KAZHDAN–LUSZTIG POLYNOMIALS OF BRAID MATROIDS

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Abstract. We provide a purely combinatorial interpretation of the Kazhdan–Lusztig polynomial of the matroid arising from the braid arrangement of type A_{n-1}, which gives an interpretation of the intersection cohomology Betti numbers of the reciprocal plane of the braid arrangement. Moreover, we prove an equivariant version of this result. The key combinatorial object is a class of matroids arising from series-parallel networks. As a consequence, we prove a conjecture of Elias, Proudfoot and Wakefield on the top coefficient of Kazhdan–Lusztig polynomials of braid matroids, and provide explicit generating functions for their Kazhdan–Lusztig and Z-polynomials.

1. Introduction

1.1. **Overview.** Motivated by the classical theory of Kazhdan–Lusztig polynomials for Coxeter groups in representation theory, Elias, Proudfoot and Wakefield initiated the study of an analogous theory for matroids in [EPW16]. There is a unique way to associate to all loopless matroids M a polynomial $P_{\mathsf{M}}(t) \in \mathbb{Z}[t]$ satisfying the following properties:

(i) If $\operatorname{rk}(\mathsf{M}) = 0$, then $P_{\mathsf{M}}(t) = 1$.
(ii) If $\operatorname{rk}(\mathsf{M}) > 0$, then $\deg P_{\mathsf{M}}(t) < \frac{1}{2}\operatorname{rk}(\mathsf{M})$.
(iii) For every matroid M, the polynomial

$$Z_{\mathsf{M}}(t) = \sum_{F \in \mathscr{L}(\mathsf{M})} t^{\operatorname{rk}(F)} P_{\mathsf{M}/F}(t)$$

is palindromic[1] of degree $\operatorname{rk}(\mathsf{M})$.

Here $\mathscr{L}(\mathsf{M})$ is the lattice of flats of M. The polynomial $P_{\mathsf{M}}(t)$ is called the *Kazhdan–Lusztig polynomial* of M, and the polynomial $Z_{\mathsf{M}}(t)$ is called the *Z-polynomial* of M. It is not difficult to prove that the three properties above uniquely specify $P_{\mathsf{M}}(t)$ and $Z_{\mathsf{M}}(t)$; the existence of these polynomials is established by Proudfoot, Xu and Young [PXY18].

Kazhdan–Lusztig polynomials and Z-polynomials of matroids display remarkable properties. For example, the coefficients of the Kazhdan–Lusztig polynomial and Z-polynomial of a matroid are non-negative, and the coefficients of the Z-polynomial are unimodal [BHM+20, Theorem 1.2]. The lattice of flats of a matroid is modular if and only if its Kazhdan–Lusztig polynomial is 1 [EPW16, Proposition 2.14]. Kazhdan–Lusztig polynomials and Z-polynomials were conjectured in [GPY17b, Conjecture 3.2] and [PXY18, Conjecture 5.1] to be real rooted. Much work in the literature is devoted to the study of these polynomials for specific classes of matroids, such as uniform matroids [GLX+21, GX21, Pro19], matroids of corank 2 [FS22], sparse paving [LNR21, FV22] and paving matroids [FNV22].

When a matroid M is realized by a linear subspace $L \subseteq \mathbb{C}^n$, the Kazhdan–Lusztig polynomial and Z-polynomial of M admit algebro-geometric interpretations. The Z-polynomial of M is the

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[1] A polynomial $f(t)$ is *palindromic* if $f(t) = t^d f(t^{-1})$, where $d = \deg f$.

Poincaré polynomial of the intersection cohomology of the closure of L in $(\mathbb{P}^1)^n$, and the Kazhdan–Lusztig polynomial is the Poincaré polynomial of the stalk of the intersection cohomology at the point $(\infty, \dots, \infty)$ of $(\mathbb{P}^1)^n$. Equivalently, the Kazhdan–Lusztig polynomial is the Poincaré polynomial of the intersection cohomology of the *reciprocal plane* of L: the closure of the image of $\mathbb{P}L$ under the Cremona transform on $\mathbb{P}^{n-1}$. This proves the non-negativity of the coefficients when M is realizable; the non-negativity for arbitrary matroids was proved by Braden, Huh, Matherne, Proudfoot and Wang [BHM+20] by combinatorializing this description of the Kazhdan–Lusztig polynomial and Z-polynomial.

When the action of a subgroup Γ of $\mathfrak{S}_n$ preserves $L \subseteq \mathbb{C}^n$, there is an action of Γ on the intersection cohomology of the closure of L in $(\mathbb{P}^1)^n$ and on the intersection cohomology of the reciprocal plane of L. By taking the Γ-equivariant Poincaré polynomial, we may then construct versions of the Kazhdan–Lusztig polynomial and Z-polynomial which have coefficients in the ring of virtual representations of Γ, $\mathrm{VRep}(\Gamma)$. This can be extended to all loopless matroids with an action of a finite group: There is a unique way of associating to each action $\Gamma \curvearrowright \mathsf{M}$ of a group on a loopless matroid[2] a polynomial $P_{\mathsf{M}}^{\Gamma}(t) \in \mathrm{VRep}(\Gamma)[t]$ in such a way that

(i) If $\mathrm{rk}(\mathsf{M}) = 0$, then $P_{\mathsf{M}}^{\Gamma}(t) = \mathrm{tr}$ is the trivial representation.
(ii) If $\mathrm{rk}(\mathsf{M}) > 0$, then $\deg P_{\mathsf{M}}^{\Gamma}(t) < \frac{1}{2}\mathrm{rk}(\mathsf{M})$.
(iii) For every M, the polynomial

$$Z_{\mathsf{M}}^{\Gamma}(t) = \sum_{[F] \in \mathscr{L}(\mathsf{M})/\Gamma} t^{\mathrm{rk}(F)} \operatorname{Ind}_{\Gamma_F}^{\Gamma} P_{\mathsf{M}/F}^{\Gamma_F}(t)$$

is palindromic of degree $\mathrm{rk}(\mathsf{M})$.

Here, Γ_F is the stabilizer of F and $\mathscr{L}(\mathsf{M})/\Gamma$ denotes the quotient of the lattice of flats of M by the group action Γ. $P_{\mathsf{M}}^{\Gamma}(t)$ is called the *equivariant Kazhdan–Lusztig polynomial*, and $Z_{\mathsf{M}}^{\Gamma}(t)$ is called the *equivariant Z-polynomial*. See [GPY17a, Pro21].

1.2. **The Kazhdan–Lusztig theory of braid matroids.** One notable example of matroids is given by the *braid matroids*. The n-th braid matroid K_n can be defined as the graphical matroid associated to the complete graph on n vertices; as such, the symmetric group $\mathfrak{S}_n$ acts on K_n by permuting the vertices of the complete graph. The n-th braid matroid has rank $n-1$. The n-th braid matroid is realized by the *braid arrangement* of hyperplanes perpendicular to the roots of type A_{n-1}. In light of this, K_n is closely related to many combinatorial and geometric objects, such as the partition lattice Π_n and the moduli space of genus zero curves with marked points [Kap93].

Kazhdan–Lusztig polynomials of braid matroids have been extensively studied, both equivariantly and non-equivariantly. For other graphic matroids, such as the matroid induced by a cycle on n edges $\mathsf{U}_{n-1,n}$ [PWY16], wheel and fan graphs [LXY22], partial saw graphs [BV20], and thagomizer graphs [Ged17, XZ19], there are fairly simple formulas for the Kazhdan–Lusztig polynomial.

Precisely this lack of understanding of braid matroids has been the driving force for many significant developments in the theory; for example, the authors of [PXY18] state that their "main motivation" for introducing the Z-polynomials of matroids was to compute the Kazhdan–Lusztig polynomials of braid matroids.

To calculate $P_{\mathsf{K}_n}(t)$ and $Z_{\mathsf{K}_n}(t)$ using the definitions is in general a heavy computational task. The main result of Karn and Wakefield in [KW19] provides an explicit (i.e., non-recursive), but

[2]The action is required to send flats to flats.

fairly complicated expression for the coefficients of $P_{\mathsf{K}_n}(t)$. In practice, the fastest known way to compute both $P_{\mathsf{K}_n}(t)$ and $Z_{\mathsf{K}_n}(t)$ is provided by the recurrence derived in [PXY18, Corollary 4.2].

The $\mathfrak{S}_n$-equivariant Kazhdan–Lusztig polynomial of K_n has also been studied. In [GPY17a], the authors computed the linear term of $P^{\mathfrak{S}_n}_{\mathsf{K}_n}(t)$ and gave a function equation which is satisfied by the generating function of $P^{\mathfrak{S}_n}_{\mathsf{K}_n}(t)$. In [PY17], the authors studied $P^{\mathfrak{S}_n}_{\mathsf{K}_n}(t)$ using tools from representation stability; this machinery allowed them to give a partial description of the poles of the generating function and bound which irreducible representations of $\mathfrak{S}_n$ can appear. See [Tos22] for a strengthening of these results.

A further conjecture posed originally by Elias, Proudfoot and Wakefield [EPW16, Section A] asserts that the leading coefficient of $P_{\mathsf{K}_{2n}}(t)$ counts labelled triangular cacti on $2n-1$ nodes. Prior to the present paper, not only did this conjecture remained elusive; the problem of formulating an analogous statement for the leading coefficient for $P_{\mathsf{K}_{2n-1}}(t)$ or other coefficients remained open.

Theorem 1.1 below provides a concrete combinatorial interpretation for all the coefficients of both $P_{\mathsf{K}_n}(t)$ and $Z_{\mathsf{K}_n}(t)$ for all values of n.

1.3. **Main results.** A matroid is *quasi series-parallel* if it does not contain $\mathsf{U}_{2,4}$ or K_4 as a minor. Equivalently, a matroid is quasi series-parallel if it is a direct sum of loops, coloops, and series-parallel matroids on a ground set of size at least 2. The first of our main theorems can be stated as follows.

Theorem 1.1 *Let $\mathscr{A}(n,r)$ denote the set of all quasi series-parallel matroids on $[n]$ of rank r and, let $\mathscr{S}(n,r)$ denote the set of simple quasi series-parallel matroids. Then*

$$[t^i]\, P_{\mathsf{K}_n}(t) = |\mathscr{S}(n-1, n-1-i)|,$$
$$[t^i]\, Z_{\mathsf{K}_n}(t) = |\mathscr{A}(n-1, n-1-i)|.$$

In fact, we will derive the preceding statement from a stronger result. We provide a description of the equivariant Kazhdan–Lusztig polynomial and the equivariant Z-polynomial of all braid matroids with respect any $\mathfrak{S}_{n-1}$ subgroup of $\mathfrak{S}_n$.

Theorem 1.2 *Choose an $\mathfrak{S}_{n-1}$ subgroup of $\mathfrak{S}_n$. Then*

(i) *The $\mathfrak{S}_{n-1}$ equivariant Kazhdan–Lusztig polynomial of K_n has t^i coefficient given by the permutation representation of $\mathfrak{S}_{n-1}$ on $\mathscr{S}(n-1, n-1-i)$.*
(ii) *The $\mathfrak{S}_{n-1}$ equivariant Z-polynomial of K_n has t^i coefficient given by the permutation representation of $\mathfrak{S}_{n-1}$ on $\mathscr{A}(n-1, n-1-i)$.*

An immediate consequence of our main results and the hard Lefschetz theorem for intersection cohomology is that the rank-indexed sequence of the number of quasi series-parallel matroids on $[n]$ is unimodal. More so, it is a γ-positive sequence, by [FMSV22, Theorem 1.8].

We also use Theorem 1.1 to confirm the conjecture in [EPW16, Section A] on the leading coefficient of $P_{\mathsf{K}_{2n}}(t)$; see Proposition 2.11. In [PY17, Remark 6.3], Proudfoot and Young note that this conjecture shows the existence of a certain pole of the generating function for the i-th coefficient of $P_{\mathsf{K}_n(t)}$. We also give an expression for the leading term of $P_{\mathsf{K}_{2k+1}}(t)$; see Proposition 2.13.

Since the study of several objects related to (quasi) series-parallel matroids is a recurring problem in enumerative combinatorics, we are able to present explicit (but complicated) generating functions for Kazhdan–Lusztig and Z-polynomials of braid matroids. In particular, this provides an additional tool to study asymptotics of the Betti numbers and the total dimensions of intersection cohomologies of braid matroids.

2. Background

Throughout this paper we shall assume familiarity with the basic properties of matroids. In particular, we mostly follow the notation and terminology of Oxley [Oxl11].

2.1. Series-parallel matroids.

Series-parallel matroids are a prominent family of matroids which pervade graph theory and the theory of electrical networks. Given the variety of sources, coming from both graph theory and enumerative combinatorics, that define similar but different objects under the name "series-parallel", we will include a recapitulation of the basic terminology and background of this topic in matroid theory, following [Oxl11, Section 5.4].

Two operations play an important role in the construction of these matroids. Assume that M and M' are matroids on E and E' respectively. We say that M' is a *parallel extension* of M if there is an element $e \in E'$ which belongs to a circuit of size 2 in M' and satisfies that $\mathsf{M}'/e = \mathsf{M}$. Dually, we say that M' is a *series extension* of M if there is an element $e \in E'$ which belongs to a cocircuit of size 2 in M' and satisfies $\mathsf{M}'/e = \mathsf{M}$. In particular, notice that, if we take $\mathsf{M} = \mathsf{U}_{1,1}$ and $\mathsf{M}' = \mathsf{U}_{2,2}$, then M' is *not* a series extension of M.

By definition, a *series-parallel matroid* is a matroid that is obtained from a single loop or a single coloop via a finite sequence of series or parallel extensions[3]. We can deduce the following elementary conclusions from the definition of these matroids.

- Series-parallel matroids are connected matroids.
- The rank of a series-parallel matroid M is equal to one plus the number of series extensions performed in the construction of M.

A classical paper by Brylawski [Bry71] establishes the following list of equivalences, reformulated also by Oxley in [Oxl82, Theorem 2.1].

Proposition 2.1 ([Bry71, Theorem 7.6]) *Let* M *be a matroid. The following are equivalent.*

(i) M *is a series-parallel matroid.*
(ii) $\mathsf{M} \cong \mathsf{U}_{0,1}$ *or* $\beta(\mathsf{M}) = 1$.
(iii) M *is connected and does not contain any minors isomorphic to* K_4 *or* $\mathsf{U}_{2,4}$.

Here $\beta(\mathsf{M})$ is the β*-invariant* of M introduced by Crapo [Cra67]. Notice that $\beta(\mathsf{M}) = \beta(\mathsf{M}^*)$ whenever $\mathsf{M} \not\cong \mathsf{U}_{0,1}, \mathsf{U}_{1,1}$. On the other hand, it follows either by the definition or the above result that M is series-parallel if and only if M^* is series-parallel — even in the case in which M is either a loop or a coloop.

Example 2.2 There are exactly 4 isomorphism classes of series-parallel matroids on 6 elements of rank 3. They are induced by the graphs depicted in Figure 1. Notice that the first of the four matroids is isomorphic to 20 matroids on $[6]$, the second one is isomorphic to 180 matroids on $[6]$,

[3] This follows Oxley's construction. Notice that starting from a loop it is actually impossible to perform any series or parallel extension. In particular, all series-parallel matroids on a ground set of size at least 2 are actually obtained via series and parallel extensions starting from the matroid $\mathsf{U}_{1,2}$.

(A) Series-parallel matroids on $[n]$ of rank k

$k\backslash n =$	1	2	3	4	5	6	7
0	1	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2		0	1	6	25	90	301
3			0	1	25	290	2450
4				0	1	90	2450
5					0	1	301
6						0	1
7							0

(B) Quasi series-parallel matroids on $[n]$ of rank k

$k\backslash n =$	1	2	3	4	5	6	7
0	1	1	1	1	1	1	1
1	1	3	7	15	31	63	127
2		1	7	35	155	651	2667
3			1	15	155	1365	10941
4				1	31	651	10941
5					1	63	2667
6						1	127
7							1

TABLE 1. Enumeration of series-parallel and quasi series-parallel matroids.

the last two are each isomorphic to 45 matroids on $[6]$. These add up to $20 + 180 + 45 + 45 = 290$ series-parallel matroids with ground set $\{1, 2, 3, 4, 5, 6\}$ of rank 3 (see Table 1).

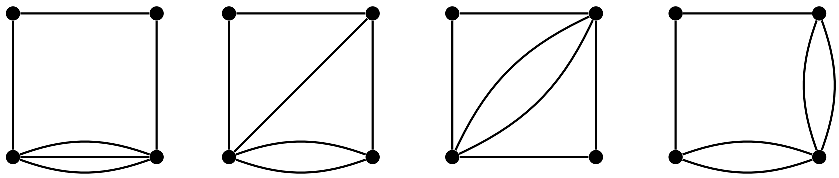

FIGURE 1. Isomorphism classes of series-parallel matroids on $[6]$ of rank 3.

Proposition 2.3 *Let $\mathscr{C}(n, k)$ denote the set of series-parallel matroids on $[n]$ of rank k. Then, the bivariate generating function*

$$C(x, y) := \sum_{n=1}^{\infty} \sum_{k=0}^{n} |\mathscr{C}(n, k)| \frac{x^n}{n!} y^k,$$

is given by

$$C(x, y) = x(y + 1) + y \int \left[\left(\frac{1}{y} \log(1 + xy) + \log(1 + x) - x \right)^{\langle -1 \rangle} \right] dx. \tag{1}$$

Here the symbol $\langle -1 \rangle$ stands for the compositional inverse of the function inside the parenthesis with respect to the variable x, i.e., treating y as a constant.

Proof. The case $y = 1$ of the above formula reduces to a classical problem studied extensively in the literature; see [Sta99, Exercise 5.40] and references mentioned therein. The case of our interest was essentially carried out by Drake in his PhD thesis [Dra08, Example 1.5.1]. By relying on a combinatorial interpretation for Lagrange's inversion theorem, he establishes a generating function for "series-parallel networks" on a labelled ground set of edges according to the number of series extensions. Explicitly:

$$\left(\frac{1}{\alpha} \log(1 + \alpha x) + \frac{1}{\beta} \log(1 + \beta x) - x \right)^{\langle -1 \rangle} = x + (\alpha + \beta) \frac{x^2}{2!} + (\alpha^2 + 6\alpha\beta + \beta^2) \frac{x^3}{3!} +$$

$$(\alpha^3 + 25\alpha^2\beta + 25\alpha\beta^2 + \beta^3)\frac{x^4}{4!} + \cdots$$

By taking $\alpha = y$ and $\beta = 1$ and integrating, we obtain:

$$\int \left(\frac{1}{y}\log(1+xy) + \log(1+x) - x\right)^{\langle -1\rangle} dx = \frac{x^2}{2!} + (y+1)\frac{x^3}{3!} + (y^2+6y+1)\frac{x^4}{4!} + (y^3+25y^2+25y+1)\frac{x^5}{5!} + \cdots$$

We must make a small correction by multiplying by y and adding $x(y+1)$. The term $x(y+1)$ comes from considering the two matroids with ground set of size 1. The reason these do not appear in Drake's formula is because, under the definition he uses for "series-parallel networks" as parenthesized expressions under an equivalence relation, these two matroids would come from "empty" expressions. $\square$

Remark 2.4 We do not know a way to simplify (1). We point out that the use of formulas for the antiderivative of an inverse function does not help to get rid of the integral symbol. On the other hand, equally complicated expressions can be deduced by using Lambert's W-function.

2.2. **Quasi series-parallel matroids.** Although the class of series-parallel matroids is closed under taking duals, it is not closed under taking direct sums or taking minors. The best way to resolve this issue is through the class of quasi series-parallel matroids.

Definition 2.5 Let M be a matroid. We say that M is a *quasi series-parallel matroid* if all the connected components of M are series-parallel matroids.

Although the name "quasi series-parallel matroid" is new, this class of matroids has actually appeared before in the literature[4].

Proposition 2.6 *Let* M *be a matroid. The following are equivalent:*

(i) M *is a quasi series-parallel matroid.*
(ii) M^* *is a quasi series-parallel matroid.*
(iii) M *does not contain any minor isomorphic to* K_4 *or* $\mathsf{U}_{2,4}$.
(iv) M *is a binary gammoid.*
(v) M *is a regular gammoid.*
(vi) M *has branch-width smaller than or equal to* 2.

The above result is essentially a restatement of [Oxl11, Theorem 10.4.8] and [Oxl11, Proposition 14.2.6]; we refer to Oxley's book for the undefined terminology.

Example 2.7 There are exactly 6 isomorphism classes of quasi series-parallel matroids on 4 elements with rank 2. They are depicted in Figure 2. The number of different matroids on $[4]$ isomorphic to each of these matroids are $6, 12, 6, 4, 3$, and 4 respectively.

This amounts to $6 + 12 + 6 + 4 + 3 + 4 = 35$ quasi series-parallel matroids with ground set $\{1, 2, 3, 4\}$ of rank 2 (see Table 1).

[4]In fact, Seymour [Sey95] calls "series-parallel matroids" what we have called "quasi series-parallel matroids". We emphasize once again that we are following Oxley's convention.

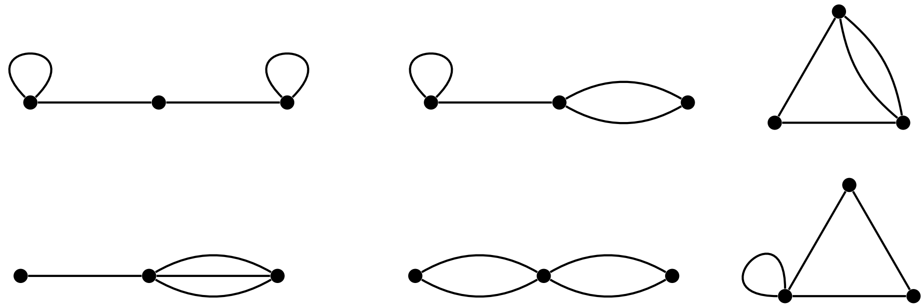

FIGURE 2. Isomorphism classes of quasi series-parallel matroids on [4] of rank 2.

A recurring theme within combinatorics consists of passing from the enumeration of connected labelled structures to the enumeration of all labelled structures (whether connected or not). In particular, the theory of exponential generating functions is useful to this end. Moreover, in the case of our interest, the formula of equation (1) enumerates all connected quasi series-parallel matroids. If we want to enumerate all quasi series-parallel matroids, by a standard technique (see [Sta99, Section 5.1] or [FS09, p. 148]) it suffices to compose this function with an exponential.

Proposition 2.8 *Let $\mathscr{A}(n, k)$ denote the set of all quasi series-parallel matroids on $[n]$ of rank k. Then, the bivariate generating function*

$$A(x, y) := \sum_{n=1}^{\infty} \sum_{k=0}^{n} |\mathscr{A}(n, k)| \frac{x^n}{n!} y^k,$$

is given by

$$A(x, y) = \exp(C(x, y)), \tag{2}$$

where $C(x, y)$ is given as in equation (1).

2.3. **Simple quasi series-parallel matroids.** The last family of matroids playing a role in the statement of Theorem 1.1 is that of simple quasi series-parallel matroids.

Recall that a matroid M is *simple* if it loopless and contains no circuits of size 2. Two element i, j are said to be *parallel* if they are not loops and $\mathrm{rk}_{\mathsf{M}}(\{i, j\}) = 1$. The ground set of M is partitioned into subsets called *parallel classes*, which are the maximal subsets that consist of elements which are parallel to each other. The *simplification* of M, denoted $\overline{\mathsf{M}}$, is the matroid on the set of parallel classes of M with rank function

$$\mathrm{rk}_{\overline{\mathsf{M}}}(S) = \mathrm{rk}_{\mathsf{M}}\left(\bigcup_{T \in S} T\right).$$

Example 2.9 There are exactly two isomorphism classes of simple quasi series-parallel matroids on 7 elements of rank 4. They are depicted in Figure 3. There are 630 matroids on [7] isomorphic to the matroid on the left, and there are 105 for the matroid on the right. These add up to 735 simple quasi series-parallel matroids on [7] of rank 4 (see Table 2).

Proposition 2.10 *A simple quasi series-parallel matroid on $[n-1]$ has rank greater than $\frac{n-1}{2}$.*

Proof. We may reduce to the case of simple (connected) series-parallel matroids. Then the result follows from the observation that in order to obtain a simple matroid from a sequence of series

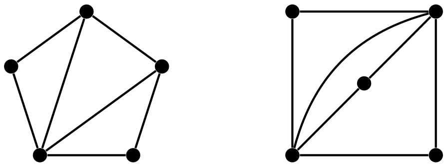

FIGURE 3. Isomorphism classes of simple quasi series-parallel matroids on [7] of rank 4.

and parallel extensions of $\mathsf{U}_{1,1}$, we must have done at least as many series extensions as parallel extensions. □

(A) Simple quasi series-parallel matroids on $[n]$ of rank k

$k\backslash n=$	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0
2		1	1	0	0	0	0	0
3			1	5	15	0	0	0
4				1	16	175	735	0
5					1	42	1225	16065
6						1	99	6769
7							1	219
8								1

TABLE 2. Enumeration of simple quasi series-parallel matroids.

We now study simple quasi series-parallel matroids of smallest possible rank. We begin with the case of ground sets of odd size. By Proposition 2.10, a simple quasi series-parallel matroid of rank k on $[2k-1]$ is connected, and thus a series-parallel matroid. A *triangular cactus* is a connected graph where every edge is contained in a unique cycle which is a triangle.

Proposition 2.11 *There is a bijection between simple series-parallel matroids of rank k on* $[2k-1]$ *and triangular cacti with vertex set* $[2k-1]$.

Before giving the bijection, we observe that both triangular cacti and rank k simple series-parallel matroids on $[2k-1]$ are built inductively. The triangles in a triangular cactus are arranged in a tree-like fashion, so every triangular cactus contains a triangle which has at least two vertices of degree 2. When we delete those two vertices, we get a triangular cactus on a smaller ground set. Every rank k simple series-parallel matroid on $[2k-1]$ is a series extension of a series-parallel matroid of rank $k-1$ on ground set of size $2k-2$. By Proposition 2.10, such a matroid cannot be simple, and so it is a parallel extension of a rank $k-1$ series-parallel matroid on a ground set of size $2k-3$.

Proof. To a simple series-parallel matroid M of rank k on $[2k-1]$, we associate the graph $T(\mathsf{M})$ with vertex set $[2k-1]$ and edges (a,b) if $\{a,b\}$ is contained in a 3-element circuit of M. Note that if we do a parallel extension and then a series extension at $i \in [2k-1]$ to M to obtain $\widetilde{\mathsf{M}}$, we add a triangle to $T(\mathsf{M})$ with vertices $i, 2k$, and $2k+1$. In particular, $T(\widetilde{\mathsf{M}})$ is a triangular cactus by induction.

To a triangular cactus G with vertex set $[2k-1]$, we build a graph whose edge set is $[2k-1]$ by adding a triangle with edges $\{a, b, c\}$ for each triangle with vertices $\{a, b, c\}$ in G. We obtain a matroid $S(\mathsf{M})$ by taking the graphical matroid of this graph. Note that G is obtain from a triangular cactus G' on a vertex set of size $2k-3$ by adding a triangle, and $S(G)$ is obtained from $S(G')$ by doing a parallel extension and then a series extension. So $S(G)$ is a rank k simple series-parallel matroid by induction. Similarly, we see by induction that $T(S(G)) = G$.

To check that $S(T(\mathsf{M})) = \mathsf{M}$, it suffices to check that every element of M is contained in a 3-element circuit. If $a \in [2k-1]$ is not contained a 3-element circuit, then M/a is a simple series-parallel matroid of rank $k-1$ on $[2k-2]$, which is impossible by Proposition 2.10. □

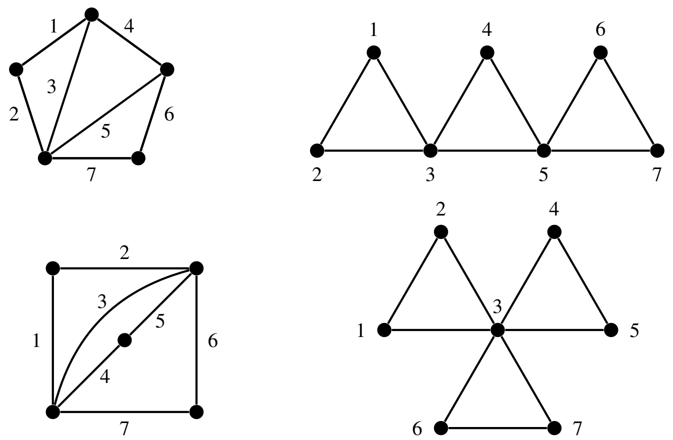

FIGURE 4. Two simple series-parallel matroids of rank 4 on [7] and their corresponding triangular cacti.

In [BL18], the authors show that the number of triangular cacti with vertex set $[2k-1]$ is $(2k-3)!!(2k-1)^{k-2}$. We therefore have the following result, which together with Theorem 1.1 proves a conjecture stated in [EPW16, Section A].

Corollary 2.12 *The number of simple quasi series-parallel matroids of rank k on $[2k-1]$ is equal to $(2k-3)!!(2k-1)^{k-2}$.*

Let E_k be the number of simple series-parallel matroids of rank $k+1$ on $[2k]$. Then

$$(E_1, E_2, \dots) = (0, 1, 75, 9345, 1865745, 554479695, 231052877055, 128938132548225, \dots).$$

We do not know a simple expression for E_k; the E_k can have large prime factors. After we prove Theorem 1.1, the following result will give an expression for the leading term of $P_{\mathsf{K}_{2k+1}}(t)$ in terms of E_k.

Proposition 2.13 *The number of simple quasi series-parallel matroids of rank $k+1$ on $[2k]$ is equal to*

$$E_k + \frac{1}{2}\sum_{a=0}^{k-1}\binom{2k}{2a+1}(2a-1)!!(2k-2a-3)!!(2a+1)^{a-1}(2k-2a-1)^{k-a-2}.$$

Proof. By Proposition 2.10, a simple quasi series-parallel matroid of rank $k+1$ on a ground set of size $2k$ can have at most two connected components. The number with one connected component is exactly E_k. If there are two connected components, then Proposition 2.10 implies that one connected component must have size $2a+1$ and rank $a+1$ for some a, and the other component must have size $2(k-a)-1$ and rank $k-a$. Proposition 2.11 then implies the result. □

Whenever $\mathscr{F}$ is a minor-closed family of isomorphism classes of matroids, one may establish elementary relationships between the number of loopless, simple and arbitrary elements in $\mathscr{F}$ having a fixed ground set and rank. For example if $\mathscr{F}_{\mathrm{simp}}$ and $\mathscr{F}_{\mathrm{loop}}$ denote the classes of simple and loopless matroids in $\mathscr{F}$ respectively, then

$$|\mathscr{F}(n,k)| = \sum_{i=k}^{n} \binom{n}{i} |\mathscr{F}_{\mathrm{loop}}(i,k)|,$$
$$|\mathscr{F}_{\mathrm{loop}}(n,k)| = \sum_{i=k}^{n} \left\{ {n \atop i} \right\} |\mathscr{F}_{\mathrm{simp}}(i,k)|.$$

where $\left\{ {n \atop i} \right\}$ denotes the Stirling number of the second kind, $\mathscr{F}(n,r)$ denotes the number of matroids in $\mathscr{F}$ on $[n]$ of rank r, and analogously for $\mathscr{F}_{\mathrm{simp}}(n,r)$ and $\mathscr{F}_{\mathrm{loop}}(n,r)$.

Proposition 2.14 *Let $\mathcal{S}(n,k)$ denote the set of all simple quasi series-parallel matroids on $[n]$ of rank k. Then, the bivariate generating function*

$$S(x,y) := \sum_{n=1}^{\infty} \sum_{k=0}^{n} |\mathcal{S}(n,k)| \frac{x^n}{n!} y^k,$$

is given by

$$S(x,y) = \frac{1}{x+1} A(\log(x+1), y) - 1, \tag{3}$$

where $A(x,y)$ is given as in equation (2).

Proof. This is a formal consequence of the fact that

$$|\mathscr{A}(n,k)| = \sum_{i=k}^{n} \binom{n}{i} \sum_{j=k}^{i} \left\{ {i \atop j} \right\} |\mathcal{S}(j,k)|.$$

It is possible to translate this expression in terms of generating function. More precisely, one obtains:

$$A(x,y) = e^{-x} S(e^x - 1, y).$$

Using the change of variable $u = \log(x+1)$ yields the result of the statement. □

Remark 2.15 There is a large literature on enumerating various objects which are closely related to (quasi) series-parallel matroids. In the work of Moon [Moo87], an asymptotic estimate of the general term of the single variable series $C(x,1)$, defined in equation (1), was obtained. In particular, one may apply standard techniques to obtain asymptotic estimates for the coefficients of the series $A(x,1)$ and $S(x,1)$. Notice that our Theorem 1.1 implies that the coefficients of these series are described by the Z-polynomial and the Kazhdan–Lusztig polynomial, both evaluated at $t=1$. In particular, that provides asymptotics for the total dimensions of the intersection cohomology of K_n and its stalk at the empty flat.

3. Proof of the main results

The flats of K_n are in bijection with partitions of $[n]$: a partition Q given by $[n] = S_1 \sqcup \cdots \sqcup S_k$ is identified with the flat F_Q corresponding to the subgraph that consists of all edges where both vertices lie in S_i for some i. The simplification of the contraction K_n/F_Q is the braid matroid on $[k]$, the set of parts in Q. The stabilizer in $\mathfrak{S}_n$ of F_Q is the set of permutations which preserve Q.

Proposition 3.1 *Let Γ be a group acting on a matroid. Then the subgroup N of Γ fixing all of the parallel classes is normal, and $P^{\Gamma}_{\mathsf{M}}(t)$ is the pullback from* $\mathrm{VRep}(\Gamma/N)[t]$ *of $P^{\Gamma/N}_{\overline{\mathsf{M}}}(t)$.*

Proof. Any automorphism of a matroid sends parallel classes to parallel classes, so there is a map from Γ to the symmetric group on the set of parallel classes whose kernel is N.

The second part follows from induction on the size of the ground set and the fact that simplifying a matroid does not change its lattice of flats. □

We now prove Theorem 1.2 from which Theorem 1.1 follows. Our strategy is to use the following observation (see, e.g., [BHM+20, Corollary A.5]): Let Γ be a finite group acting on a matroid M, and let $P^{\Gamma}(t) \in \mathrm{VRep}(\Gamma)$ be a polynomial of degree less than $\frac{1}{2}\operatorname{rk}(\mathsf{M})$. Suppose that

$$Z^{\Gamma}(t) := P^{\Gamma}(t) + \sum_{\varnothing \neq [F] \in \mathscr{L}(\mathsf{M})/\Gamma} t^{\operatorname{rk}(F)} \operatorname{Ind}^{\Gamma}_{\Gamma_F} P^{\Gamma_F}_{\mathsf{M}/F}(t)$$

is palindromic. Then $Z^{\Gamma}(t) = Z^{\Gamma}_{\mathsf{M}}(t)$ and $P^{\Gamma}(t) = P^{\Gamma}_{\mathsf{M}}(t)$.

Proof of Theorem 1.2. We induct on n. Let $\Gamma = \mathfrak{S}_{n-1}$ be the subgroup of $\mathfrak{S}_n$ that fixes 1. Let $P^{\Gamma} \in \mathrm{VRep}(\Gamma)[t]$ be the generating function of the permutation representation of simple quasi series-parallel matroids of rank $n-1-i$ on $[n-1]$, and let Z^{Γ} be the generating function of the permutation representation of quasi series-parallel matroids of rank $n-1-i$ on $[n-1]$. By Proposition 2.10, the degree of P^{Γ} is less than $\frac{n-1}{2}$. Because the dual of a quasi series-parallel matroid is quasi series-parallel, Z^{Γ} is palindromic. Therefore it suffices to show that

$$Z^{\Gamma}(t) = P^{\Gamma}(t) + \sum_{\varnothing \neq [F] \in \mathscr{L}(\mathsf{K}_n)/\mathfrak{S}_{n-1}} t^{\operatorname{rk}(F)} \operatorname{Ind}^{\Gamma}_{\Gamma_F} P^{\Gamma_F}_{\mathsf{K}_n/F}(t). \tag{4}$$

Let $Q = S_1 \sqcup \cdots \sqcup S_k$ be a partition of $[n]$ with $n \in S_i$, and let M be a simple quasi series-parallel matroid on $[k] \setminus i$ of rank $k-1-j$. The quotient of Γ_{F_Q} by the normal subgroup which fixes each parallel class of K_n/F_Q is a subgroup of $\mathfrak{S}_{[k]\setminus i}$. Let V_{M} be the pullback of the permutation representation on the set of matroids on $[k] \setminus i$ isomorphic to M to Γ_{F_Q} from $\mathfrak{S}_{[k]\setminus i}$ By induction and Proposition 3.1, V_{M} is a summand of $[t^j]P^{\Gamma_{F_Q}}_{\mathsf{K}_n/F_Q}(t)$. The stabilizer of M is the inverse image in Γ_{F_Q} of the automorphism group of M.

Let $\widetilde{\mathsf{M}}$ denote the matroid on $[n-1]$ which has $S_i \setminus n$ as a set of loops, each S_ℓ is a parallel class ($\ell \neq i$), and simplifies to M. The automorphism group of $\widetilde{\mathsf{M}}$ is the inverse image in Γ_{F_Q} of the automorphism group of M, so the induction of V_{M} to $\mathfrak{S}_{n-1}$ can be identified with the permutation representation on the set of matroids on $[n-1]$ which are isomorphic to $\widetilde{\mathsf{M}}$. Note that $\operatorname{rk}(F_Q) + k - 1 - j = n - 1 - j$, so the permutation representation on the set of matroids on $[n-1]$ isomorphic to $\widetilde{\mathsf{M}}$ appears as a summand in degree $n-1-\operatorname{rk}(\widetilde{\mathsf{M}})$. Furthermore, every quasi series-parallel matroid is either simple or simplifies to a simple quasi series-parallel matroid on a smaller ground set, which proves (4). □

As a consequence of the main result, we can use the generating functions described in Propositions 2.8 and 2.14 to compute Z-polynomials and Kazhdan–Lusztig polynomials of braid matroids.

Corollary 3.2 *The bivariate generating functions $A(x, y)$ and $S(x, y)$ defined by equations (2) and (3) satisfy*

$$\frac{1}{n!}[x^n]\,A(x,y) = Z_{\mathsf{K}_{n+1}}(y),$$
$$\frac{1}{n!}[x^n]\,S(x,y) = y^n P_{\mathsf{K}_{n+1}}(1/y).$$

A very short code on SAGE implementing these two generating functions is included on the website of the first author[5].

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