K-THEORETIC POSITIVITY FOR MATROIDS

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ABSTRACT. Hilbert polynomials have positivity properties under favorable conditions. We establish a similar "K-theoretic positivity" for matroids. As an application, for a multiplicity-free subvariety of a product of projective spaces such that the projection onto one of the factors has birational image, we show that a transformation of its K-polynomial is Lorentzian. This partially answers a conjecture of Castillo, Cid-Ruiz, Mohammadi, and Montaño. As another application, we show that the h^*-vector of a simplicially positive divisor on a matroid is a Macaulay vector, affirmatively answering a question of Speyer for a new infinite family of matroids.

1. INTRODUCTION

For a d-dimensional lattice polytope Q, Stanley [Sta80] showed that the h^*-vector $(h^*_0(Q), \dots, h^*_d(Q))$ defined by

$$\sum_{k\geq 0} |\{\text{lattice points in } kQ\}| t^k = \frac{h^*_0(Q) + h^*_1(Q)t + \cdots + h^*_d(Q)t^d}{(1-t)^{d+1}}$$

is nonnegative, and it is further a Macaulay vector (Definition 4.2) if, for every k, all lattice points in kQ are sums of lattice points in Q. Via standard results in toric geometry [CLS11, Chapter 9], this result can be formulated geometrically as a "K-theoretic positivity" in the following way.

Let X be a smooth projective toric variety with fan Σ, and let $\chi\colon K(X) \to \mathbb{Z}$ be the sheaf Euler characteristic map on the Grothendieck ring $K(X)$ of vector bundles sheaves on X. For a nef line bundle $\mathcal{L}$ associated to a lattice polytope Q whose normal fan coarsens Σ, toric vanishing theorems imply that $\chi(X, \mathcal{L}^{\otimes k}) = \dim H^0(X, \mathcal{L}^{\otimes k}) = |\{\text{lattice points in } kQ\}|$ (for $k \geq 0$), and that the graded ring $R^\bullet_{\mathcal{L}} := \bigoplus_{k\geq 0} H^0(X, \mathcal{L}^{\otimes k})$ is Cohen–Macaulay. See Proposition 4.3 for a detailed review. Quotienting $R^\bullet_{\mathcal{L}}$ by a linear system of parameters, the vector $(h^*_0(\mathcal{L}), \dots, h^*_d(\mathcal{L}))$ defined by

$$\sum_{k\geq 0} \chi(X, \mathcal{L}^{\otimes k}) t^k = \text{Hilbert series of } R^\bullet_{\mathcal{L}} = \frac{h^*_0(\mathcal{L}) + h^*_1(\mathcal{L})t + \cdots + h^*_d(\mathcal{L})t^d}{(1-t)^{\dim Q+1}}$$

is the Hilbert function of a graded artinian ring. In particular, the vector $(h^*_0(\mathcal{L}), \dots, h^*_d(\mathcal{L}))$ is nonnegative, and it is further a Macaulay vector if $R^\bullet_{\mathcal{L}}$ is generated in degree 1.

Here, we establish a similar positivity property for matroids. We begin in the more general setting of polymatroids. For a nonnegative integer m, let $[m] = \{1, \dots, m\}$, and let $\mathbf{a} = (a_1, \dots, a_m)$ be a sequence of nonnegative integers.

Definition 1.1. A *polymatroid* P on $[m]$ with *cage* $\mathbf{a}$ is a function $\mathrm{rk}_{\mathrm{P}}\colon 2^{[m]} \to \mathbb{Z}_{\geq 0}$ satisfying

(1) (Submodularity) $\mathrm{rk}_{\mathrm{P}}(I_1) + \mathrm{rk}_{\mathrm{P}}(I_2) \geq \mathrm{rk}_{\mathrm{P}}(I_1 \cap I_2) + \mathrm{rk}_{\mathrm{P}}(I_1 \cup I_2)$ for any $I_1, I_2 \subseteq [m]$,
(2) (Monotonicity) $\mathrm{rk}_{\mathrm{P}}(I_1) \leq \mathrm{rk}_{\mathrm{P}}(I_2)$ for any $I_1 \subseteq I_2 \subseteq [m]$,
(3) (Normalization) $\mathrm{rk}_{\mathrm{P}}(\emptyset) = 0$, and
(4) (Cage) $\mathrm{rk}_{\mathrm{P}}(i) \leq a_i$ for any $i \in [m]$.

We say that rk_{P} is the *rank function* of the polymatroid P, and that P has *rank* $r = \mathrm{rk}_{\mathrm{P}}([m])$.

A *matroid* is a polymatroid with cage $(1,\dots,1)$. See [Wel76] for the fundamentals of matroid theory. In [LLPP], analogues of K-rings for matroids were introduced, modeled after the following geometry of realizable matroids. Let $\Bbbk$ be a field. A *realization* of a matroid M on a finite set E is a linear subspace $L \subseteq \Bbbk^E$ such that $\mathrm{rk}_{\mathrm{M}}(S) = \dim\big(\text{image of } L \text{ under the projection } \Bbbk^E \to \Bbbk^S\big)$ for all $S \subseteq E$. A realization $L \subseteq \Bbbk^E$ defines a smooth projective irreducible variety W_L called the *augmented wonderful variety* [BHM+22], defined by

$$W_L = \text{the closure of the image of } L \text{ in } \prod_{\emptyset \subsetneq S \subseteq E} \mathbb{P}(\Bbbk^S \oplus \Bbbk),$$

where the map $L \to \mathbb{P}(\Bbbk^S \oplus \Bbbk)$ is the composition of the projection $L \to \Bbbk^S$ with the projective completion $\Bbbk^S \hookrightarrow \mathbb{P}(\Bbbk^S \oplus \Bbbk)$. For $\emptyset \subsetneq S \subseteq E$, let $\mathcal{L}_S$ be the line bundle on W_L obtained by pulling back $\mathcal{O}(1)$ from $\mathbb{P}(\Bbbk^S \oplus \Bbbk)$. These line bundles $\{\mathcal{L}_S\}_{\emptyset \subsetneq S \subseteq E}$ generate the Picard group of W_L, and their K-classes $\{[\mathcal{L}_S]\}_{\emptyset \subsetneq S \subseteq E}$ generate the Grothendieck ring of vector bundles $K(W_L)$ as a ring [LLPP, Corollary 1.9].

For an arbitrary (not necessarily realizable) matroid M, the authors of [LLPP] introduced the *augmented K-ring* $K(\mathrm{M})$ of M. The following are its key properties:

(i) It is equipped with an "Euler characteristic map" $\chi(\mathrm{M}, -)\colon K(\mathrm{M}) \to \mathbb{Z}$.
(ii) Each nonempty subset $S \subseteq E$ defines an element $[\mathcal{L}_S] \in K(\mathrm{M})$ such that $\{[\mathcal{L}_S]\}_{\emptyset \subsetneq S \subseteq E}$ generates $K(\mathrm{M})$ as a ring. A *line bundle* in $K(\mathrm{M})$ is a Laurent monomial in the $[\mathcal{L}_S]$.
(iii) When M has a realization $L \subseteq \Bbbk^E$, identifying the $[\mathcal{L}_S]$ in $K(\mathrm{M})$ and $K(W_L)$ gives an isomorphism $K(\mathrm{M}) \simeq K(W_L)$ such that $\chi(\mathrm{M}, -) = \chi(W_L, -)$.

See Section 2.2 for the definition of $K(\mathrm{M})$ and further properties of of $K(\mathrm{M})$ and $\chi(\mathrm{M}, -)$.

To state our main theorem about $K(\mathrm{M})$, we prepare with the following constructions:

- For a matroid M on a finite set E and subsets $S_1, \dots, S_m \subseteq E$, the function $\mathrm{rk}\colon 2^{[m]} \to \mathbb{Z}$ defined by $\mathrm{rk}(I) = \mathrm{rk}_{\mathrm{M}}(\bigcup_{i \in I} S_i)$ is a polymatroid, which we call the *restriction polymatroid of* M *to* $S_1, \dots, S_m$. Every polymatroid is a restriction polymatroid of a matroid; see Definition 2.3.
- For a polymatroid P with cage $(a_1, \dots, a_m)$, define a subvariety $Y_{\mathrm{P}} \subseteq \mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_m}$ as follows. For $i \in [m]$ and an integer $0 \le j \le a_i$, let $L_i(j)$ be the j-dimensional linear subvariety $\{[x_0, \dots, x_{a_i}] \in \mathbb{P}^{a_i} : x_k = 0 \text{ if } k > j\}$ of $\mathbb{P}^{a_i}$. We define

$$Y_{\mathrm{P}} = \bigcup_{(b_1,\dots,b_m) \in B(\mathrm{P})} L_1(b_1) \times \cdots \times L_m(b_m)$$

  where the union runs over all lattice points $(b_1, \dots, b_m) \in \mathbb{Z}^m$ in the *base polytope* of P defined as $B(\mathrm{P}) = \{(x_1, \dots, x_m) \in \mathbb{R}^m_{\ge 0} : \sum_{i \in [m]} x_i = \mathrm{rk}_{\mathrm{P}}([m]) \text{ and } \sum_{i \in I} x_i \le \mathrm{rk}_{\mathrm{P}}(I) \text{ for all } I \subseteq [m]\}$. Note that the variety Y_{P} and the restrictions to Y_{P} of the line bundles $\mathcal{O}(k_1, \dots, k_m)$ on $\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_m}$ does not depend on the choice of the cage $\mathbf{a}$.

Theorem 1.2. For a polymatroid P and a matroid M on E with subsets $S_1, \dots, S_m \subseteq E$ such that the restriction polymatroid is P, one has

$$\chi(\mathrm{M}, \mathcal{L}_{S_1}^{\otimes k_1} \otimes \cdots \otimes \mathcal{L}_{S_m}^{\otimes k_m}) = \chi(Y_{\mathrm{P}}, \mathcal{O}(k_1, \dots, k_m)) \quad \text{for all } k_1, \dots, k_m.$$

The theorem originates from the following geometry. Let $X \subseteq \mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_m}$ be an irreducible *multiplicity-free subvariety* (i.e., the coefficients of its multidegree are 0 or 1). By [BH20, Corollary 4.7], the function $\mathrm{rk}_{\mathrm{P}}\colon 2^{[m]} \to \mathbb{Z}$ defined by $\mathrm{rk}_{\mathrm{P}}(I) = \dim\big(\text{image of } X \text{ under the projection to } \prod_{i\in I}\mathbb{P}^{a_i}\big)$ is a polymatroid P. Brion [Bri03] showed that any such X has a flat degeneration to Y_{P}.

For example, if a matroid M has a realization $L \subseteq \Bbbk^E$, and if the dimension of the projection of W_L to $\mathbb{P}(\Bbbk^{S_1} \oplus \Bbbk) \times \cdots \times \mathbb{P}(\Bbbk^{S_m} \oplus \Bbbk)$ is equal to the dimension of L, then the projection is an irreducible multiplicity-free subvariety X whose polymatroid P is the restriction polymatroid of M to $S_1, \ldots, S_m$. Thus, if further the projection is an isomorphism $W_L \simeq X$, Brion's flat degeneration implies Theorem 1.2 in this special case. We prove Theorem 1.2 in general by using properties of *polymatroid valuativity* [DF10, EL] (Definition 2.1) and the fact that multiplicity-free subvarieties have rational singularities in characteristic 0 [BF22, Theorem 4.3].

On the other hand, combining Brion's flat degeneration with Theorem 1.2 implies the following.

Corollary 1.3. Let P, M, and $(S_1, \ldots, S_m)$ be as above. If $X \subseteq \mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_m}$ is any irreducible multiplicity-free subvariety with $\mathrm{rk}_{\mathrm{P}}(I) = \dim\big(\mathrm{Image}(X \to \prod_{i\in I}\mathbb{P}^{a_i})\big)$ for all $I \subseteq [m]$, then one has

$$\chi(\mathrm{M}, \mathcal{L}_{S_1}^{\otimes k_1} \otimes \cdots \otimes \mathcal{L}_{S_m}^{\otimes k_m}) = \chi(X, \mathcal{O}(k_1, \ldots, k_m))$$

for all line bundles $\mathcal{O}(k_1, \ldots, k_m)$ on $\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_m}$.

As an application, we use Corollary 1.3 to study Snapper polynomials of multiplicity-free subvarieties via matroid theory. For a projective variety X and line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_m$ on X, the function assigning to each tuple of integers $(k_1, \ldots, k_m)$ the Euler characteristic $\chi(X, \mathcal{L}_1^{\otimes k_1} \otimes \cdots \otimes \mathcal{L}_m^{\otimes k_m})$ is a polynomial [Sna59], often called the *Snapper polynomial*. This property also holds for $\chi(\mathrm{M}, -)$ to give Snapper polynomials of matroids, for which we establish the following.

For a sequence $\mathbf{k} = (k_1, \ldots, k_m)$ of nonnegative integers, set $|\mathbf{k}| = \sum_i k_i$, and denote $\mathbf{t}^{\mathbf{k}} = t_1^{k_1} \cdots t_m^{k_m}$ and $\mathbf{t}^{[\mathbf{k}]} = \binom{t_1+k_1}{k_1} \cdots \binom{t_m+k_m}{k_m}$, where $\binom{t+k}{k} = \frac{t(t-1)\cdots(t-k+1)}{k!}$.

Theorem 1.4. For a matroid M on E and subsets $S_1, \ldots, S_m$ of E, whose restriction polymatroid has rank r, define a polynomial $H(t_1, \ldots, t_m)$ by

$$H(t_1, \ldots, t_m) = \sum_{\mathbf{k}} a_{\mathbf{k}} \mathbf{t}^{\mathbf{k}} \quad \text{such that} \quad \chi\Big(\mathrm{M}, \mathcal{L}_{S_1}^{\otimes t_1} \otimes \cdots \otimes \mathcal{L}_{S_m}^{\otimes t_m}\Big) = \sum_{\mathbf{k}} (-1)^{r-|\mathbf{k}|} a_{\mathbf{k}} \mathbf{t}^{[\mathbf{k}]}.$$

Suppose at least one of $S_1, \ldots, S_m$ satisfies $\mathrm{rk}_{\mathrm{M}}(S_i) = r$. Then, the homogenization $\widetilde{H}(\mathbf{t}, t_0) = \sum_{\mathbf{k}} a_{\mathbf{k}} \mathbf{t}^{\mathbf{k}} t_0^{r-|\mathbf{k}|}$ by an auxiliary variable t_0 is *denormalized Lorentzian* in the sense of [BH20].

When combined with Corollary 1.3, Theorem 1.4 answers [CCRMM, Conjecture 7.18 and Question 7.21] about the "twisted K-polynomial" of a multiplicity-free subvariety $X \subseteq \mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_m}$ when $X \to \mathbb{P}^{a_i}$ is birational for some i. See Section 3.

As another application, we deduce properties of matroids by using geometric properties of Y_{P}, namely, that Y_{P} is Cohen–Macaulay [CCRC23] and is a compatibly Frobenius split subvariety in the product of projective spaces [BK05]. To connect to previous questions in matroid theory, it is convenient to phrase our statements in terms of the *non-augmented K-ring* $\underline{K}(\mathrm{M})$ of a loopless matroid M on E (see Section 2.4), for which an analogue of Theorem 1.2 holds (Corollary 2.15). Like $K(\mathrm{M})$, the ring $\underline{K}(\mathrm{M})$ is equipped with a map $\underline{\chi}(\mathrm{M}, -)\colon \underline{K}(\mathrm{M}) \to \mathbb{Z}$, and each nonempty subset $S \subseteq E$ defines an element $[\underline{\mathcal{L}}_S] \in \underline{K}(\mathrm{M})$ such that $\{[\underline{\mathcal{L}}_S]\}$ generates $\underline{K}(\mathrm{M})$. When M has a realization $L \subseteq \Bbbk^E$, these objects again coincide with those of the (non-augmented) *wonderful variety* $\underline{W}_L$ of [DCP95].

Theorem 1.5. For a loopless matroid M and a *line bundle* $\mathcal{L}$ *in* $\underline{K}(\mathrm{M})$ (i.e., a Laurent monomial in the $\underline{\mathcal{L}}_S$), define the h^**-vector* $(h^*_0(\mathrm{M}, \mathcal{L}), \dots, h^*_d(\mathrm{M}, \mathcal{L}))$ to be the coefficients of the polynomial

$$h^*(\mathrm{M}, \mathcal{L}; t) = \sum_{k=0}^{d} h^*_0(\mathrm{M}, \mathcal{L}) t^k \quad \text{such that} \quad \sum_{k \geq 0} \underline{\chi}(\mathrm{M}, \mathcal{L}^{\otimes k}) t^k = \frac{h^*(\mathrm{M}, \mathcal{L}; t)}{(1-t)^{d+1}}$$

where $d =$ degree of the polynomial $\underline{\chi}(\mathrm{M}, \mathcal{L}^{\otimes t})$. If $\mathcal{L}$ is *simplicially positive* (i.e., $\mathcal{L} = \bigotimes_S \underline{\mathcal{L}}_S^{\otimes k_S}$ for some nonnegative integers k_S), then the h^*-vector $(h^*_0(\mathrm{M}, \mathcal{L}), \dots, h^*_d(\mathrm{M}, \mathcal{L}))$ is a Macaulay vector and is in particular nonnegative.

One verifies that the h^*-vector is equivalently defined by the equation

$$\underline{\chi}(\mathrm{M}, \mathcal{L}^{\otimes t}) = \sum_{k=0}^{d} h^*_k(\mathrm{M}, \mathcal{L}) \binom{t+d-k}{d},$$

from which one sees that $(-1)^d \underline{\chi}(\mathrm{M}, \mathcal{L}^{-1}) = h^*_d(\mathrm{M}, \mathcal{L})$. We apply Theorem 1.5 in this form to answer affirmatively a question of Speyer [Spe09] for new infinite families of matroids using a result of Fink, Shaw, and Speyer; see Section 5.1.

When M has a realization $L \subseteq \Bbbk^E$, the simplicially positive line bundles form a full dimensional subcone of the nef cone of $\underline{W}_L$, but it is usually strictly smaller than the nef cone. In Section 4.3, we conjecture that the conclusion of Theorem 1.5 holds for a larger family of line bundles. This would answer Speyer's question affirmatively for all matroids. We also establish and conjecture some other properties of h^*-vectors of matroids.

Organization. In Section 2, we recall properties of polymatroids and (augmented) K-rings of matroids, and we use them to prove Theorem 1.2. In Section 3, we prove Theorem 1.4 and discuss its consequences. In Section 4, we prove Theorem 1.5. In Section 5, we discuss some applications and some further properties.

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2. THE COMPARISON THEOREM

We give background on polymatroids in Section 2.1, and we collect properties of the augmented K-ring of a matroid in Section 2.2. Then, in Section 2.3, we prove Theorem 1.2 comparing the Euler characteristic maps χ on $K(\mathrm{M})$ and Y_{P}. Analogues for the non-augmented K-ring of a matroid are given in Section 2.4.

2.1. **Polymatroids.** We review realizability, valuativity, and lifts for polymatroids. We begin with realizations. Let P be a polymatroid with cage $(a_1, \ldots, a_m)$. A *realization* of P over $\Bbbk$ is a subspace $L \subseteq V_1 \oplus \cdots \oplus V_m$, where V_i is a vector space over $\Bbbk$ of dimension a_i, such that

$$\mathrm{rk}_P(S) = \dim \left(\text{the image of } L \text{ under the projection to } \bigoplus_{i \in S} V_i \right)$$

for all $S \subseteq [m]$. When such an L exists, we say P is *realizable* over $\Bbbk$. When P is a matroid (i.e., a polymatroid of cage $(1, \ldots, 1)$), this specializes to realizability of matroids as discussed in the introduction.

We will obtain Theorem 1.2 by reducing to the realizable case. This reduction step will be facilitated by the notion of valuativity [AFR10, DF10].

Definition 2.1. For a polymatroid P on $[m]$, let $\mathbf{1}_{\mathrm{P}} \colon \mathbb{R}^m \to \mathbb{Z}$ be the indicator function of its base polytope $B(\mathrm{P})$. The *valuative group* of polymatroids with cage $\mathbf{a} = (a_1, \ldots, a_m)$, denoted $\mathrm{Val}_{\mathbf{a}}$, is the subgroup of $\mathbb{Z}^{(\mathbb{R}^m)}$ generated by $\mathbf{1}_{\mathrm{P}}$ for P a polymatroid on $[m]$ with cage $\mathbf{a}$.

A function from the set of polymatroids with cage $\mathbf{a}$ to an abelian group is said to be *valuative* if it factors through $\mathrm{Val}_{\mathbf{a}}$.

By [DF10] or [EL, Remark 3.16], $\mathrm{Val}_{\mathbf{a}}$ is generated by polymatroids which are realizable over $\mathbb{C}$. In particular, this gives the following useful result.

Corollary 2.2. Let f_1 and f_2 be functions from the set of polymatroids with cage $\mathbf{a}$ to an abelian group G. If f_1 and f_2 are valuative, and if $f_1(\mathrm{P}) = f_2(\mathrm{P})$ for any polymatroid P with cage $\mathbf{a}$ that is realizable over $\mathbb{C}$, then $f_1(\mathrm{P}) = f_2(\mathrm{P})$ for all polymatroids P with cage $\mathbf{a}$.

Lastly, we recall multisymmetric lifts of polymatroids, a construction which has appeared many times in the literature [Hel72, McD75, Lov77, Ngu86, BCF] with many different names. We use the terminology and description given in [CHL$^+$, EL].

Definition 2.3. Let P be a polymatroid with cage $\mathbf{a} = (a_1, \ldots, a_m)$ on $[m]$. The *multisymmetric lift* of P is a matroid M on a ground set E of size $a_1 + \cdots + a_m$ which is equipped with a distinguished partition $E = S_1 \sqcup \cdots \sqcup S_m$ into parts of size $a_1, \ldots, a_m$ with the following characterizing property: rk_{M} is preserved by the action of the product of symmetric groups $\mathfrak{S}_{S_1} \times \cdots \times \mathfrak{S}_{S_m}$, and

$$\mathrm{rk}_{\mathrm{P}}(I) = \mathrm{rk}_{\mathrm{M}} \left(\cup_{i \in I} S_i \right) \text{ for all } I \subseteq [m].$$

Note that the multisymmetric lift depends on the choice of cage $\mathbf{a}$, and that the restriction polymatroid of the multisymmetric lift M to the subsets $S_1, \ldots, S_m$ appearing in the distinguished partition is the polymatroid P.

The construction of the multisymmetric lift respects realizability. When P is realized by $L \subseteq \bigoplus_{i \in [m]} V_i$ over an infinite field $\Bbbk$, the multisymmetric lift of P can be realized by generically choosing a basis for each V_i to identify $\bigoplus_{i \in [m]} V_i$ with $\Bbbk^{a_1 + \cdots + a_m}$.

2.2. **Augmented K-rings of matroids.** Let M be a matroid on a ground set E.

Definition 2.4. The *augmented K-ring* $K(\mathrm{M})$ of M is the Grothendieck ring of vector bundles on the toric variety $X_{\Sigma_{\mathrm{M}}}$ of the *augmented Bergman fan* Σ_{M} of M.

The definition of the augmented Bergman fan can be found in [BHM+22], but won't be needed. Let us record the properties of $K(\mathrm{M})$ that we will need here.

Proposition 2.5. The augmented K-ring $K(\mathrm{M})$ of M satisfies the following properties.

(i) It is equipped with an "Euler characteristic map" $\chi(\mathrm{M}, -)\colon K(\mathrm{M}) \to \mathbb{Z}$.
(ii) Each nonempty subset $S \subseteq E$ defines an element $[\mathcal{L}_S] \in K(\mathrm{M})$ such that $\{[\mathcal{L}_S]\}_{\emptyset \subsetneq S \subseteq E}$ generates $K(\mathrm{M})$ as a ring. A *line bundle* in $K(\mathrm{M})$ is a Laurent monomial in the $[\mathcal{L}_S]$.
(iii) When M has a realization $L \subseteq \Bbbk^E$, identifying the $[\mathcal{L}_S]$ in $K(\mathrm{M})$ and $K(W_L)$ gives an isomorphism $K(\mathrm{M}) \simeq K(W_L)$ such that $\chi(\mathrm{M}, -) = \chi(W_L, -)$.

Proof. These statements follow from [LLPP, Corollary 1.9 and Proposition 1.13]. For (ii), the original statement in [LLPP] is in terms of the $\mathcal{L}_F$ for F a nonempty flat of M, but we set $\mathcal{L}_S = \mathcal{L}_{\mathrm{cl}_{\mathrm{M}}(S)}$ where cl_{M} denotes the closure operator of the matroid M. □

We caution that the map $\chi(\mathrm{M}, -)$ is generally different from the sheaf Euler characteristic map $\chi(X_{\Sigma_{\mathrm{M}}}, -)$ of the toric variety $X_{\Sigma_{\mathrm{M}}}$. Next, we review a formula for $\chi(\mathrm{M}, -)$ given in [LLPP], stated in terms of the following definition.

Definition 2.6. We say that a sequence $(S_1, \dots, S_m)$ of nonempty subsets of E satisfies the *Hall–Rado* condition (with respect to M) if

$$\mathrm{rk}_{\mathrm{M}}\left(\bigcup_{i \in I} S_i\right) \geq |I| \quad \text{for every } I \subseteq [m].$$

Moreover, we say that $\mathbf{k} = (k_1, \dots, k_m) \in \mathbb{Z}^m_{\geq 0}$ satisfies the *Hall–Rado* condition if the sequence $(S_1^{k_1}, \dots, S_m^{k_m})$, where $S_i^{k_i}$ denotes S_i repeated k_i times, satisfies the condition, or, equivalently if

$$\mathrm{rk}_{\mathrm{M}}\left(\bigcup_{i \in I} S_i\right) \geq \sum_{i \in I} k_i \quad \text{for every } I \subseteq [m].$$

If P denotes the restriction polymatroid of M to $S_1, \dots, S_m$, note then that $\mathbf{k}$ satisfies the Hall–Rado condition if and only if it is a lattice point in the *independence polytope of* P, defined as

$$I(\mathrm{P}) = \{(x_1, \dots, x_m) \in \mathbb{R}^m_{\geq 0} : \sum_{i \in I} x_i \leq \mathrm{rk}_{\mathrm{P}}(I) \text{ for all } I \subseteq [m]\}.$$

The independence polytope $I(\mathrm{P})$ relates to the base polytope $B(\mathrm{P})$ by

$$I(\mathrm{P}) = \{\mathbf{x} \in \mathbb{R}^m_{\geq 0} : \mathbf{y} - \mathbf{x} \in \mathbb{R}^m_{\geq 0} \text{ for some } \mathbf{y} \in B(\mathrm{P})\},$$

or, equivalently, $I(\mathrm{P}) = (B(\mathrm{P}) + \mathbb{R}^m_{\leq 0}) \cap \mathbb{R}^m_{\geq 0}$, where the $+$ denotes Minkowski sum.

Proposition 2.7. [LLPP, Corollary 7.5] For a nonnegative integer k, let $t^{(d)}$ denote the polynomial $\binom{t+d-1}{d} = \frac{t(t+1)\cdots(t+d-1)}{d!}$. For a sequence $\mathbf{k} = (k_S)_{S\in\mathcal{S}}$ of nonnegative integers indexed by a collection $\mathcal{S}$ of nonempty subsets of E, denote $\mathbf{t}^{(\mathbf{k})} = \prod t_S^{(k_S)}$. We have that

$$\chi(\mathrm{M}, \bigotimes_S \mathcal{L}_S^{\otimes t_S}) = \sum_{\mathbf{k} \text{ satisfies Hall–Rado}} \mathbf{t}^{(\mathbf{k})}.$$

In particular, if $\mathcal{L}$ is a line bundle which is the tensor product of line bundles of the form $\mathcal{L}_{S_i}$ for some subsets $S_1, \ldots, S_k$ of the ground set of M, then $\chi(\mathrm{M}, \mathcal{L})$ only depends on the restriction polymatroid of $S_1, \ldots, S_k$. We record this observation as the following corollary, which will allow us to reduce the proof of Theorem 1.2 to the case when M is the multisymmetric lift of P.

Corollary 2.8. Let M_1 and M_2 be matroids, let $S_1, \ldots, S_k$ be subsets of the ground set of M_1, and let $T_1, \ldots, T_k$ be subsets of the ground set of M_2. Suppose that the restriction polymatroid of M_1 to $S_1, \ldots, S_k$ is the same as the restriction polymatroid of M_2 to $T_1, \ldots, T_k$. Then, for any $a_1, \ldots, a_k$,

$$\chi(\mathrm{M}_1, \mathcal{L}_{S_1}^{\otimes a_1} \otimes \cdots \otimes \mathcal{L}_{S_k}^{\otimes a_k}) = \chi(\mathrm{M}_2, \mathcal{L}_{T_1}^{\otimes a_1} \otimes \cdots \otimes \mathcal{L}_{T_k}^{\otimes a_k}).$$

Another crucial feature of the Snapper polynomials of matroids is their valuativity, which will allow us to reduce Theorem 1.2 to the case of realizable polymatroids.

Proposition 2.9. Let $\mathbf{a} = (a_1, \ldots, a_m)$ and $\mathbf{b} = (b_1, \ldots, b_m)$ be sequence of integers, with $a_i \geq 0$. For a polymatroid P with cage $\mathbf{a}$, let M be its multisymmetric lift with distinguished partition $S_1 \sqcup \cdots \sqcup S_m$ of its ground set. Then the function which assigns to a polymatroid P with cage $\mathbf{a}$ the quantity $\chi(\mathrm{M}, \mathcal{L}_{S_1}^{\otimes b_1} \otimes \cdots \otimes \mathcal{L}_{S_m}^{\otimes b_m})$ is valuative.

Proof. By [EL, Lemma 3.2], the function that sends a polymatroid of cage $\mathbf{a}$ to the class of its multisymmetric lift in the valuative group of matroids is valuative. By [LLPP, Lemma 6.4], for fixed S_i and b_i, the function that sends a matroid M to $\chi(\mathrm{M}, \mathcal{L}_{S_1}^{\otimes b_1} \otimes \cdots \otimes \mathcal{L}_{S_m}^{\otimes b_m})$ is a valuative invariant of matroids. Putting these together implies the result. □

2.3. **Multiplicity-free subvarieties and the proof of Theorem 1.2.** An integral subvariety X of a product of projective spaces $\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_m}$ is said to be *multiplicity-free* if the intersection number of any monomial in the hyperplane classes of the factors with the fundamental class of X is either 0 or 1. By [BH20, Corollary 4.7], the function $\mathrm{rk}_{\mathrm{P}} \colon 2^{[m]} \to \mathbb{Z}$ defined by $\mathrm{rk}_{\mathrm{P}}(I) = \dim\big(\text{image of } X \text{ under the projection to } \prod_{i\in I} \mathbb{P}$ is a polymatroid P. The K-class of the structure sheaf $[\mathcal{O}_X] \in K(\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_m})$ is then determined by the following theorem.

Proposition 2.10. [Bri03] There is a flat degeneration of X to Y_{P} inside of $\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_m}$. In particular, $[\mathcal{O}_X] = [\mathcal{O}_{Y_{\mathrm{P}}}]$.

The second statement follows from the first because the pairing

$$K(\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_m}) \times K(\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_m}) \to \mathbb{Z} \text{ given by } (a, b) \mapsto \chi(\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_m}, ab)$$

is nondegenerate, and Euler characteristics are locally constant in proper flat families. This implies that the class of a subvariety in $K(\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_m})$ is locally constant in proper flat families.

We now state a formula for $[\mathcal{O}_{Y_{\mathrm{P}}}] \in K(\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_m})$. This formula originates in the work of Knutson, who studied the more general problem of calculating the K-class of a reduced union of Schubert varieties inside a homogeneous space. He showed that one can compute the K-class in terms of Möbius inversion on the poset of Schubert varieties. The special case of products of projective spaces was also proven in [CCRMM, Theorem 7.12]. For each tuple $\mathbf{b} = (b_1, \ldots, b_m)$ with $b_i \leq a_i$, let $Y_{\mathbf{b}}$ be a $\mathbb{P}^{b_1} \times \cdots \times \mathbb{P}^{b_m}$ embedded linearly into $\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_m}$; the class $[\mathcal{O}_{Y_{\mathbf{b}}}]$ does not depend on the choice of an embedding. The classes $\{[\mathcal{O}_{Y_{\mathbf{b}}}]\}$ form a basis for $K(\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_m})$.

Proposition 2.11. [Knu] Write $[\mathcal{O}_{Y_{\mathrm{P}}}] = \sum_{\mathbf{b}} c_{\mathbf{b}} [\mathcal{O}_{Y_{\mathbf{b}}}]$. If $\sum b_i > \mathrm{rk}(\mathrm{P})$, then $c_{\mathbf{b}} = 0$. If $\sum b_i = \mathrm{rk}(\mathrm{P})$, then

$$c_{\mathbf{b}} = \begin{cases} 1 & \text{if } \mathbf{b} \in B(\mathrm{P}) \\ 0 & \text{otherwise.} \end{cases}$$

If $\sum b_i < \mathrm{rk}(\mathrm{P})$, then $c_{\mathbf{b}} = 1 - \sum_{\mathbf{b}' > \mathbf{b}} c_{\mathbf{b}'}$.

Proposition 2.12. The function which assigns a polymatroid P with cage $(a_1, \ldots, a_m)$ to $[\mathcal{O}_{Y_{\mathrm{P}}}] \in K(\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_m})$ is valuative.

Proof. We show that, for every $\mathbf{b} = (b_1, \ldots, b_m)$ with $b_i \leq a_i$, the function assigns a polymatroid P with cage $(a_1, \ldots, a_m)$ to $c_{\mathbf{b}}$ is valuative. This is clear if $\sum b_i \geq \mathrm{rk}(\mathrm{P})$. The recursive formula $c_{\mathbf{b}} = -\sum_{\mathbf{b}' > \mathbf{b}} c_{\mathbf{b}'}$ then implies that it holds in general. □

We first prove Theorem 1.2 in the case when P is realizable over $\mathbb{C}$.

Proposition 2.13. Let $V_1, \ldots, V_m$ be vector spaces over $\mathbb{C}$ of dimension $a_1, \ldots, a_m$, and let $L \subseteq V_1 \oplus \cdots \oplus V_m$ be a realization of a polymatroid P with cage $(a_1, \ldots, a_m)$. Let M be the multisymmetric lift of P, whose ground set is equipped with the distinguished partition $S_1 \sqcup \cdots \sqcup S_m$. Let W_L be the augmented wonderful variety of a realization of M. Then, for any $(b_1, \ldots, b_m)$,

$$\chi(\mathrm{M}, \mathcal{L}_{S_1}^{\otimes b_1} \otimes \cdots \otimes \mathcal{L}_{S_m}^{\otimes b_m}) = \chi(Y_{\mathrm{P}}, \mathcal{O}(b_1, \ldots, b_m)).$$

Proof. Let Y be the image of W_L under the projection p to $\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_m}$, or, equivalently, Y is the closure of L inside $\mathbb{P}(V_1 \oplus \Bbbk) \times \cdots \times \mathbb{P}(V_m \oplus \Bbbk)$. As W_L is also a compactification of L, the map $W_L \to Y$ is birational. By [BF22, Theorem 4.3], which is based on [Bri01, Theorem 5], Y has rational singularities. As $p \colon W_L \to Y$ is a resolution of singularities, we therefore have that $Rp_* \mathcal{O}_{W_L} = \mathcal{O}_Y$. By the projection formula, we have that

$$Rp_*(\mathcal{L}_{S_1}^{\otimes b_1} \otimes \cdots \otimes \mathcal{L}_{S_m}^{\otimes b_m}) = \mathcal{O}(b_1, \ldots, b_m).$$

Because $\chi(\mathrm{M}, -)$ agrees with $\chi(W_L, -)$, we have that

$$\chi(\mathrm{M}, \mathcal{L}_{S_1}^{\otimes b_1} \otimes \cdots \otimes \mathcal{L}_{S_m}^{\otimes b_m}) = \chi(W_L, \mathcal{L}_{S_1}^{\otimes b_1} \otimes \cdots \otimes \mathcal{L}_{S_m}^{\otimes b_m}) = \chi(Y, \mathcal{O}(b_1, \ldots, b_m)).$$

To conclude, we note that Y is an irreducible multiplicity-free subvariety by [Li18] or [EL, Corollary 1.4]. By Proposition 2.10, $[\mathcal{O}_Y] = [\mathcal{O}_{Y_{\mathrm{P}}}] \in K(\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_m})$ as Y is multiplicity-free, which implies the result. □

Proof of Theorem 1.2. Fix $(b_1, \ldots, b_m)$. We may assume M is the multisymmetric lift of P by Corollary 2.8. When P is realizable over $\mathbb{C}$, the statement follows from Proposition 2.13. By Proposition 2.9, the function that assigns a polymatroid P with cage $(a_1, \ldots, a_m)$ to $\chi(\mathrm{M}, \mathcal{L}_{S_1}^{\otimes b_1} \otimes \cdots \otimes \mathcal{L}_{S_m}^{\otimes b_m})$ is valuative, and by Proposition 2.12, the same is true with the function that assigns P to $\chi(Y_{\mathrm{P}}, \mathcal{O}(b_1, \ldots, b_m))$. Corollary 2.2 thus implies the desired equality. □

2.4. **Non-augmented K-rings.** We now discuss the analogue of Theorem 1.2 for (non-augmented) K-rings of matroid. This section is not used until Section 4.3. Let M be a loopless matroid. The (non-augmented) *Bergman fan* $\underline{\Sigma}_{\mathrm{M}}$ of a matroid M is the star fan of a particular ray in the augmented Bergman fan Σ_{M}; see [EHL23, Definition 5.12] for details. In other words, its toric variety $X_{\underline{\Sigma}_{\mathrm{M}}}$ is a toric divisor on $X_{\Sigma_{\mathrm{M}}}$. We define the *(non-augmented) K-ring of* M, denoted $\underline{K}(\mathrm{M})$, to be the K-ring of $X_{\underline{\Sigma}_{\mathrm{M}}}$. As $X_{\underline{\Sigma}_{\mathrm{M}}}$ is a divisor on $X_{\Sigma_{\mathrm{M}}}$, there is a restriction map $K(\mathrm{M}) \to \underline{K}(\mathrm{M})$. The restriction of $[\mathcal{L}_S]$ is denoted $[\underline{\mathcal{L}}_S]$.

The facts about the augmented K-ring (Proposition 2.5) have analogues for the non-augmented K-ring $\underline{K}(\mathrm{M})$ [LLPP]. More precisely, we have:

(i) $\underline{K}(\mathrm{M})$ is equipped with an "Euler characteristic map" $\underline{\chi}(\mathrm{M}, -)\colon \underline{K}(\mathrm{M}) \to \mathbb{Z}$.
(ii) $\underline{K}(\mathrm{M})$ is spanned by the restrictions $[\underline{\mathcal{L}}_S]$ of the classes $[\mathcal{L}_S]$.
(iii) When M has a realization $L \subseteq \Bbbk^E$, let $\underline{W}_L$ be the *wonderful variety* [DCP95] defined as

$$\underline{W}_L = \text{the closure of the image of } \mathbb{P}L \text{ in } \prod_{\emptyset \subsetneq S \subseteq E} \mathbb{P}(\Bbbk^S)$$

where $\mathbb{P}L \to \mathbb{P}(\Bbbk^S)$ is the projectivization of the projection $L \to \Bbbk^S$, and let $\underline{\mathcal{L}}_S$ be the pullback of $\mathcal{O}(1)$ from $\mathbb{P}(\Bbbk^S)$. Then, identifying the $[\underline{\mathcal{L}}_S]$ in $\underline{K}(\mathrm{M})$ and $K(\underline{W}_L)$ gives an isomorphism $\underline{K}(\mathrm{M}) \simeq K(\underline{W}_L)$ such that $\underline{\chi}(\mathrm{M}, -) = \chi(\underline{W}_L, -)$.

We also have a formula for the Euler characteristic map $\underline{\chi}(\mathrm{M}, -)\colon \underline{K}(\mathrm{M}) \to \mathbb{Z}$ analogous to Proposition 2.7. We say that a sequence $(S_1, \ldots, S_m)$ of nonempty subsets of E satisfies the *dragon Hall–Rado* condition (with respect to M) if

$$\mathrm{rk}_{\mathrm{M}}\left(\bigcup_{i \in I} S_i\right) \geq 1 + |I| \quad \text{for every } I \subseteq [m].$$

Moreover, we say that $\mathbf{k} = (k_1, \ldots, k_m) \in \mathbb{Z}_{\geq 0}^m$ satisfies the *dragon Hall–Rado* condition if the sequence $(S_1^{k_1}, \ldots, S_m^{k_m})$, where $S_i^{k_i}$ denotes S_i repeated k_i times, satisfies the condition, or, equivalently if

$$\mathrm{rk}_{\mathrm{M}}\left(\bigcup_{i \in I} S_i\right) \geq 1 + \sum_{i \in I} k_i \quad \text{for every } I \subseteq [m].$$

This defines a polymatroid on $\{S : \emptyset \subsetneq S \subseteq E\}$ whose bases are the $\mathbf{k}$ satisfying the dragon-Hall–Rado condition with $\sum k_S = \mathrm{rk}(\mathrm{M}) - 1$. We call this the *dragon-Hall–Rado polymatroid*. The significance of the dragon-Hall–Rado condition for us comes from the following formula for $\underline{\chi}(\mathrm{M}, -)$.

Proposition 2.14. [LLPP, Collary 7.5] We have that

$$\underline{\chi}(\mathrm{M}, \bigotimes_S \underline{\mathcal{L}}_S^{\otimes t_S}) = \sum_{\mathbf{k} \text{ satisfies dragon-Hall–Rado}} \mathbf{t}^{(\mathbf{k})}.$$

By comparing this with Proposition 2.7 and using Theorem 1.2, we obtain the following non-augmented analogue of the theorem.

Corollary 2.15. Let M be a matroid with subsets $S_1, \dots, S_m$ of the ground set, and let P be the restriction of the dragon-Hall–Rado polymatroid to $S_1, \dots, S_m$. Then, for any line bundle $\mathcal{L}$ which is a tensor product of the $\underline{\mathcal{L}}_{S_i}$, we have $\underline{\chi}(\mathrm{M}, \mathcal{L}) = \chi(Y_{\mathrm{P}}, \mathcal{L})$.

3. Lorentzian property

We briefly summarize Lorentzian polynomials and then prove Theorem 1.4. Then, we explain the application to K-polynomials of multiplicity-free subvarieties.

3.1. **Lorentzian Snapper polynomials.**

Lorentzian polynomials were introduced in [BH20] as a generalization of stable polynomials in optimization theory and volume polynomials in algebraic geometry.

Definition 3.1. A homogeneous polynomial $f = \sum_{\mathbf{k}} c_{\mathbf{k}} \mathbf{t}^{\mathbf{k}} \in \mathbb{R}[t_1, \dots, t_m]$ of degree d with nonnegative coefficients is *Lorentzian* if

(1) the support $\{\mathbf{k} \in \mathbb{Z}^m_{\geq 0} : c_{\mathbf{k}} > 0\}$ of f equals $B(\mathrm{P}) \cap \mathbb{Z}^m$ for some polymatroid P on $[m]$, and
(2) any $(d-2)$-th partial derivative of f is a quadratic form with at most one positive eigenvalue.

The *normalization* $N(f)$ of a polynomial $f \in \mathbb{R}[t_1, \dots, t_m]$ is the polynomial obtained by replacing each term $c_{\mathbf{k}} \mathbf{t}^{\mathbf{k}}$ in f with $c_{\mathbf{k}} \frac{\mathbf{t}^{\mathbf{k}}}{\mathbf{k}!}$ where $\mathbf{k}! = k_1! \cdots k_m!$. We say that f is *denormalized Lorentzian* if $N(f)$ is Lorentzian.

For an irreducible complete variety, the volume polynomial of a collection of nef divisors is Lorentzian [BH20, Theorem 4.6]. We now prove Theorem 1.4, which states that the Snapper polynomial of the line bundles $\{\mathcal{L}_S\}$ on a matroid is also Lorentzian after a minor transformation.

As before, for $k \in \mathbb{Z}_{\geq 0}$, denote $t^{(k)} = \binom{t+k-1}{k}$ and $t^{[k]} = \binom{x+k}{k}$, and for $\mathbf{k} \in \mathbb{Z}^m_{\geq 0}$, denote $\mathbf{t}^{(\mathbf{k})} = t_1^{(k_1)} \cdots t_m^{(k_m)}$ and $\mathbf{t}^{[\mathbf{k}]} = t_1^{[k_m]} \cdots t_m^{[k_m]}$. Let us recall the notations in Theorem 1.4 that $\widetilde{H}(\mathbf{t}, t_0)$ is the homogenization of the polynomial $H(\mathbf{t})$ defined by

$$H(t_1, \dots, t_m) = \sum_{\mathbf{k}} a_{\mathbf{k}} \mathbf{t}^{\mathbf{k}} \quad \text{such that} \quad \chi\Big(\mathrm{M}, \mathcal{L}_{S_1}^{\otimes t_1} \otimes \cdots \otimes \mathcal{L}_{S_m}^{\otimes t_m}\Big) = \sum_{\mathbf{k}} (-1)^{r-|\mathbf{k}|} a_{\mathbf{k}} \mathbf{t}^{[\mathbf{k}]}$$

for a matroid M on E and $S_1, \dots, S_m \subseteq E$ whose restriction polymatroid has rank r.

Proof of Theorem 1.4. By Corollary 2.8 and the multisymmetric lift, we may assume that the matroid M on E has rank r also. When one of $S_1, \dots, S_m$ has full rank, say S_m, the restriction polymatroid of M to $S_1, \dots, S_m$ is the same as if $S_m = E$. So, we may set $S_m = E$. The polynomial of interest is

$$H(\mathbf{t}, t_E) = \sum_{\mathbf{k}, \ell} a_{\mathbf{k}, \ell} \mathbf{t}^{\mathbf{k}} t_E^{\ell} \quad \text{such that} \quad \chi\Big(\mathrm{M}, \mathcal{L}_{S_1}^{\otimes t_1} \otimes \cdots \otimes \mathcal{L}_{S_{m-1}}^{\otimes t_{m-1}} \otimes \mathcal{L}_E^{\otimes t_E}\Big) = \sum_{\mathbf{k}, \ell} (-1)^{r-|\mathbf{k}|-\ell} a_{\mathbf{k}, \ell} \mathbf{t}^{[\mathbf{k}]} t_E^{[\ell]}$$

where the summation is over $(\mathbf{k}, \ell) \in \mathbb{Z}^{m-1}_{\geq 0} \times \mathbb{Z}_{\geq 0}$. Let $\widetilde{H}(\mathbf{t}, t_E, t_0)$ be its homogenization. We need show that $\widetilde{H}$ is denormalized Lorentzian.

For $(\mathbf{k},\ell) \in \mathbb{Z}_{\geq 0}^{m-1} \times \mathbb{Z}_{\geq 0}$ with $|\mathbf{k}| \leq r$, note that $(\mathbf{k},\ell)$ satisfies Hall–Rado condition, i.e., $(\mathbf{k},\ell) \in I(\mathrm{P})$, if and only if $(\mathbf{k},\ell') \in I(\mathrm{P})$ for all ℓ' such that $|\mathbf{k}| + \ell' \leq r$. Thus, from Proposition 2.7, we compute

$$\chi\Big(\mathrm{M}, \mathcal{L}_{S_1}^{\otimes t_1} \otimes \cdots \otimes \mathcal{L}_{S_{m-1}}^{\otimes t_{m-1}} \otimes \mathcal{L}_E^{\otimes t_E}\Big) = \sum_{(\mathbf{k},\ell)\in I(\mathrm{P})} \mathbf{t}^{(\mathbf{k})} t_E^{(\ell)}$$
$$= \sum_{(\mathbf{k},\ell)\in B(\mathrm{P})} \mathbf{t}^{(\mathbf{k})} t_E^{[\ell]} = \sum_{(\mathbf{k},\ell)\in B(\mathrm{P})} t_E^{[\ell]} \prod_{i=1}^{m-1} \big(t_i^{[k_i]} - t_i^{[k_i-1]}\big),$$

where we used the binomial identity $t^{[k]} = t^{(k)} + t^{(k-1)} + \cdots + t^{(1)} + 1$ for the second equality, and the binomial identity $t^{(k)} = t^{[k]} - t^{[k-1]}$ for the third. In other words, we find

$$H(\mathbf{t}, t_E) = \sum_{(\mathbf{k},\ell)\in B(\mathrm{P})} t_E^{\ell} \prod_{i=1}^{m-1} \big(t_i^{k_i} + t_i^{k_i-1}\big),$$

(1)

$$\text{so that} \quad \widetilde{H}(\mathbf{t}, t_E, t_0) = \sum_{(\mathbf{k},\ell)\in B(\mathrm{P})} t_E^{\ell} \prod_{i=1}^{m-1} \big(t_i^{k_i} + t_0 t_i^{k_i-1}\big).$$

Normalizing, we thus have

$$N(\widetilde{H})(\mathbf{t}, t_E, t_0) = \sum_{(\mathbf{k},\ell)\in B(\mathrm{P})} \frac{t_E^{\ell}}{\ell!} \cdot \left(\prod_{i=1}^{m-1} \Big(1 + t_0 \frac{\partial}{\partial t_i}\Big)\right) \left(\frac{\mathbf{t}^{\mathbf{k}}}{\mathbf{k}!}\right).$$

The exponential generating function over the lattice points of a base polytope of a polymatroid is Lorentzian [BH20, Theorem 3.10], and the operator $(1 + t_0 \frac{\partial}{\partial t_i})$ preserves Lorentzian polynomials [BH20, Proposition 2.7]. Hence, $N(\widetilde{H})$ is Lorentzian, i.e., $\widetilde{H}$ is denormalized Lorentzian. □

Remark 3.2. In general, the Snapper polynomial of very ample divisors on an irreducible projective variety may not similarly give a denormalized Lorentzian polynomial. For example, on $\mathbb{P}^1 \times \mathbb{P}^1$, consider the line bundles $\mathcal{L}_1 = \mathcal{O}(2,2)$ and $\mathcal{L}_2 = \mathcal{O}(1,1)$. We have that

$$\chi(X, \mathcal{L}_1^{\otimes t_1} \otimes \mathcal{L}_2^{\otimes t_2}) = (2t_1 + t_2 + 1)^2 = 8t_1^{[2]} + 4t_1^{[1]}t_2^{[1]} + 2t_2^{[2]} - 12t_1^{[1]} - 5t_2^{[1]} + 4.$$

The normalization of the homogenization of this polynomial (after removing the alternating signs) is

$$4t_1^2 + 4t_1t_2 + t_2^2 + 12t_0t_1 + 5t_0t_2 + 2t_0^2,$$

whose Hessian matrix has signature $(+,+,-)$. See [FH, Section 5.2] for a related example.

3.2. **Applications.** We now explain applications of Theorem 1.4 to K-polynomials. The connection stems from the following formal consequences of some binomial identities, whose proofs we omit.

For a polynomial $\chi(t_1,\ldots,t_m) \in \mathbb{Q}[t_1,\ldots,t_m]$ where each monomial has degree at most $(a_1,\ldots,a_m)$, we have

$$\sum_{\mathbf{k}\geq 0} \chi(k_1,\ldots,k_m)\mathbf{t}^{\mathbf{k}} = \frac{\mathcal{K}(\chi,\mathbf{t})}{(1-t_1)^{a_1}\cdots(1-t_k)^{a_m}}$$

for some polynomial $\mathcal{K}(\chi;\mathbf{t})$ of degree at most $(a_1,\ldots,a_m)$. The polynomial $\mathcal{K}(\chi;1-t_1,\ldots,1-t_m)$, denoted $\mathcal{K}(\chi;\mathbf{1}-\mathbf{t})$, is equivalently described as

$$\mathcal{K}(\chi,\mathbf{1}-\mathbf{t})=\sum_{\mathbf{k}} c_{\mathbf{a}-\mathbf{k}}\mathbf{t}^{\mathbf{k}} \qquad \text{where} \qquad \chi(t_1,\ldots,t_m)=\sum_{\mathbf{k}} c_{\mathbf{k}}\mathbf{t}^{[\mathbf{k}]}.$$

Now, suppose a subvariety $X\subseteq \mathbb{P}^{a_1}\times\cdots\times\mathbb{P}^{a_m}$ has the property that $\chi(X,\mathcal{O}_X(\mathbf{k}))=h^0(X,\mathcal{O}(\mathbf{k}))$ for all $\mathbf{k}\in\mathbb{Z}^m_{\geq 0}$. For instance, an irreducible multiplicity-free subvariety satisfies this property [Bri03]. In this case, with $\chi(t_1,\ldots,t_m)$ as the polynomial $\chi(X,\mathcal{O}(t_1,\ldots,t_m))$, the polynomial $\mathcal{K}(\chi,\mathbf{1}-\mathbf{t})=\sum_{\mathbf{k}} c_{\mathbf{a}-\mathbf{k}}\mathbf{t}^{\mathbf{k}}$ encodes the K-class $[\mathcal{O}_X]\in K(\prod_{i=1}^m \mathbb{P}^{a_i})$ of the structure sheaf of X, that is,

$$[\mathcal{O}_X]=\sum_{\mathbf{k}} c_{\mathbf{a}-\mathbf{k}}[\mathcal{O}_{H_1}]^{k_1}\cdots[\mathcal{O}_{H_m}]^{k_m}$$

where $\mathcal{O}_{H_i}$ denotes the structure sheaf of $\mathbb{P}^{a_1}\times\cdots\times\mathbb{P}^{a_{i-1}}\times H_i\times\mathbb{P}^{a_{i+1}}\times\cdots\times\mathbb{P}^{a_m}$ for a hyperplane $H_i\subset\mathbb{P}^{a_i}$.

The polynomial $\mathcal{K}(\chi,\mathbf{1}-\mathbf{t})$ is sometimes called the *twisted K-polynomial*. The authors of [CCRMM] showed that, for an irreducible multiplicity-free subvariety $X\subseteq\prod_i\mathbb{P}^{a_i}$, its coefficients have alternating signs, i.e., $(-1)^{\dim X-|\mathbf{k}|}c_{\mathbf{k}}>0$. Over $\mathbb{C}$, Brion [Bri02] showed more generally that an irreducible subvariety X with rational singularities in a flag variety G/P has the property that the expansion of $[\mathcal{O}_X]\in K(G/P)$ in terms of the structure sheaves of Schubert subvarieties has alternating signs.

Here, for any polymatroid P, by comparing the definition of $H(\mathbf{t})$ in Theorem 1.4 with $\mathcal{K}(\chi,\mathbf{1}-\mathbf{t})$, and noting Proposition 2.11, one sees that $\widetilde{H}$ is exactly the polynomial denoted g_{P} in [CCRMM]. In particular, Theorem 1.4 implies the following.

Corollary 3.3. Let P be a polymatroid on $[m]$ with cage $\mathbf{a}$ such that $\mathrm{rk}_{\mathrm{P}}(i)=\mathrm{rk}_{\mathrm{P}}([m])$ for some $i\in[m]$. Then, the normalization of g_{P} is Lorentzian, and hence, the support of $\mathbf{t}^{\mathbf{a}}g_{\mathrm{P}}(\mathbf{t}^{-1})$ is the set of lattice points of the base polytope of a polymatroid.[1].

In particular, if X is an irreducible multiplicity-free subvariety in a product of projective spaces such that the projection onto one of the factors has birational image, then its twisted K-polynomial is *dually Lorentzian* in the sense of [RSW]. The corollary partially answers [CCRMM, Question 7.21] and [CCRMM, Conjecture 7.18], which posit that the corollary holds without the condition "$\mathrm{rk}_{\mathrm{P}}(i)=\mathrm{rk}_{\mathrm{P}}([m])$ for some $i\in[m]$."

Remark 3.4. Let P be the restriction polymatroid of a matroid M on E to the collection $S_1,\ldots,S_m$, and let $\widetilde{H}$ be the polynomial defined in Theorem 1.4. Using results in [EHL23], one can show that when P is a matroid, Theorem 1.4 holds without the assumption "$\mathrm{rk}_{\mathrm{P}}(i)=\mathrm{rk}_{\mathrm{P}}([m])$ for some $i\in[m]$." We sketch a proof here. Let r be the rank of P.

Replacing E by $S_1\cup\cdots\cup S_m$, we assume $\mathrm{rk}_{\mathrm{M}}(E)=\mathrm{rk}_{\mathrm{P}}([m])=r$. Consider the polynomial

$$H'(\mathbf{t},t_E)=\sum_{\mathbf{k},\ell}a_{\mathbf{k},\ell}\mathbf{t}^{\mathbf{k}}t_E^{\ell} \quad \text{such that} \quad \chi\Big(\mathrm{M},\mathcal{L}_{S_1}^{\otimes t_1}\otimes\cdots\otimes\mathcal{L}_{S_m}^{\otimes t_m}\otimes\mathcal{L}_E^{\otimes t_E}\Big)=\sum_{\mathbf{k},\ell}(-1)^{r-|\mathbf{k}|-\ell}a_{\mathbf{k},\ell}\mathbf{t}^{[\mathbf{k}]}t_E^{[\ell]}$$

[1]When the support of the homogenization of an inhomogeneous polynomial satisfies this property, the authors of [CCRMM] say that the support of the polynomial is a *generalized polymatroid*.

where the summation is over $(\mathbf{k}, \ell) \in \mathbb{Z}^m_{\geq 0} \times \mathbb{Z}_{\geq 0}$. Let $\widetilde{H}'(\mathbf{t}, t_E, t_0)$ be its homogenization. Because $0^{[\ell]} = 1$, and $\widetilde{H}$ and $\widetilde{H}'$ both have degree r, setting $-t_E = t_0$ in $\widetilde{H}'$ gives the originally desired $\widetilde{H}(\mathbf{t}, t_0)$. Combining this with the formula (1), we find that

$$\text{the coefficient of } \mathbf{t}^{\mathbf{k}} t_0^{r-|\mathbf{k}|} \text{ in } \widetilde{H}(\mathbf{t}, t_0) \text{ is } \sum_{\substack{J \subseteq [m] \text{ such that} \\ \mathbf{k}+\mathbf{e}_J \in B(\mathrm{P})}} (-1)^{|J|}$$

where $\mathbf{e}_J = \sum_{J \in J} \mathbf{e}_j \in \mathbb{R}^m$ denotes the sum of the standard basis vectors of J.

Now, if P is a matroid N on $[m]$, the displayed equation implies that

$$\widetilde{H}(\mathbf{t}, t_0) = \sum_{\substack{I \subseteq [m] \\ \mathbf{e}_I \in I(\mathrm{N})}} T_{\mathrm{N}/I}(0,1) \mathbf{t}^{\mathbf{e}_I} t_0^{r-|I|}$$

where $T_{\mathrm{N}/I}$ is the Tutte polynomial of the contraction matroid N/I. The right-hand-side polynomial $\sum_{I \in I(\mathrm{N})} T_{\mathrm{N}/I}(0,1)\mathbf{t}^{\mathbf{e}_I} t_0^{r-|I|}$ is obtained from a denormalized Lorentzian polynomial in variables $x, z, w, u_1, \ldots, u_m$ provided in [EHL23, Theorem 1.4 and Remark 8.9] via the following two steps. One keeps only the terms exactly divisible by w^{m-r}, and sets $x = 0, z = t_0, u_i = t_i$. Both steps preserve denormalized Lorentzian polynomials, and hence $\widetilde{H}(\mathbf{t}, t_0)$ is denormalized Lorentzian when P is a matroid N.

4. h^*-VECTORS FOR MATROIDS

In this section, we define and study h^*-vectors of line bundles in $\underline{K}(\mathrm{M})$. Let M be a loopless matroid. For a line bundle $\mathcal{L}$ on $\underline{K}(\mathrm{M})$, it follows from Proposition 2.7 that the function $t \mapsto \underline{\chi}(\mathrm{M}, \mathcal{L}^{\otimes t})$ is a polynomial in t, which we call the *Snapper polynomial* of $\mathcal{L}$ on M.

Definition 4.1. For a loopless matroid M on a ground set E and a line bundle $\mathcal{L}$ in $\underline{K}(\mathrm{M})$, we define its h^*-*vector* $(h^*_0(\mathrm{M}, \mathcal{L}), \ldots, h^*_d(\mathrm{M}, \mathcal{L}))$ by

$$\sum_{k \geq 0} \underline{\chi}(\mathrm{M}, \mathcal{L}^{\otimes k}) t^k = \frac{h^*(\mathrm{M}, \mathcal{L}; t)}{(1-t)^{d+1}} \quad \text{where} \quad h^*(\mathrm{M}, \mathcal{L}; t) = \sum_{k=0}^{d} h^*_k(\mathrm{M}, \mathcal{L}) t^k,$$

and d is the degree of the Snapper polynomial of $\mathcal{L}$.

Theorem 1.5 states that the h^*-vector is a Macaulay vector when $\mathcal{L} = \bigotimes_{S \subseteq E} \mathcal{L}_S^{\otimes k_S}$ with $k_S \geq 0$ for all S. In this section, we prove this theorem.

In Section 4.1, we review Macaulay vectors and their relation to Cohen–Macaulayness and cohomology vanishing. In Section 4.2, we use properties of Y_{P} to prove Theorem 1.5. A generalization of Theorem 1.5 is conjectured in Section 4.3. Results on the degree of Snapper polynomials, necessary for studying h^*-vectors, are given in Section 4.4.

4.1. **Macaulay vectors.** Recall that the Hilbert function of a graded algebra over a field $\Bbbk$ is the sequence of the $\Bbbk$-dimensions of the graded pieces. For the numerical properties we consider, we may extend scalars to an extension of $\Bbbk$, so we may assume $\Bbbk$ is infinite as needed.

Definition 4.2. A sequence $(h_0, h_1, \ldots, h_d)$ is a *Macaulay vector* if $(h_0, h_1, \ldots, h_d, 0, 0, \ldots)$ is the Hilbert function of a graded artinian $\Bbbk$-algebra $A^\bullet$ which is generated in degree 1 and has $A^0 = \Bbbk$.

Macaulay vectors are also called M-vectors and O-sequences. Macaulay gave an explicit description of these vectors as follows [BH93, Theorem 4.2.10]. Given positive integers n and d, there is a unique expression

$$n = \binom{k_d}{d} + \binom{k_{d-1}}{d-1} + \cdots + \binom{k_\delta}{\delta}, \quad k_d > k_{d-1} > \cdots > k_\delta \geq 1.$$

Set $n^{\langle d \rangle} = \binom{k_d+1}{d+1} + \cdots + \binom{k_\delta+1}{\delta+1}$. Then $(1, a_1, \ldots, a_d)$ is a Macaulay vector if and only if $0 \leq a_{t+1} \leq a_t^{\langle t \rangle}$ for all $t \geq 1$.

Macaulay vectors often appear in the following way. Suppose $R^\bullet$ is a graded Cohen–Macaulay algebra of Krull dimension $d+1$ with $R^0 = \Bbbk$. If the quotient of $R^\bullet$ by the ideal generated by R^1 is artinian, then $R^\bullet$ admits a linear system of parameters [BH93, Propositions 1.5.11 and 1.5.12]. In this case, the quotient by a linear system of parameters is a graded artinian algebra $A^\bullet$ with the property that

$$\sum_{k \geq 0} (\dim_\Bbbk R^k) t^k = \frac{\dim_\Bbbk A^0 + (\dim_\Bbbk A^1) t^1 + \cdots + (\dim_\Bbbk A^d) t^d}{(1-t)^{d+1}}.$$

See for instance [BH93, Remark 4.1.11]. In particular, if $R^\bullet$ is generated in degree 1, then the numerator of its Hilbert series $\sum_{k \geq 0} (\dim_\Bbbk R^k) t^k$ is a polynomial whose coefficients form a Macaulay vector. For the proof of Theorem 1.5, we record the following cohomological criterion for a section ring to be Cohen–Macaulay.

Proposition 4.3. Let $\mathcal{L}$ be an ample line bundle on a geometrically connected and geometrically reduced projective $\Bbbk$-variety X of dimension d. Suppose that $H^i(X, \mathcal{L}^{\otimes k}) = 0$ for all $i > 0$ when $k \geq 0$, and $H^i(X, \mathcal{L}^{\otimes k}) = 0$ for all $i < d$ when $k < 0$. Then, the section ring

$$R^\bullet_\mathcal{L} := \bigoplus_{k \geq 0} H^0(X, \mathcal{L}^{\otimes k})$$

is a graded Cohen–Macaulay $\Bbbk$-algebra with $R^0_\mathcal{L} = \Bbbk$. If further $R^\bullet_\mathcal{L}$ is generated in degree 1, then the sequence $(h_0, \ldots, h_d)$ defined by

$$\sum_{k \geq 0} \chi(X, \mathcal{L}^{\otimes k}) t^k = \frac{h_0 + h_1 t + \cdots + h_d t^d}{(1-t)^{d+1}} \tag{2}$$

is a Macaulay vector.

Proof. The sequence $(h_0, \ldots, h_d)$ is well-defined via (2) because $\chi(X, \mathcal{L}^{\otimes k})$ is a polynomial in k (see [Sta12, Section 4.3]). Because X is geometrically connected, geometrically reduced, and proper over $\operatorname{Spec} \Bbbk$, we have $R^0_\mathcal{L} = \Bbbk$. Because all of the higher cohomology vanishes, we have $\chi(X, \mathcal{L}^{\otimes k}) = \dim H^0(X, \mathcal{L}^{\otimes k})$ for $k \geq 0$. Therefore the second statement follows from the first by our discussion above about Macaulay vectors.

It remains to show that $R^\bullet_\mathcal{L}$ is a Cohen–Macaulay graded ring. That is, we show that the local cohomology $H^i_\mathfrak{m}(R^\bullet_\mathcal{L}; R^\bullet_\mathcal{L})$ with respect to the irrelevant ideal $\mathfrak{m}$ of $R^\bullet_\mathcal{L}$ vanishes for $i < d+1$. The vanishing when $i = 0, 1$ is automatic since $R^\bullet_\mathcal{L}$ is the section ring of $\mathcal{O}(1)$ on $X = \operatorname{Proj} R^\bullet_\mathcal{L}$. For $i \geq 2$, we have $H^i_\mathfrak{m}(R^\bullet_\mathcal{L}; R^\bullet_\mathcal{L}) = \bigoplus_{k \in \mathbb{Z}} H^{i-1}(\operatorname{Proj} R^\bullet_\mathcal{L}, \mathcal{L}^{\otimes k})$ by [BS98, Theorem 20.4.4]. As $X = \operatorname{Proj} R^\bullet_\mathcal{L}$, the sheaf cohomology vanishing hypothesis gives desired vanishing of local cohomology. □

4.2. **Properties of Y_{P} and Theorem 1.5.** Let P be a polymatroid with cage $(a_1, \ldots, a_m)$, and let $Y_{\mathrm{P}} \subseteq \mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_m}$ be the subvariety defined in the introduction. We note that Y_{P} is Cohen–Macaulay and compatibly Frobenius split, and we use these properties of prove Theorem 1.5.

Proposition 4.4. The variety Y_{P} is Cohen–Macaulay.

Proof. When there is a multiplicity-free subvariety $X \subseteq \mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_m}$ whose polymatroid is P, the Cohen–Macaulayness of Y_{P} is proven in [Bri03] via a geometric argument. For arbitrary P, the proposition is [CCRC23, Proof of Theorem 5.6], which was obtained by using properties of "polymatroid ideal" in [HH11, Chapter 12.6]. □

Note that Y_{P} is defined over $\operatorname{Spec}\mathbb{Z}$, with an embedding in a product of projective spaces over $\operatorname{Spec}\mathbb{Z}$. Viewing the product of projective spaces as a homogeneous space, Y_{P} is a reduced union of Schubert varieties, and hence it is a compatibly Frobenius split subvariety of the product of projective spaces when base changed to any positive characteristic field $\Bbbk$ [BK05, Proposition 1.2.1, Theorem 2.3.10]. Together with Proposition 4.4, this gives the following strong cohomology vanishing results on Y_{P}.

Proposition 4.5. Let $\mathcal{L}$ be the restriction of a very ample line bundle from the product of projective spaces to Y_{P}. Then, we have $H^i(Y_{\mathrm{P}}, \mathcal{L}^{\otimes k}) = 0$ for all $i > 0$ when $k \geq 0$, and $H^i(Y_{\mathrm{P}}, \mathcal{L}^{\otimes k}) = 0$ for all $i < \mathrm{rk}(P)$ when $k < 0$. Moreover, we have that $H^0(Y_{\mathrm{P}}, \mathcal{O}_{Y_{\mathrm{P}}})$ is the ground field $\Bbbk$, and the section ring $R^\bullet_{\mathcal{L}} = \bigoplus_{k \geq 0} H^0(Y_{\mathrm{P}}, \mathcal{L}^{\otimes k})$ is generated in degree 1.

Proof. The cohomology vanishing follows from [BK05, Theorem 1.2.8(ii), Theorem 1.2.9] because Y_{P} is Cohen–Macaulay. By [BK05, Theorem 1.2.8(ii)], Y_{P} is projectively normal in the embedding given by $\mathcal{L}$, so $R^\bullet_{\mathcal{L}}$ is generated in degree 1. To check that $H^0(Y_{\mathrm{P}}, \mathcal{O}_{Y_{\mathrm{P}}})$ is 1-dimensional, we check that Y_{P} is geometrically reduced and geometrically connected. That it is geometrically reduced is obvious; it is geometrically connected because each component of Y_{P} contains the point $[1, 0, \ldots, 0] \times [1, 0, \ldots, 0] \times \cdots \times [1, 0, \ldots, 0]$. □

Proof of Theorem 1.5. Let $\mathcal{L} = \bigotimes_{i=1}^m \underline{\mathcal{L}}_{S_i}^{\otimes b_i}$ for some subsets $S_1, \ldots, S_m$ of the ground set E of the matroid M and integers $b_i > 0$. Let P be the restriction of the dragon-Hall–Rado polymatroid to the subsets $S_1, \ldots, S_m$. By Corollary 2.15, we have that $\underline{\chi}(\mathrm{M}, \mathcal{L}^{\otimes k}) = \chi(Y_{\mathrm{P}}, \mathcal{O}(b_1, \ldots, b_m)^{\otimes k})$. Note that $\mathcal{O}(b_1, \ldots, b_m)$ is the restriction of an ample divisor from the product of projective spaces to Y_{P}. By Proposition 4.5, we have that Y_{P} and $\mathcal{O}(b_1, \ldots, b_m)$ satisfy the conditions of Proposition 4.3, including the generation of $\bigoplus_{k \geq 0} H^0(Y_{\mathrm{P}}, \mathcal{O}(b_1, \ldots, b_m)^{\otimes k})$ in degree 1. Hence, we conclude that $h^*(\mathrm{M}, \mathcal{L})$ is a Macaulay vector. □

4.3. **Line bundles from polymatroids.** We conjecture a generalization of Theorem 1.5. In Section 5, we explain how the conjecture contains a question of Speyer [Spe09] as a special case, and how Theorem 1.5 answers the question for a new family of cases. To do so, it is convenient to phrase the line bundles in $\underline{K}(\mathrm{M})$ in terms of divisors in the non-augmented Chow ring $\underline{A}^\bullet(\mathrm{M})$.

Definition 4.6. [FY04] Let M be a loopless matroid on a ground set E. The *non-augmented Chow ring* of M is the graded ring

$$\underline{A}^\bullet(\mathrm{M}) = \frac{\mathbb{Z}[z_F : F \text{ a nonempty flat of } \mathrm{M}]}{\langle z_F z_{F'} : F \subsetneq F' \text{ and } F \supsetneq F' \rangle + \langle \sum_{F \ni i} z_F : i \in E \rangle}.$$

An element of $\underline{A}^1(\mathrm{M})$ is called a *divisor class* on M. Equivalently, $\underline{A}^\bullet(\mathrm{M})$ is the Chow ring of the toric variety $X_{\underline{\Sigma}_\mathrm{M}}$ of the non-augmented Bergman fan $\underline{\Sigma}_\mathrm{M}$ of M.

For a nonempty subset $S \subseteq E$, define an element $\underline{h}_S \in \underline{A}^1(\mathrm{M})$ by

$$\underline{h}_S = \sum_{F \supseteq S} -z_F.$$

Because $\underline{K}(\mathrm{M}) = K(X_{\underline{\Sigma}_\mathrm{M}})$ and $\underline{A}^\bullet(\mathrm{M}) = A^\bullet(X_{\underline{\Sigma}_\mathrm{M}})$, one has a *Chern class map* $c \colon \underline{K}(\mathrm{M}) \to \underline{A}^\bullet(\mathrm{M})^\times$, a homomorphism from the additive group of $K(\mathrm{M})$ to the units in $\underline{A}^\bullet(\mathrm{M})$, see [Ful98, Section 15.3]. It has the characterizing property that

$$c(\underline{\mathcal{L}}_S) = 1 + \underline{h}_S \quad \text{for all nonempty } S \subseteq E.$$

That is, we have $c_1(\underline{\mathcal{L}}_S) = \underline{h}_S$. More generally, for a polymatroid P on E, let us define the line bundle $\underline{\mathcal{L}}_{B(\mathrm{P})}$ in $\underline{K}(\mathrm{M})$ via the property

$$c_1(\underline{\mathcal{L}}_{B(\mathrm{P})}) = \sum_F \big(\mathrm{rk}_\mathrm{P}(E \setminus F) - \mathrm{rk}_\mathrm{P}(E)\big) z_F.$$

One recovers $\underline{\mathcal{L}}_S$ via the polymatroid whose rank function is $\mathrm{rk}(I) = 1$ if $I \cap S \neq \emptyset$ and 0 otherwise.

Remark 4.7. These constructions have the following geometric origin. When M has a realization $L \subseteq \Bbbk^E$, the Chow ring $\underline{A}^\bullet(\mathrm{M})$ coincides with the Chow ring of the wonderful variety $\underline{W}_L$ [DCP95], and the Chern class map $\underline{K}(\mathrm{M}) \to \underline{A}^\bullet(\mathrm{M})$ coincides with the Chern class map $K(\underline{W}_L) \to A^\bullet(\underline{W}_L)$.

When further M is the Boolean matroid $\mathrm{U}_{|E|,E}$, whose realization is $L = \Bbbk^E$, the wonderful variety $\underline{W}_L$ is a toric variety $\underline{X}_E$ known as the *permutohedral variety*. In this case, under the standard correspondence between nef divisor classes on toric varieties and polytopes [CLS11, Chapter 6], the divisor class $c_1(\underline{\mathcal{L}}_{B(\mathrm{P})})$ corresponds to the base polytope $B(\mathrm{P})$. Moreover, every nef divisor class is equal to $c_1(\underline{\mathcal{L}}_{B(\mathrm{P})})$ for some polymatroid P. See [BEST23, Section 2.7] and references therein.

We conjecture the following positivity for h^*-vectors of line bundles from polymatroids.

Conjecture 4.8. Let M be a loopless matroid on E, and let P be a polymatroid on E. Then, the h^*-vector $h^*(\mathrm{M}, \underline{\mathcal{L}}_{B(\mathrm{P})})$ is a Macaulay vector, and is in particular nonnegative.

Theorem 1.5 states that Conjecture 4.8 holds when $c_1(\underline{\mathcal{L}}_{B(\mathrm{P})})$ is a nonnegative linear combination of the $\underline{h}_S$. Several other cases in which Conjecture 4.8 holds are discussed in Section 5.2.

4.4. **Degree of Snapper polynomials and numerical dimension.** To study h^*-vectors arising from line bundles $\underline{\mathcal{L}}_{B(\mathrm{P})}$ in Conjecture 4.8, one needs some tools to understand the degree of the Snapper polynomial, since the degree is essential in the definition of $h^*(\mathrm{M}, \underline{\mathcal{L}}_{B(\mathrm{P})})$. One such tool is given in terms of the following.

Definition 4.9. The *numerical dimension* of a line bundle $\mathcal{L}$ in $\underline{K}(\mathrm{M})$ is the largest nonnegative integer k such that $c_1(\mathcal{L})^k \neq 0$ in $\underline{A}^\bullet(\mathrm{M})$.

Our main result for numerical dimensions is the following.

Theorem 4.10. Let M be a loopless matroid or rank r on a ground set E.

(1) For $\mathcal{L}$ a line bundle in $\underline{K}(\mathrm{M})$, the degree of the Snapper polynomial $\underline{\chi}(\mathrm{M}, \mathcal{L}^{\otimes t})$ is at most the numerical dimension of $c_1(\mathcal{L})$. Moreover, the degree equals $r-1$ if and only if the numerical dimension is $r-1$.
(2) For P a polymatroid on E such that the base polytope $B(\mathrm{P})$ is full dimensional (i.e., $(|E|-1)$-dimensional), then its numerical degree is $r-1$, so the degree of the Snapper polynomial of $\underline{\mathcal{L}}_{B(\mathrm{P})}$ is $r-1$.

To prove Theorem 4.10(1), we develop a version of the Hirzebruch–Riemann–Roch theorem for K and Chow rings of matroids. For this, we recall that the Chow ring $\underline{A}^\bullet(\mathrm{M})$ is equipped with a degree map $\underline{\deg}_{\mathrm{M}} \colon \underline{A}^{r-1} \overset{\sim}{\to} \mathbb{Z}$ that satisfies Poincaré duality. See [AHK18, Section 6] for details.

Proposition 4.11. There is a ring homomorphism $\mathrm{ch}\colon \underline{K}(\mathrm{M}) \to \underline{A}(\mathrm{M})_{\mathbb{Q}}$ which induces an isomorphism $\underline{K}(\mathrm{M})_{\mathbb{Q}} \to \underline{A}^\bullet(\mathrm{M})$ defined by

$$\mathrm{ch}([\mathcal{L}]) = \exp(c_1(\mathcal{L})) = 1 + c_1(\mathcal{L}) + c_1(\mathcal{L})^2/2! + \cdots .$$

There is a class $\underline{\mathrm{Todd}}_{\mathrm{M}} \in \underline{A}^\bullet(\mathrm{M})_{\mathbb{Q}}$ such that, for any $\xi \in \underline{K}(\mathrm{M})_{\mathbb{Q}}$,

$$\underline{\chi}(\mathrm{M}, \xi) = \underline{\deg}_{\mathrm{M}}\big(\mathrm{ch}(\xi) \cdot \underline{\mathrm{Todd}}_{\mathrm{M}}\big).$$

Moreover, the degree 0 part of $\underline{\mathrm{Todd}}_{\mathrm{M}}$ is 1.

Proof. We first recall $\underline{K}(\mathrm{M}) = K(X_{\underline{\Sigma}_{\mathrm{M}}})$ and $\underline{A}^\bullet(\mathrm{M}) = A^\bullet(X_{\underline{\Sigma}_{\mathrm{M}}})$, i.e., the K and Chow rings of the toric variety $X_{\underline{\Sigma}_{\mathrm{M}}}$ (respectively). Hence, that Chern character map ch is well-defined and is an isomorphism after tensoring with $\mathbb{Q}$ is a general fact about algebraic varieties [Ful98, Example 15.2.16]. Because $\underline{K}(\mathrm{M})$ is spanned by classes of line bundles [LLPP, Theorem 1.15], the formula $\mathrm{ch}([\mathcal{L}]) = \exp(c_1(\mathcal{L}))$ determines ch. By [AHK18, Theorem 6.19], the pairing $\underline{A}^\bullet(\mathrm{M})_{\mathbb{Q}} \otimes \underline{A}^\bullet(\mathrm{M})_{\mathbb{Q}} \to \mathbb{Q}$ given by $(x,y) \mapsto \underline{\deg}_{\mathrm{M}}(x \cdot y)$ is a perfect pairing. Therefore there is some class $\underline{\mathrm{Todd}}_{\mathrm{M}} \in \underline{A}^\bullet(\mathrm{M})_{\mathbb{Q}}$ such that the linear functional $x \mapsto \underline{\chi}(\mathrm{M}, \mathrm{ch}^{-1}(x))$ on $\underline{A}^\bullet(\mathrm{M})_{\mathbb{Q}}$ is given by $x \mapsto \underline{\deg}_{\mathrm{M}}(x \cdot \underline{\mathrm{Todd}}_{\mathrm{M}})$. Lastly, the degree 0 part of $\underline{\mathrm{Todd}}_{\mathrm{M}}$, which is some number in $\mathbb{Q}$, must be 1 because Proposition 2.14 implies that the leading term of the polynomial $\underline{\chi}(\mathrm{M}, \underline{\mathcal{L}}_E^{\otimes t})$ is $t^{r-1}/(r-1)!$, whereas

$$\begin{aligned}\underline{\deg}_{\mathrm{M}}\big(c_1(\underline{\mathcal{L}}_E)^{r-1}\big) = \underline{\deg}_{\mathrm{M}}\big((-z_E)^{r-1}\big) &= \underline{\deg}_{\mathrm{M}}\Big(\big(\sum_{i \in F \subsetneq E} z_F\big)^{r-1}\Big) \quad \text{for any fixed } i \in E \\ &= 1 \quad \text{by [AHK18, Proposition 5.8].}\end{aligned}$$

□

Proof of Theorem 4.10. Let $\mathcal{L}$ be a line bundle of numerical dimension d. Because $c_1(\mathcal{L}^{\otimes t}) = tc_1(\mathcal{L})$, we have that

$$\underline{\chi}(\mathrm{M}, \mathcal{L}^{\otimes t}) = \underline{\deg}_{\mathrm{M}}((1 + tc_1(\mathcal{L}) + t^2 c_1(\mathcal{L})^2/2! + \cdots) \cdot \underline{\mathrm{Todd}}_{\mathrm{M}}).$$

Since $c_1(\mathcal{L})^{d+1} = 0$, we see that the right-hand side is a polynomial in t whose leading term is $t^\ell \underline{\deg}_{\mathrm{M}}(c_1(\mathcal{L})^\ell \cdot \underline{\mathrm{Todd}}_{\mathrm{M}})/\ell!$ for the largest $0 \le \ell \le d$ such that $\underline{\deg}_{\mathrm{M}}(c_1(\mathcal{L})^\ell \cdot \underline{\mathrm{Todd}}_{\mathrm{M}}) \neq 0$. Moreover, because the degree 0 part of $\underline{\mathrm{Todd}}_{\mathrm{M}}$ is 1, we have

$$\underline{\chi}(\mathrm{M}, \mathcal{L}^{\otimes t}) = \underline{\deg}_{\mathrm{M}}(c_1(\mathcal{L})^{r-1})\frac{t^{r-1}}{(r-1)!} + O(t^{r-2}).$$

Thus, $\mathcal{L}$ has numerical dimension $r-1$ if and only if the Snapper polynomial has degree $r-1$. We have proven the first statement (1).

For second statement (2), we only need show that the numerical degree of $\underline{\mathcal{L}}_{B(\mathrm{P})}$ is $r-1$ if $B(\mathrm{P})$ is full dimensional. When $B(\mathrm{P})$ is full dimensional, the line bundle $\underline{\mathcal{L}}_{B(\mathrm{P})}$ in $K(\mathrm{U}_{|E|,E})$ of the boolean matroid corresponds to a nef and big line bundle on the projective toric variety $\underline{X}_E$ (see Remark 4.7). By [Laz04, Corollary 2.2.7], we can write the first Chern class as the sum of an ample class and an effective divisor class (inside $A^\bullet(\underline{X}_E) \otimes \mathbb{Q}$). Restricting this to $\underline{A}^\bullet(\mathrm{M})$, we get that $c_1(\underline{\mathcal{L}}_{B(\mathrm{P})}) = A + E$, where A is the restriction of an ample class from $\underline{X}_E$ and E is the restriction of an effective class.

We now prove by induction on k that $\underline{\deg}_{\mathrm{M}}(c_1(\underline{\mathcal{L}}_{B(\mathrm{P})})^k A^{r-1-k}) > 0$, using Proposition 4.12 stated below. The case $k = 0$ is Proposition 4.12(1). For $k > 0$, Proposition 4.12(2) gives that

$$\begin{aligned}\underline{\deg}_{\mathrm{M}}(c_1(\underline{\mathcal{L}}_{B(\mathrm{P})})^k A^{r-1-k}) &= \underline{\deg}_{\mathrm{M}}(c_1(\underline{\mathcal{L}}_{B(\mathrm{P})})^{k-1} A^{r-k}) + \underline{\deg}_{\mathrm{M}}(c_1(\underline{\mathcal{L}}_{B(\mathrm{P})})^{k-1} E A^{r-1-k}) \\ &\ge \underline{\deg}_{\mathrm{M}}(c_1(\underline{\mathcal{L}}_{B(\mathrm{P})})^{k-1} A^{r-k}),\end{aligned}$$

which is positive by induction. □

Proposition 4.12. Let M be a loopless matroid of rank r.

(1) Let $A \in \underline{A}^1(\mathrm{M})$ be the restriction of an ample class from $\underline{X}_E$. Then $\underline{\deg}_{\mathrm{M}}(A^{r-1}) > 0$.
(2) Let $\mathrm{P}_1, \ldots, \mathrm{P}_{r-2}$ be polymatroids. Then, for any class $E \in \underline{A}^1(\mathrm{M})$ which is a restriction of an effective divisor class on $\underline{X}_E$, $\underline{\deg}_{\mathrm{M}}(c_1(\underline{\mathcal{L}}_{\mathrm{P}_1}) \cdots c_1(\underline{\mathcal{L}}_{\mathrm{P}_{r-2}}) \cdot E) \ge 0$.

One can deduce the proposition as a general statement about combinatorially nef divisors on a fan with nonnegative Minkowski weights. To avoid developing such notions here, we indicate a proof in terms of the Hodge–Riemann relations for $\underline{A}(\mathrm{M})$ proven in [AHK18].

Proof. The first statement is the Hodge–Riemann relations in degree 0 for $\underline{A}(\mathrm{M})$ [AHK18, Theorem 1.4]. The second statement is a consequence of the mixed Hodge–Riemann relations in degree 0 [AHK18, Theorem 8.9], when one notes that the divisors $c_1(\underline{\mathcal{L}}_{B(\mathrm{P}_i)})$ are nef (thus, a limit of ample classes), and that the star of a ray in the Bergman fan of the matroid is a product of Bergman fans of matroids [AHK18, Proposition 3.5]. □

5. Applications, Examples, and Problems

In Section 5.1, we study a question of Speyer [Spe09] as an application of results developed in the previous section. It is for this application that we have focused on the non-augmented setting, although analogous statements for the augmented setting also hold. Examples for Conjecture 4.8 and some further general properties of h^*-vectors of matroids are given in Section 5.2, along with future directions.

5.1. **Application to Speyer's g-polynomial.** In this section, we apply Theorem 1.5 to study Speyer's *g-polynomial* of a matroid [Spe09]. For a loopless and coloopless matroid M of rank r on $[n]$, the g-polynomial $g_{\mathrm{M}}(t)$ is a polynomial of degree at most r defined in terms of the K-theory of the Grassmannian $Gr(r,n)$, first defined for matroids realizable over a field of characteristic 0 in [Spe09] and then for all matroids in [FS12].

An outstanding problem about the g-polynomial is to show that it always has nonnegative coefficients. In [Spe09], Speyer used the Kawamata–Viehweg vanishing theorem to show the nonnegativity for matroids realizable over a field of characteristic 0. This allowed him to bound the number of cells of each dimension in a subdivision of the hypersimplex into matroid polytopes when all of the cells correspond to matroids realizable in characteristic 0. Nonnegativity of $g_{\mathrm{M}}(t)$ for all matroids would bound the complexity of any such subdivision in general. The nonnegativity was proved for all sparse paving matroids in [FS, Theorem 13.16]. Using Theorem 1.5 we give a new infinite family of matroids for which the nonnegativity holds.

We begin by explaining how the nonnegativity of the coefficients of the g-polynomial is a special case of Conjecture 4.8. For a loopless and coloopless matroid M of rank r, let $\omega(\mathrm{M})$ be the t^r coefficient of $g_{\mathrm{M}}(t)$. In forthcoming work, Alex Fink, Kris Shaw, and David Speyer show the following result.

Proposition 5.1. Suppose that $\omega(\mathrm{M}) \geq 0$ for all connected matroids. Then all coefficients of $g_{\mathrm{M}}(t)$ are nonnegative for all loopless and coloopless matroids.

The following result was communicated to the authors by Alex Fink, Kris Shaw, and David Speyer.

Proposition 5.2. Let M be a matroid of rank r with c connected components, and denote by $B(\mathrm{M}^\perp)$ the base polytope of the dual matroid $\mathrm{M}^\perp$ of M. Then, we have

$$\omega(\mathrm{M}) = (-1)^{r-c}\underline{\chi}(\mathrm{M}, \underline{\mathcal{L}}_{B(\mathrm{M}^\perp)}^{-1}).$$

Proof. We sketch a proof using results from [LLPP] and [BEST23]. By [LLPP, Theorem 1.6], we have

$$\underline{\chi}(\mathrm{M}, \underline{\mathcal{L}}_{B(\mathrm{M}^\perp)}^{-1}) = \underline{\deg}_{\mathrm{M}}(\zeta_{\mathrm{M}}([\underline{\mathcal{L}}_{B(\mathrm{M}^\perp)}^{-1}]) \cdot (1 + \underline{h}_E + \underline{h}_E^2 + \cdots)),$$

where ζ_{M} is defined in [LLPP]. Computing in the equivariant Chow groups of the permutohedral variety $\underline{X}_E$ using [LLPP, Proposition 5.3] and [BEST23, Theorem 10.1] (see [EHL23, Corollary 6.5]), we have that $\zeta_{\mathrm{M}}([\underline{\mathcal{L}}_{B(\mathrm{M}^\perp)}^{-1}])$ is the restriction to $\underline{A}^\bullet(\mathrm{M})$ of the class denoted $c(\mathcal{Q}_{\mathrm{M}}^\vee)$ in [BEST23]. Then the result follows from [BEST23, Theorem 10.12]. □

Now, recall the formal identity satisfied by the h^*-vector

$$h_d^*(\mathrm{M}, \mathcal{L}) = (-1)^d \underline{\chi}(\mathrm{M}, \mathcal{L}^{-1}) \tag{3}$$

(see for instance [Sta12, Proposition 1.4.4]). Moreover, when M is connected, the polytope $B(\mathrm{M}^\perp)$ is full dimensional, so Theorem 4.10 implies that the degree d of the Snapper polynomial is $r-1$. Therefore, the two preceding propositions show that the nonnegativity of the coefficients of $g_{\mathrm{M}}(t)$ is a special case of Conjecture 4.8 with $\mathrm{P} = \mathrm{M}^\perp$.

We now make explicit how Theorem 1.5 proves the positivity of $\omega(\mathrm{M})$ in some special cases. The first step is to express $\underline{\mathcal{L}}_{B(\mathrm{M}^\perp)}$ as a Laurent monomial in the $[\underline{\mathcal{L}}_S]$, or, equivalently, write $c_1(\underline{\mathcal{L}}_{B(\mathrm{M}^\perp)})$

as a linear combination of the $\underline{h}_S$. To do so, for a matroid M, we recall its *β-invariant* [Cra67], defined by two properties:

- $\beta(U_{0,1}) = 0$, $\beta(U_{1,1}) = 1$, and $\beta(\mathrm{M}) = 0$ if M is disconnected, and
- the recursive relation: for any i which is not a loop or coloop of M ,

$$\beta(\mathrm{M}) = \beta(\mathrm{M}/i) + \beta(\mathrm{M} \setminus i).$$

Equivalently, the β-invariant is the coefficient of x in the Tutte polynomial of M.

Proposition 5.3. Let M be a matroid on $[n]$. Then, the polytope $B(\mathrm{M}^\perp)$ satisfies

$$c_1(\underline{\mathcal{L}}_{B(\mathrm{M}^\perp)}) = \sum_{\substack{F \text{ connected flat} \\ \text{of } \operatorname{rk}_\mathrm{M}(F) \geq 2}} \sum_{\operatorname{cl}_\mathrm{M}(S) = F} (-1)^{|S| - \operatorname{rk}_\mathrm{M}(S) + 1} \beta(\mathrm{M}|_S) \underline{h}_F \in \underline{A}^\bullet(\mathrm{M}).$$

Proof. Let $\underline{\Delta}_S$ be the simplex $\operatorname{Conv}(\{\mathbf{e}_i : i \in S\})$. [ABD10, Theorem 2.6] expressed $B(\mathrm{M}^\perp)$ as a signed Minkowski sum of these simplices as follows:

$$B(\mathrm{M}^\perp) = \sum_{S \subseteq [n], |S| \geq 2} (-1)^{|S| - \operatorname{rk}_\mathrm{M}(S) + 1} \beta(\mathrm{M}|_S) \underline{\Delta}_S + \sum_{i \text{ loop of } \mathrm{M}} \underline{\Delta}_i.$$

This gives an expression for $c_1(\underline{\mathcal{L}}_{B(\mathrm{M}^\perp)})$ on the permutohedral toric variety as a sum of the simplicial generators $\underline{h}_S$. As $\underline{h}_S = \underline{h}_{\operatorname{cl}_\mathrm{M}(S)}$ in $\underline{A}^\bullet(\mathrm{M})$ and $\underline{h}_i = 0$, we obtain the desired expression. □

Theorem 5.4. Let M be a loopless and coloopless matroid of rank r such that, for all connected flats F of M of rank at least 2, we have $\sum_{\operatorname{cl}_\mathrm{M}(S) = F} (-1)^{|S| - \operatorname{rk}_\mathrm{M}(S) + 1} \beta(\mathrm{M}|_S) \geq 0$. Then $\omega(\mathrm{M}) \geq 0$.

Proof. First suppose that M is connected, so the polytope $B(\mathrm{M}^\perp)$ is full dimensional. By Theorem 4.10(2), the degree of the Snapper polynomial of $\underline{\mathcal{L}}_{B(\mathrm{M}^\perp)}$ is $r - 1$ in this case. By Theorem 1.5 along with (3), we thus have $\omega(\mathrm{M}) = (-1)^{r-1} \underline{\chi}(\mathrm{M}, \underline{\mathcal{L}}_{B(\mathrm{M}^\perp)}^{-1}) = h^*_{r-1}(\mathrm{M}, \underline{\mathcal{L}}_{B(\mathrm{M}^\perp)}) \geq 0$.

Now suppose that $\mathrm{M} = \mathrm{M}_1 \oplus \cdots \oplus \mathrm{M}_c$, with each M_i connected. The hypothesis implies that each $\underline{\mathcal{L}}_{B(\mathrm{M}_i^\perp)}$ is simplicially positive for each i, and so $\omega(\mathrm{M}_i) \geq 0$. By [FS12, Proposition 7.2], $g_\mathrm{M}(t) = g_{\mathrm{M}_1}(t) \cdots g_{\mathrm{M}_c}(t)$. Because the t^i coefficient of $g_\mathrm{M}(t)$ vanishes for $i > \operatorname{rk}(\mathrm{M})$,

$$\omega(\mathrm{M}) = \omega(\mathrm{M}_1) \cdot \cdots \cdot \omega(\mathrm{M}_c) \geq 0.$$ □

While it appears that the above condition is not often satisfied, Theorem 5.4 does show that $\omega(\mathrm{M}) \geq 0$ for many matroids. We give two examples.

Example 5.5. For a nonempty subset $S \subseteq E$, let H_S be the corank 1 matroid on E with S as its unique circuit. A *co-transversal matroid* is a matroid M that arises as the matroid intersection $\mathrm{M} = \mathrm{H}_{S_1} \wedge \cdots \wedge \mathrm{H}_{S_c}$ for some (not necessarily distinct) subsets $S_1, \ldots, S_c$. In this case, one verifies that $c_1(\underline{\mathcal{L}}_{B(\mathrm{M}^\perp)}) = \sum_{i=1}^c \underline{h}_{S_i} \in \underline{A}^\bullet(\mathrm{M})$. Thus, Theorem 5.4 applies to all co-transversal matroids.

Co-transversal matroids are realizable over an infinite field of arbitrary characteristic, so we could have used [Spe09, Proposition 3.3] or Remark 5.9 below. We now construct an infinite family of matroids to which Theorem 5.4 applies but which are not realizable over a field of characteristic 0, as follows. We will use the notion of *principal extensions*, whose definition and properties can be found in [Oxl92, §7.2].

Lemma 5.6. Let M be a loopless matroid on E, and fix a nonempty flat G. Denote by $\mathrm{M}' = \mathrm{M} +_G \star$ the principal extension of M by G. Then, writing

$$c_1(\underline{\mathcal{L}}_{B(\mathrm{M}^\perp)}) = \sum_{F \text{ a flat of } \mathrm{M}} c_F \underline{h}_F \in \underline{A}^\bullet(\mathrm{M}),$$

we have that the expression for $c_1(\underline{\mathcal{L}}_{B((\mathrm{M}')^\perp)}) \in \underline{A}(\mathrm{M}')$ is roughly "obtained by increasing the coefficient of c_G by 1," or precisely,

$$c_1(\underline{\mathcal{L}}_{B((\mathrm{M}')^\perp)}) = \underline{h}_{G\cup\star} + \sum_{F\supseteq G} c_F \underline{h}_{F\cup\star} + \sum_{F\not\supseteq G} c_F \underline{h}_F.$$

Proof. We use the fact that $c_1(\underline{\mathcal{L}}_{B(\mathrm{M}^\perp)}) = \sum_F \mathrm{rk}_\mathrm{M}(F) z_F$ (see [BEST23, Section 2.7 and Remark III.1]), so that the coefficients c_F are defined by the property that $\sum_{F'\subseteq F} c_{F'} = \mathrm{rk}_\mathrm{M}(F)$ for all flats F of M.

Now, we recall that the set of flats of M' is partitioned into three categories [Oxl92, Corollary 7.2.5]:

(i) $\{F : F \text{ a flat of } \mathrm{M} \text{ such that } F \not\supseteq G\}$, in which case $\mathrm{rk}_{\mathrm{M}'}(F) = \mathrm{rk}_\mathrm{M}(F)$,
(ii) $\{F\cup\star : F \text{ a flat of } \mathrm{M} \text{ such that } F \supseteq G\}$, in which case $\mathrm{rk}_{\mathrm{M}'}(F\cup\star) = \mathrm{rk}_\mathrm{M}(F)$, and
(iii) $\{F\cup\star : F \text{ a flat of } \mathrm{M} \text{ such that } F \not\supseteq G$ and F is not covered by an element in $[G,E]\}$, in which case $\mathrm{rk}_{\mathrm{M}'}(F\cup\star) = \mathrm{rk}_\mathrm{M}(F) + 1$.

Thus, on $\underline{A}^\bullet(\mathrm{M}')$, since $\underline{h}_\star = 0$ so that $-z_{E\cup\star} = \sum_{\emptyset\subseteq F\subsetneq E} z_{F\cup\star}$, we have

$$\underline{h}_{G\cup\star} = \sum_{\emptyset\subseteq F\subsetneq E} z_{F\cup\star} + \sum_{G\subseteq F\subsetneq E} -z_{F\cup\star} = \sum_{F\not\supseteq G} z_{F\cup\star}.$$

The claimed expression for $c_1(\underline{\mathcal{L}}_{B((\mathrm{M}')^\perp)})$ in all three cases of flats now follows, as the above expression for $\underline{h}_{G\cup\star}$ contributes only to the case (iii) and not to cases (i) or (ii). Explicitly, we have:

(i) In this case, the coefficient of z_F is $\sum_{F'\subseteq F} c_{F'} = \mathrm{rk}_\mathrm{M}(F) = \mathrm{rk}_{\mathrm{M}'}(F)$.
(ii) In this case, the coefficient of $z_{F\cup\star}$ is again $\sum_{F'\subseteq F} c_{F'} = \mathrm{rk}_\mathrm{M}(F) = \mathrm{rk}_{\mathrm{M}'}(F\cup\star)$.
(iii) In this case, the coefficient of $z_{F\cup\star}$ is $1 + \sum_{F'\subseteq F} c_{F'} = 1 + \mathrm{rk}_\mathrm{M}(F) = \mathrm{rk}_{\mathrm{M}'}(F\cup\star)$. □

Given any matroid M, repeatedly applying the lemma provides a method to construct a matroid $\widetilde{\mathrm{M}}$ for which Theorem 5.4 applies. Moreover, a matroid is realizable over an infinite field $\Bbbk$ if and only if its principal extensions are realizable over the same field $\Bbbk$. Thus, the matroid $\widetilde{\mathrm{M}}$ has the same realizability property as M over infinite fields. In particular, the lemma produces infinite families of matroids that are not realizable or realizable only over certain positive characteristics for which Theorem 5.4 applies.

5.2. **Examples and problems.** We present several cases in which Conjecture 4.8 holds.

Example 5.7. When M is the boolean matroid, the discussion in Remark 4.7 implies that $h^*(\mathrm{M}, \underline{\mathcal{L}}_{B(\mathrm{P})})$ is the usual h^*-vector of the base polytope $B(\mathrm{P})$, and hence is nonnegative. Moreover, because base polytopes of polymatroids have the property that every lattice point in $kB(\mathrm{P})$ is a sum of k lattice points in $B(\mathrm{P})$ (see [Wel76, Chapter 18.6, Theorem 3]), $h^*(\mathrm{M}, \underline{\mathcal{L}}_{B(\mathrm{P})})$ is a Macaulay vector.

Example 5.8. Let $\nabla = \mathrm{Conv}(\{(0,1,\dots,1),(1,0,1,\dots,1),\dots,(1,1,\dots,1,0)\})$, the base polytope of the uniform matroid of corank 1, so $c_1(\underline{\mathcal{L}}_\nabla) \in \underline{A}^1(\mathrm{M})$ is the class usually denoted β. Then [LLPP, Lemma

8.5] implies that

$$\underline{\chi}(\mathrm{M}, \underline{\mathcal{L}}_{\nabla}^{\otimes t}) = \sum_i f_{r-1-i}(BC_>(\mathrm{M})) \binom{t}{r-1-i},$$

where $f_j(BC_>(\mathrm{M}))$ is the number of j-dimensional faces of the reduced broken circuit complex of M under any ordering $>$. As $\binom{t}{r-1-i} = \sum_{j=0}^{i} (-1)^j \binom{i}{j}\binom{t+i}{r-1}$, we may express the Snapper polynomial in terms of the h-vector of the reduced broken circuit complex:

$$\underline{\chi}(\mathrm{M}, \underline{\mathcal{L}}_{\nabla}^{\otimes t}) = \sum_i h_{r-1-i}(BC_>(\mathrm{M})) \binom{t+i}{r-1}.$$

Comparing this with the definition of $h^*(\mathrm{M}, \underline{\mathcal{L}}_\nabla)$, we see that $h_i(BC_>(\mathrm{M})) = h_i^*(\mathrm{M}, \underline{\mathcal{L}}_\nabla)$. By [Bjö92], the reduced broken circuit complex is shellable and hence Cohen–Macaulay, so its h-vector is a Macaulay vector. This argument is closely related to [PS06].

Example 5.9. Let M be a connected matroid which has a realization $L \subseteq \Bbbk^E$ over a field of characteristic 0. Then the base polytope of the dual matroid $B(\mathrm{M}^\perp)$ is full dimensional. It follows from [BF22, Theorem 5.1] and [BEST23, Theorem 7.10] that, for all $k > 0$, the restriction map $H^0(\underline{X}_E, \underline{\mathcal{L}}_{B(\mathrm{M}^\perp)}^{\otimes k}) \to H^0(W_L, \underline{\mathcal{L}}_{B(\mathrm{M}^\perp)}^{\otimes k})$ is surjective and that $H^i(W_L, \underline{\mathcal{L}}_{B(\mathrm{M}^\perp)}^{\otimes k}) = 0$ for $i > 0$. Therefore, by [Wel76, Chapter 18.6, Theorem 3], the ring

$$R^\bullet := \bigoplus_{k \geq 0} H^0(W_L, \underline{\mathcal{L}}_{B(\mathrm{M}^\perp)}^{\otimes k})$$

is generated in degree 1. This implies that $\operatorname{Proj} R^\bullet$ is the image of W_L under the complete linear system of $\underline{\mathcal{L}}_{B(\mathrm{M}^\perp)}$ which is the *space of visual contours* of L, and so $\operatorname{Proj} R^\bullet$ has rational singularities [Tev07, Theorem 1.4 and 1.5]. In particular,

$$H^i(W_L, \underline{\mathcal{L}}_{B(\mathrm{M}^\perp)}^{\otimes k}) = H^i(\operatorname{Proj} R^\bullet, \mathcal{O}(k))$$

for all i and k. Because $B(\mathrm{M}^\perp)$ is full dimensional, the line bundle $\underline{\mathcal{L}}_{B(\mathrm{M}^\perp)}$ is nef and big. By the Kawamata–Viehweg vanishing theorem, $H^i(W_L, \underline{\mathcal{L}}_{B(\mathrm{M}^\perp)}^{\otimes k}) = H^i(\operatorname{Proj} R^\bullet, \mathcal{O}(k)) = 0$ for $k < 0$ and $i < \dim W_L$. As W_L is rational, $H^i(W_L, \mathcal{O}_{W_L}) = 0$ for $i > 0$. Then Proposition 4.3 implies that $R^\bullet$ is Cohen–Macaulay, and so $h^*(\mathrm{M}, \underline{\mathcal{L}}_{B(\mathrm{M}^\perp)})$ is a Macaulay vector.

Lastly, we discuss the valuativity of h^*-vectors of matroids and conjecture a monotonicity property for them. When P is full dimensional, Theorem 4.10 implies that the degree of the Snapper polynomial of $\underline{\mathcal{L}}_{B(\mathrm{P})}$ depends only on the rank of M. From the formula for $h^*(\mathrm{M}, \underline{\mathcal{L}}_{B(\mathrm{P})})$ and the valuativity of $\underline{\chi}(\mathrm{M}, \underline{\mathcal{L}}_{B(\mathrm{P})}^{\otimes k})$ for fixed P and k [LLPP, Lemma 6.4], we obtain the following corollary.

Corollary 5.10. Let P be a polymatroid such that $B(\mathrm{P})$ is full dimensional. Then the function that assigns a loopless matroid M to $h^*(\mathrm{M}, \underline{\mathcal{L}}_{B(\mathrm{P})})$ is valuative.

However, the degree of the Snapper polynomial of $\underline{\mathcal{L}}_{B(\mathrm{P})}$ on M is not in general determined by the rank of M. The numerical dimension of $\underline{\mathcal{L}}_{B(\mathrm{P})}$ is also not determined by the rank of M.

Question 5.11. Let P be a polymatroid. What is the degree of the Snapper polynomial of $\underline{\mathcal{L}}_{B(\mathrm{P})}$ on M? Is it equal to the numerical dimension of $\underline{\mathcal{L}}_{B(\mathrm{P})}$? Is $h^*(\mathrm{M}, \underline{\mathcal{L}}_{B(\mathrm{P})})$ valuative?

We also conjecture the following monotonicity property for h^*-vectors, inspired by Stanley's monotonicity result for h^*-vectors of polytopes [Sta93], which implies the following conjecture when M is the boolean matroid.

Conjecture 5.12. Let $\mathrm{P}_1, \mathrm{P}_2$ be polymatroids with $B(\mathrm{P}_1) \subseteq B(\mathrm{P}_2)$. Then for any loopless matroid M, $h_i^*(\mathrm{M}, \underline{\mathcal{L}}_{B(\mathrm{P}_1)}) \leq h_i^*(\mathrm{M}, \underline{\mathcal{L}}_{B(\mathrm{P}_2)})$ for all i.

If the degree of the Snapper polynomial of $\mathcal{L}$ is $\mathrm{rk}(\mathrm{M}) - 1$, then $\sum h_i^*(\mathrm{M}, \mathcal{L}) = \underline{\deg}_{\mathrm{M}}(c_1(\mathcal{L})^{r-1})$, so the following result gives evidence for Conjecture 5.12.

Proposition 5.13. Let $\mathrm{P}_1, \mathrm{P}_2$ be polymatroids with $B(\mathrm{P}_1) \subseteq B(\mathrm{P}_2)$. Then

$$\underline{\deg}_{\mathrm{M}}(c_1(\underline{\mathcal{L}}_{B(\mathrm{P}_1)})^{r-1}) \leq \underline{\deg}_{\mathrm{M}}(c_1(\underline{\mathcal{L}}_{B(\mathrm{P}_2)})^{r-1}).$$

Proof. Because $B(\mathrm{P}_1) \subseteq B(\mathrm{P}_2)$, the difference of the divisor class in $A^1(\underline{X}_E)$ corresponding to $B(\mathrm{P}_2)$ with the divisor class corresponding to $B(\mathrm{P}_1)$ is an effective divisor class, see [BEST23, Section 2.7]. Then,

$$\begin{aligned} &c_1(\underline{\mathcal{L}}_{B(\mathrm{P}_2)})^{r-1} - c_1(\underline{\mathcal{L}}_{B(\mathrm{P}_1)})^{r-1} \\ &= (c_1(\underline{\mathcal{L}}_{B(\mathrm{P}_2)}) - c_1(\underline{\mathcal{L}}_{B(\mathrm{P}_1)})) \cdot (c_1(\underline{\mathcal{L}}_{B(\mathrm{P}_2)})^{r-2} + c_1(\underline{\mathcal{L}}_{B(\mathrm{P}_2)})^{r-3} c_1(\underline{\mathcal{L}}_{B(\mathrm{P}_1)}) + \cdots + c_1(\underline{\mathcal{L}}_{B(\mathrm{P}_1)})^{r-2}). \end{aligned}$$

By Proposition 4.12, the degree of this class is nonnegative. ☐

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