q'$ which also gives rise to \tilde{t} via the above construction. Then, by the observation, a bijection \tilde{m} realizing \tilde{t} must restrict to give a bijection $\{i : j_i = n-2 \text{ and } T_i \cap P' = \emptyset\} \rightarrow \{b \in Q' \setminus \{n-1, n\} : \tilde{t}(b) = n-2\}$. As $Q' \neq Q$, we thus have $\{i : j_i = n-2 \text{ and } T_i \cap P' = \emptyset\} \neq \{i : j_i = n-2 \text{ and } T_i \cap P = \emptyset\}$. Hence, by switching $(P|Q)$ with $(P'|Q')$ if necessary, we may assume that

$$W = \{i : j_i = n-2, T_i \cap P = \emptyset, \text{ and } T_i \cap P' \neq \emptyset\}$$

is nonempty. A bijection \tilde{m} realizing \tilde{t} then must restrict to a bijection

$$W \rightarrow \{b \in Q \setminus \{n-1, n\} : \tilde{t}(b) = n-2 \text{ and } b \in P'\} = \bigcup_{i \in W} T_i \cap P'.$$

In particular, we have $|\bigcup_{i \in W} T_i \cap P'| = |W|$, which contradicts that the subsets of $P' \setminus (n-2)$ in $\mathcal{T}|_{P'}$ must satisfy the dragon marriage condition. \square

Remark 5.3. For a subset $T \subseteq [n-2]$, let Δ_T be the simplex $\text{conv}\{e_i : i \in T\} \subset \mathbb{R}^{n-2}$, and let $\nabla_T = \text{conv}\{-e_i : i \in T\}$. Denote $\tilde{T} = T \cup \{n-1, n\}$. When $\{n-1, n\} \subset S_j$ and $i_j \in \{n-1, n\}$ for all j , the Kapranov degree is the mixed volume of these simplices, i.e.,

$$\int_{\overline{M}_{0,n}} X_{\tilde{T}_1, n-1} \cdots X_{\tilde{T}_k, n-1} X_{\tilde{T}_{k+1}, n} \cdots X_{\tilde{T}_{n-3}, n} = \text{mixed volume of } (\Delta_{T_1}, \dots, \Delta_{T_k}, \nabla_{T_{k+1}}, \dots, \nabla_{T_{n-3}}).$$

To see this, let LM_n be the Losev–Manin space [LM00], i.e., the moduli space of weighted stable rational curves [Has03] with n points marked by $[n]$ with weights $(\epsilon, \dots, \epsilon, 1, 1)$ for small $\epsilon > 0$. It is isomorphic to the permutohedral toric variety; under the standard correspondence between polytopes and base-point-free divisors on toric varieties, the simplices Δ_T and ∇_T correspond to divisors on LM_n which pullback to $X_{\bar{T}, n-1}$ and $X_{\bar{T}, n}$, respectively, under the birational map $\bar{M}_{0,n} \rightarrow LM_n$. See [DR22] for details.

When $k = 0$ or $n - 3$, these mixed volumes were computed in [Pos09, Theorem 5.1], and when $n - 2 \in T_j$ for all j , they were computed in [EFLS22, Theorem B], which is generalized by Theorem C. It can be deduced from [HK12, BES23] that when all of the simplices are of the form Δ_T for some T or $\nabla_{[n-2]}$, then this mixed volume can be computed in terms of the characteristic polynomial of a certain transversal matroid. It then follows from [CPV95] that computing Kapranov degrees is #P-hard.

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