K-CLASSES OF DELTA-MATROIDS AND EQUIVARIANT LOCALIZATION

CHRISTOPHER EUR, MATT LARSON, HUNTER SPINK

ABSTRACT. Delta-matroids are "type B" generalizations of matroids in the same way that maximal orthogonal Grassmannians are generalizations of Grassmannians. A delta-matroid analogue of the Tutte polynomial of a matroid is the interlace polynomial. We give a geometric interpretation for the interlace polynomial via the K-theory of maximal orthogonal Grassmannians. To do so, we develop a new Hirzebruch–Riemann–Roch-type formula for the type B permutohedral variety.

1. INTRODUCTION

For a nonnegative integer n, let $[n] = \{1, \dots, n\}$, and for a subset $S \subseteq [n]$, let $\mathbf{e}_S = \sum_{i \in S} \mathbf{e}_i$ be the sum of the corresponding standard basis vectors in $\mathbb{R}^n$. Let $[\bar{n}] = \{\bar{1}, \dots, \bar{n}\}$, and consider $[n, \bar{n}] = [n] \sqcup [\bar{n}]$ equipped with the involution $i \mapsto \bar{i}$. Writing $\mathbf{e}_{\bar{i}} = -\mathbf{e}_i$, let $\mathbf{e}_S = \sum_{i \in S} \mathbf{e}_i$ for a subset $S \subseteq [n, \bar{n}]$. A subset $S \subseteq [n, \bar{n}]$ is *admissible* if $\{i, \bar{i}\} \not\subset S$ for all $i \in [n]$. Note that a *maximal admissible subset* of $[n, \bar{n}]$ has cardinality n.

Definition 1.1. A *delta-matroid* D on $[n, \bar{n}]$ is a nonempty collection $\mathcal{F}$ of maximal admissible subsets of $[n, \bar{n}]$ such that each edge of the polytope

$$P(\mathrm{D}) = \text{the convex hull of } \{\mathbf{e}_{B \cap [n]} : B \in \mathcal{F}\} \subset \mathbb{R}^n$$

is a parallel translate of $\mathbf{e}_i$ or $\mathbf{e}_i \pm \mathbf{e}_j$ for some $i, j \in [n]$.

The collection $\mathcal{F}$ is called the *feasible sets* of D, and $P(\mathrm{D})$ is called the *base polytope* of D. One often works with the following translation of the twice-dilated base polytope

$$\widehat{P(\mathrm{D})} = 2P(\mathrm{D}) - (1, \dots, 1) = \text{the convex hull of } \{\mathbf{e}_B : B \in \mathcal{F}\} \subset \mathbb{R}^n.$$

Delta-matroids generalize matroids as the "minuscule type B matroids" in the theory of Coxeter matroids [GS87, BGW03], and as "2-matroids" in the theory of multimatroids [Bou97]. The Tutte polynomial of a matroid [Tut67, Cra69] admits a delta-matroid analogue called the *interlace polynomial*, introduced in [ABS04, BH14].

Definition 1.2. For a delta-matroid D on $[n, \bar{n}]$ with feasible sets $\mathcal{F}$ and a subset $S \subseteq [n]$, let

$$d_{\mathrm{D}}(S) = \min_{B \in \mathcal{F}} \big(|S \cup (B \cap [n])| - |S \cap B \cap [n]|\big), \text{ the lattice distance between } \mathbf{e}_S \text{ and } P(\mathrm{D}).$$

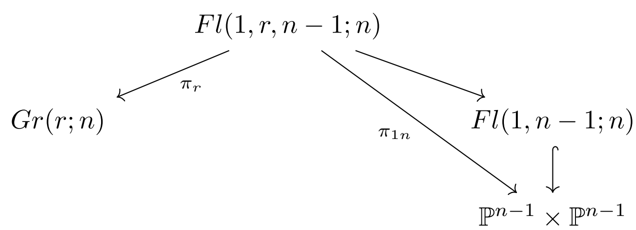
Then, the *interlace polynomial* $\mathrm{Int}_{\mathrm{D}}(v) \in \mathbb{Z}[v]$ of D is defined as

$$\mathrm{Int}_{\mathrm{D}}(v) = \sum_{S \subseteq [n]} v^{d_{\mathrm{D}}(S)}.$$

Similar to the Tutte polynomial of a matroid, the interlace polynomial has several alternative definitions: it satisfies a deletion-contraction recursion [BH14, Theorem 30], and $\mathrm{Int}_{\mathrm{D}}(v-1)$ has an activities description [Mor19]. Additionally, its evaluation at $q=0$ gives the number of feasible sets. Here, we show that Fink and Speyer's geometric interpretation of Tutte polynomials via the K-theory of Grassmannians [FS12] also generalizes to interlace polynomials. Let us first recall their result.

Each r-dimensional linear space $L \subseteq \Bbbk^n$ over a field $\Bbbk$ gives rise to a matroid M on $[n]$ and a point $[L]$ in the Grassmannian $Gr(r;n)$. The torus $T = (\Bbbk^*)^n$ acts on $Gr(r;n)$, and we consider the torus-orbit-closure $\overline{T \cdot [L]}$ of L. The K-class of the structure sheaf $[\mathcal{O}_{\overline{T\cdot[L]}}]$ in Grothendieck ring $K(Gr(r;n))$ of vector bundles on $Gr(r;n)$ depends only on M, and it admits a combinatorial formula which makes sense for any matroid M of rank r on $[n]$. This formula is used to define a class $y(\mathrm{M}) \in K(Gr(r;n))$ such that $y(\mathrm{M}) = [\mathcal{O}_{\overline{T\cdot[L]}}]$ whenever M has a realization L.

Now, consider the diagram



where π_r and π_{1n} are the natural forgetful maps. Then [FS12, Theorem 5.1] states that

$$\pi_{1n*}\pi_r^*\big(y(\mathrm{M}) \cdot [\mathcal{O}(1)]\big) = \mathrm{T}_{\mathrm{M}}(\alpha,\beta),$$

where $\mathcal{O}(1)$ is the line bundle on $Gr(r;n)$ defining the Plücker embedding, α and β are the K-classes of the structure sheaves of hyperplanes in each of the $\mathbb{P}^{n-1}$ factors, and T_{M} is the Tutte polynomial of M. This result was subsequently generalized to Tutte polynomials of morphisms of matroids in [CDMS22, DES21]. Here, we establish a similar geometric interpretation for the interlace polynomials of delta-matroids via the K-theory of maximal orthogonal Grassmanians.

Let $\Bbbk^{2n+1}$ have coordinates labelled $\bar{n},\ldots,\bar{1},0,1,\ldots,n$. Let q be the nondegenerate quadratic form on $\Bbbk^{2n+1}$ given by $q(x) = x_1x_{\bar{1}} + \cdots + x_nx_{\bar{n}} + x_0^2$. For $0 \le r \le n$, let $OGr(r;2n+1)$ be the *orthogonal Grassmannian*, which is the subvariety of $Gr(r;2n+1)$ consisting of isotropic r-dimensional subspaces, i.e.,

$$OGr(r;2n+1) = \{r\text{-dimensional linear subspaces } L \subset \Bbbk^{2n+1} \text{ such that } q|_L \text{ is identically zero}\}.$$

The action of the torus $T = (\Bbbk^*)^n$ on $\Bbbk^{2n+1}$ given by

$$(t_1,\ldots,t_n)\cdot(x_{\bar{n}},\ldots,x_{\bar{1}},x_0,x_1,\ldots,x_n) = (t_n^{-1}x_{\bar{n}},\ldots,t_1^{-1}x_{\bar{1}},x_0,t_1x_1,\ldots,t_nx_n)$$

preserves the quadratic form q, and hence induces a T-action on $OGr(r;2n+1)$. One has the T-equivariant Plücker embedding $OGr(r;2n+1) \hookrightarrow Gr(r;2n+1) \hookrightarrow \mathbb{P}(\bigwedge^r \Bbbk^{2n+1})$.

The *maximal orthogonal Grassmannian* is $OGr(n;2n+1)$. Points on $OGr(n;2n+1)$ realize delta-matroids in the same way that points on the usual Grassmannian realize matroids. More precisely, [EFLS, Proposition 6.2] [GS87] showed that the torus-orbit-closure $\overline{T \cdot [L]}$ of a point $[L] \in OGr(n; 2n+1)$, considered as a T-invariant subvariety of $\mathbb{P}(\bigwedge^n \Bbbk^{2n+1})$ via the Plücker embedding, has moment polytope $\mu(\overline{T \cdot [L]})$ equal to $\widehat{P(\mathrm{D})}$, where D is a delta-matroid with the set of feasible sets

$$\{\text{maximal admissible } B \subset [n, \bar{n}] \text{ such that the } B\text{-th Plücker coordinate of } L \text{ is nonzero}\}.$$

Using this polyhedral property, we construct for any (not necessarily realizable) delta-matroid D an element $y(\mathrm{D})$ in the Grothendieck ring $K(OGr(n; 2n+1))$ of vector bundles on $OGr(n; 2n+1)$ (see Proposition 2.2).[1]

To relate the K-class $y(\mathrm{D})$ to the the interlace polynomial, we consider the orthogonal partial flag variety $OFl(1, n; 2n+1) \subset OGr(1; 2n+1) \times OGr(n; 2n+1)$. Note that $OGr(1; 2n+1)$ is a smooth quadric inside of $Gr(1; 2n+1) = \mathbb{P}^{2n}$. We have the diagram

$$\begin{array}{ccccc} & & OFl(1,n;2n+1) & & \\ & \swarrow^{\pi_n} & & \searrow^{\pi_1} \quad \searrow & \\ OGr(n;2n+1) & & & & OGr(1;2n+1) \\ & & & & \downarrow \\ & & & & \mathbb{P}^{2n}. \end{array}$$

Let $\mathcal{O}(1)$ denote the ample line bundle that generates the Picard group of $OGr(n; 2n+1)$, i.e., its square $\mathcal{O}(2)$ defines the Plücker embedding $OGr(n; 2n+1) \hookrightarrow Gr(n; 2n+1) \hookrightarrow \mathbb{P}(\bigwedge^n \Bbbk^{2n+1})$. The line bundle $\mathcal{O}(1)$ defines the Spinor embedding of $OGr(n; 2n+1)$ into $\mathbb{P}^{2^n - 1}$. Recall that $K(\mathbb{P}^{2n}) \simeq \mathbb{Z}[u]/(u^{2n+1})$, where u is the structure sheaf of a hyperplane in $\mathbb{P}^{2n}$. So we may represent any class in $K(\mathbb{P}^{2n})$ uniquely as a polynomial in u of degree at most $2n$.

Theorem A. Let $\mathrm{Int}_{\mathrm{D}}(v) \in \mathbb{Z}[v]$ be the interlace polynomial of a delta-matroid D. We have

$$\pi_{1*}\pi_n^*\big(y(\mathrm{D}) \cdot [\mathcal{O}(1)]\big) = u \cdot \mathrm{Int}_{\mathrm{D}}(u-1) \in K(\mathbb{P}^{2n}).$$

To prove the theorem, in Proposition 4.1 we transport the pullback-pushforward $\pi_{1*}\pi_n^*(-)$ computation to a sheaf Euler characteristic $\chi(-)$ computation on a smooth projective toric variety X_{B_n} known as the *type B permutohedral variety* (Definition 2.6). Then, to carry out the sheaf Euler characteristic computation, we establish the following new Hirzebruch–Riemann–Roch-type formula for X_{B_n}. Let $A^\bullet(X_{B_n})$ be the Chow ring of X_{B_n}, with the degree map $\int_{X_{B_n}} : A^n(X_{B_n}) \xrightarrow{\sim} \mathbb{Z}$.

Theorem B. There is an injective ring homomorphism $\psi \colon K(X_{B_n}) \to A^\bullet(X_{B_n})$, which becomes an isomorphism after tensoring with $\mathbb{Z}[\frac{1}{2}]$. For any $[\mathcal{E}] \in K(X_{B_n})$, the map ψ satisfies

$$\chi(X_{B_n}, [\mathcal{E}]) = \frac{1}{2^n} \int_{X_{B_n}} \psi([\mathcal{E}]) \cdot (1 + \gamma + \gamma^2 + \cdots + \gamma^n)$$

[1]We caution that, unlike the matroid case in [FS12], the class $y(\mathrm{D})$ of a delta-matroid D with a realization $[L] \in OGr(n; 2n+1)$ may not be equal to the K-class of the structure sheaf $[\mathcal{O}_{\overline{T \cdot [L]}}]$, although it is closely related, see Proposition 2.9 and Proposition 2.3. For a detailed discussion of $[\mathcal{O}_{\overline{T \cdot [L]}}]$, see Remark 2.10 and Section 5 .

where γ is the anti-canonical divisor of X_{B_n}.

The map ψ in Theorem B is unrelated to the usual Chern character. It also differs from the Hirzebruch–Riemann–Roch-type isomorphism of [EFLS, Theorem C], which is not as suitable for proving Theorem A.

Question 1.3. The g-polynomial [Spe09] of a matroid is an invariant of matroids that can be (conjecturally) used to give strong bounds on the number of pieces in a matroid polytope subdivision. The coefficients of the g-polynomial are certain linear combinations of the coefficients that are used to express $y(\mathrm{M})$ in terms of structure sheaves of Schubert varieties in $K(Gr(r;n))$. In [FS12, Theorem 6.1], the authors express the g-polynomial in terms of a computation similar to the one in Theorem A. Is there an invariant of delta-matroids which gives strong bounds on the number of pieces in a delta-matroid polytope subdivision?

The paper is organized as follows. In Section 2, we discuss equivariant K-theory and define $y(\mathrm{D})$. In Section 3, we prove Theorem B and discuss certain class in $K(X_{B_n})$ which will be used in the proof of Theorem A. In Section 4, we prove Theorem A. In Section 5, we give some examples and questions.

Acknowledgements. We thank Alex Fink, Steven Noble, Kris Shaw, and David Speyer for helpful conversations. The first author is partially supported by the US National Science Foundation (DMS-2001854 and DMS-2246518). The second author is supported by an NDSEG graduate fellowship.

2. K-CLASSES OF DELTA-MATROIDS

Throughout, we will use localization for the torus-equivariant K-theory of toric varieties and flag varieties, for which one can consult [FS12, §2.2], [DES21, §2.2], or [CDMS22, §8] along with references therein. Let $T = (\Bbbk^*)^n$ for $\Bbbk$ an algebraically closed field, and denote by $K_T(X)$ the T-equivariant K-ring of vector bundles on a T-variety X. Identifying the character lattice of T with $\mathbb{Z}^n$, we write $K_T(\mathrm{pt}) = \mathbb{Z}[T_1^{\pm 1}, \ldots, T_n^{\pm 1}]$ for the equivariant K-ring of a point pt. For $\mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{Z}^n$, we write $T^{\mathbf{v}} = T_1^{v_1} \cdots T_n^{v_n}$.

For a countable-dimensional T-representation $V \simeq \bigoplus_i \Bbbk \cdot v_i$, where T acts on v_i by $t \cdot v_i = t^{\mathbf{m}} v_i$, the *Hilbert series* $\mathrm{Hilb}(V) = \sum_i T^{\mathbf{m}}$ is the sum of the characters of the action, which is often a rational function. For an affine semigroup $S \subseteq \mathbb{Z}^n$, we write $\mathrm{Hilb}(S) = \mathrm{Hilb}(\Bbbk[S]) = \sum_{\mathbf{m} \in S} T^{-\mathbf{m}}$. Note the minus sign, which arise because for $\chi^{\mathbf{m}} \in \Bbbk[S]$, we have $t \cdot \chi^{\mathbf{m}} = t^{-\mathbf{m}} \chi^{\mathbf{m}}$.

2.1. **K-classes on the maximal orthogonal Grassmannian.** We begin by recalling some facts about the T-action on $OGr(n; 2n+1)$, whose verification is routine and is omitted. Recall that we have set $\mathbf{e}_{\bar{i}} = -\mathbf{e}_i$.

- The T-fixed points $OGr(n; 2n+1)^T$ of $OGr(n; 2n+1)$ are in bijection with maximal admissible subsets, where such a subset $B \subset [n, \bar{n}]$ corresponds to the isotropic subspace

$$L_B = \{x \in \Bbbk^{2n+1} : x_0 = 0 \text{ and } x_j = 0 \text{ for all } j \in [n, \bar{n}] \setminus B\}.$$

Polyhedrally, by identifying $B \subset [n, \bar{n}]$ with $\mathbf{e}_{B\cap[n]} \in \mathbb{R}^n$, we may further identify the T-fixed points with the vertices of the unit cube $[0,1]^n \subset \mathbb{R}^n$.

- Each T-fixed point L_B admits a T-invariant affine chart $U_B \simeq \mathbb{A}^{n(n+1)/2}$, on which T acts with characters

$$\mathcal{T}_B = \{-\mathbf{e}_i : i \in B\} \cup \{-\mathbf{e}_i - \mathbf{e}_j : i \neq j \in B\}.$$

In particular, for $\mathbf{v} \in \mathcal{T}_B$ with $B' \subset [n, \bar{n}]$ such that $\mathbf{e}_{B'} = \mathbf{e}_B + 2\mathbf{v}$, we have an 1-dimensional T-orbit in $OGr(n; 2n+1)$ whose boundary points are L_B and $L_{B'}$. All 1-dimensional T-orbits of $OGr(n; 2n+1))$ arise in this way.

Now, the localization theorem applied to $K_T(OGr(n; 2n+1))$ states the following:

Theorem 2.1. [VV03, Corollary 5.11] The restriction map

$$K_T(OGr(n; 2n+1)) \to K_T(OGr(n; 2n+1)^T) = \prod_{L_B \in OGr(n;2n+1)^T} \mathbb{Z}[T_1^{\pm 1}, \dots T_n^{\pm 1}]$$

is injective, and its image is

$$\left\{ (f_B)_B \in \prod_{L_B \in OGr(n;2n+1)^T} \mathbb{Z}[T_1^{\pm 1}, \dots T_n^{\pm 1}] : \begin{array}{c} \text{for } \mathbf{v} \in \mathcal{T}_B \text{ with } B' \subset [n,\bar{n}] \text{ such that } \mathbf{e}_{B'} = \mathbf{e}_B + 2\mathbf{v} \\ f_B - f_{B'} \equiv 0 \bmod (1 - T^{\mathbf{v}}) \end{array} \right\}.$$

For an equivariant K-class $[\mathcal{E}] \in K_T(OGr(n; 2n+1))$ and a maximal admissible subset B, we write $[\mathcal{E}]_B \in \mathbb{Z}[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$ for the B-th factor of the image of $[\mathcal{E}]$ under the restriction map in Theorem 2.1.

For a matroid M on a ground set $[n]$, Fink and Speyer defined a T-equivariant K-class $y(\mathrm{M})$ on a Grassmannian $Gr(r; n)$. We now define an analogous T-equivariant K-class $y(\mathrm{D})$ for a delta-matroid D. For a feasible set B of D, denote by $\mathrm{cone}_B(\mathrm{D})$ the tangent cone of $P(\mathrm{D})$ at the vertex $\mathbf{e}_{B\cap[n]}$, i.e.,

$$\mathrm{cone}_B(\mathrm{D}) = \mathbb{R}_{\geq 0}\{P(\mathrm{D}) - \mathbf{e}_{B\cap[n]}\}.$$

Since $\mathrm{cone}_B(\mathrm{D})$ is a rational strongly convex cone whose set of primitive rays is a subset of $\mathcal{T}_B$, the multigraded Hilbert series

$$\mathrm{Hilb}(\mathrm{cone}_B(\mathrm{D}) \cap \mathbb{Z}^n) = \sum_{\mathbf{m} \in \mathrm{cone}_B(\mathrm{D}) \cap \mathbb{Z}^n} T^{-\mathbf{m}}$$

is a rational function whose denominator divides $\prod_{\mathbf{v} \in \mathcal{T}_B}(1 - T^{-\mathbf{v}})$ [Sta12, Theorem 4.5.11].

Proposition-Definition 2.2. For a delta-matroid D on $[n, \bar{n}]$, define $y(\mathrm{D}) \in K_T(OGr(n; 2n+1)^T)$ by

$$y(\mathrm{D})_B = \begin{cases} \mathrm{Hilb}(\mathrm{cone}_B(\mathrm{D}) \cap \mathbb{Z}^n) \cdot \prod_{\mathbf{v} \in \mathcal{T}_B}(1 - T^{-\mathbf{v}}) & \text{if } B \text{ a feasible set of } \mathrm{D} \\ 0 & \text{otherwise} \end{cases}$$

for any maximal admissible subset $B \subset [n, \bar{n}]$. Then $y(\mathrm{D})$ lies in the subring $K_T(OGr(n; 2n+1))$.

We omit the proof of the proposition, as it is essentially identical to the proof of the analogous statement [FS12, Proposition 3.2] for matroids. Alternatively, it can be deduced from Theorem 2.8 and Proposition 2.9. Let us note however the following difference from the matroid case. For a matroid M on $[n]$, the class $y(\mathrm{M})$ in [FS12] has the property that if $[L] \in Gr(r;n)$ realizes M, then $y(\mathrm{M})$ equals $[\mathcal{O}_{\overline{T\cdot[L]}}]$, the K-class of the structure sheaf of the torus-orbit closure. This property often fails for delta-matroids because delta-matroid base polytopes often do not enjoy certain polyhedral properties enjoyed by matroid base polytopes, namely normality and very ampleness.

Recall that a lattice polytope $P \subset \mathbb{R}^n$ (with respect to the lattice $\mathbb{Z}^n$) is *normal* if for all positive integer ℓ one has $(\ell P) \cap \mathbb{Z}^n = \{\mathbf{m}_1 + \cdots + \mathbf{m}_\ell : \mathbf{m}_i \in P \cap \mathbb{Z}^n \text{ for all } i = 1, \ldots, \ell\}$. If P is normal, then it is *very ample*, meaning that for every vertex $\mathbf{v}$ of P, one has

$$\big(\mathbb{R}_{\geq 0}\{P - \mathbf{v}\}\big) \cap \mathbb{Z}^n = \mathbb{Z}_{\geq 0}\{(P - \mathbf{v}) \cap \mathbb{Z}^n\}.$$

Proposition 2.3. For a delta-matroid D realized by $[L] \in OGr(n; 2n+1)$, the T-equivariant K-class $[\mathcal{O}_{\overline{T\cdot[L]}}]$ of the structure sheaf of the torus-orbit-closure of L satisfies

$$[\mathcal{O}_{\overline{T\cdot[L]}}]_B = \begin{cases} \mathrm{Hilb}\left(\mathbb{Z}_{\geq 0}\{(P(\mathrm{D}) - \mathbf{e}_{B\cap[n]}) \cap \mathbb{Z}^n\}\right) \prod_{\mathbf{v}\in\mathcal{T}_B}(1 - T^{-\mathbf{v}}) & \text{if } B \text{ a feasible subset of } \mathrm{D} \\ 0 & \text{otherwise} \end{cases}$$

for any maximal admissible subset B. In particular, the T-equivariant K-class $y(\mathrm{D})$ equals $[\mathcal{O}_{\overline{T\cdot[L]}}]$ if and only if $P(\mathrm{D})$ is very ample.

Proof. For a finite subset $\mathscr{A} \subset \mathbb{Z}^n$, let $Y_{\mathscr{A}}$ be the projective toric variety defined as the closure of the image of the map $T \to \mathbb{P}^{|\mathscr{A}|-1}$ given by $\mathbf{t} \mapsto (\mathbf{t}^{\mathbf{m}})_{\mathbf{m}\in\mathscr{A}}$. Writing $\mathbf{e}_0 = 0 \in \mathbb{Z}^n$, let us consider

$$\mathscr{A}(L) = \left\{ \mathbf{e}_S : \begin{array}{c} S \subset [n,\bar{n}] \cup \{0\} \text{ with } |S| = n \text{ such that} \\ \text{the } S\text{-th Plücker coordinate of } L \text{ is nonzero} \end{array} \right\}.$$

There is an embedding of $\mathbb{P}^{|\mathscr{A}|-1}$ into $\mathbb{P}(\bigwedge^n \Bbbk^{2n+1})$ which identifies the orbit closure $\overline{T\cdot[L]} \subset \mathbb{P}(\bigwedge^n \Bbbk^{2n+1})$ with $Y_{\mathscr{A}(L)}$. We now claim that

$$\mathscr{A}(L) = \{\mathbf{m} + \mathbf{m}' - (1,\ldots,1) : \mathbf{m}, \mathbf{m}' \in P(\mathrm{D}) \cap \mathbb{Z}^n\} \subset \widehat{P(\mathrm{D})}.$$

That is, up to translation by $-(1,\ldots,1)$, the set $\mathscr{A}(L)$ is the set of all sums of two (not necessarily distinct) lattice points in $P(\mathrm{D})$. When B is a feasible set of D, in the T-invariant affine chart U_B around L_B, the coordinate ring $\mathcal{O}_{\overline{T\cdot[L]}}(U_B)$ equals the semigroup algebra $\Bbbk[\mathbb{Z}_{\geq 0}\{\mathbf{m} - \mathbf{e}_B : \mathbf{m} \in \mathscr{A}(L)\}]$, which the claim implies equals $\Bbbk[\mathbb{Z}_{\geq 0}\{(P(\mathrm{D}) - \mathbf{e}_{B\cap[n]}) \cap \mathbb{Z}^n\}]$, and thus the proposition follows from [MS05, Theorem 8.34] (see also [FS10, Theorem 2.6]).

For the claim, we first note that $\mathscr{A}(L)$ is contained in $\widehat{P(\mathrm{D})} \cap \mathbb{Z}^n$ and contains all vertices of $\widehat{P(\mathrm{D})}$ because the moment polytope $\mu(\overline{T\cdot[L]})$ equals $\widehat{P(\mathrm{D})}$ by [EFLS, Proposition 6.2]. The Plücker embedding $OGr(n; 2n+1) \hookrightarrow \mathbb{P}(\bigwedge^n \Bbbk^{2n+1})$ is given by the square $\mathcal{O}(2)$ of the very ample generator $\mathcal{O}(1)$ of the Picard group of $OGr(n; 2n+1)$. Because homogeneous spaces are projectively normal, we find that $\overline{T\cdot[L]}$ is isomorphic to $Y_{\mathscr{A}}$ for some subset $\mathscr{A} \subseteq P(\mathrm{D}) \cap \mathbb{Z}^n$ that includes all vertices of $P(\mathrm{D})$. But all lattices points of $P(\mathrm{D})$ are its vertices, so $\mathscr{A} = P(\mathrm{D}) \cap \mathbb{Z}^n$.

Therefore, the projective embedding of $\overline{T \cdot [L]}$ given by $\mathcal{O}(2)$ is isomorphic to $Y_{2\mathscr{A}}$ where $2\mathscr{A} = \{\mathbf{m} + \mathbf{m}' : \mathbf{m}, \mathbf{m}' \in \mathscr{A}\}$, which after translating each element by $-(1, \ldots, 1)$ is exactly $\mathscr{A}(L)$. $\square$

The polytope $P(\mathrm{D})$ can fail to be very ample in various degrees. See Section 5 for a series of examples. In particular, the class $y(\mathrm{D})$ may not equal $[\mathcal{O}_{\overline{T \cdot [L]}}]$ when L realizes D.

Remark 2.4. Proposition 2.3 also implies that the class $[\mathcal{O}_{\overline{T \cdot [L]}}]$ depends only on the delta-matroid D, independently of the realization L of D. The analogous statement fails when delta-matroids are considered as "type C Coxeter matroids," a.k.a. symplectic matroids. More precisely, in [BGW98], realizations of delta-matroids are points on the Lagrangian Grassmannian $LGr(n; 2n)$ consisting of maximal isotropic subspaces with respect to the standard symplectic form on $\mathbb{k}^{2n}$. However, in this case, the K-class of the torus-orbit-closure of a point $[L] \in LGr(n; 2n)$ may not depend only on the delta-matroid that L realizes. See the following example. This is related to the fact that the parabolic corresponding to $OGr(n; 2n+1)$ is *minuscule*, but the parabolic corresponding to $LGr(n; 2n)$ is not.

Example 2.5. Let $\mathbb{C}^4$ (with coordinates labeled by $(1, 2, \bar{1}, \bar{2})$ be equipped with the standard symplectic form. The torus $T = (\mathbb{C}^*)^2$ acts on $\mathbb{C}^4$ by $(t_1, t_2) \cdot (x_1, x_2, x_{\bar{1}}, x_{\bar{2}}) = (t_1 x_1, t_2 x_2, t_1^{-1} x_{\bar{1}}, t_2^{-1} x_{\bar{2}})$. For each $z \in \mathbb{C}$, consider the 2-dimensional subspace L_z spanned by $(1, 0, 1, z)$ and $(0, 1, z, 1)$, which is Lagrangian. For all $z \neq \pm 1$, every Plücker coordinate corresponding to a maximal admissible subset is nonzero. Thus, the moment polytope $\mu(\overline{T \cdot [L_z]})$ always equals $[-1, 1]^2 \subset \mathbb{R}^2$ as long as $z \neq \pm 1$. However, when $z = 0$, one computes that $\overline{T \cdot [L_z]} \simeq \mathbb{P}^1 \times \mathbb{P}^1$, whereas $\overline{T \cdot [L_z]}$ is a toric surface with four conical singularities when $z \neq \pm 1$ and $z \neq 0$. As a result, one verifies that the $[\mathcal{O}_{\overline{T \cdot [L_0]}}] \neq [\mathcal{O}_{\overline{T \cdot [L_3]}}]$, even as non-equivariant K-classes.

2.2. **K-classes on the type B permutohedral variety.** We explain how the geometry of the type B permutohedral variety X_{B_n} relates to the class $y(\mathrm{D})$ on $OGr(n; 2n+1)$, which we will use to prove Theorem A. We begin by briefly reviewing the relation between delta-matroids and X_{B_n}, details of which can be found in [EFLS, Section 2].

Definition 2.6. Let W be the *signed permutation group* on $[n, \bar{n}]$, which is the subgroup of the permutation group $\mathfrak{S}_{[n,\bar{n}]}$ defined as

$$W = \{w \in \mathfrak{S}_{[n,\bar{n}]} : w(\bar{i}) = \overline{w(i)} \text{ for all } i \in [n]\}.$$

The B_n *permutohedral fan* Σ_{B_n} is the complete fan in $\mathbb{R}^n$, unimodular with respect to the lattice $\mathbb{Z}^n$, whose maximal cones are labeled by elements of W, with the maximal cone σ_w being

$$\mathbb{R}_{\geq 0}\{\mathbf{e}_{w(1)}, \mathbf{e}_{w(1)} + \mathbf{e}_{w(2)}, \ldots, \mathbf{e}_{w(1)} + \mathbf{e}_{w(2)} + \cdots + \mathbf{e}_{w(n)}\} \quad \text{for each } w \in W.$$

Let X_{B_n} be the (smooth projective) toric variety of the fan Σ_{B_n}, which contains T as its open dense torus. For each $w \in W$, let pt_w be the T-fixed point of X_{B_n} corresponding to the maximal cone σ_w. For toric variety conventions, we follow [Ful93, CLS11].

The normal fan of a delta-matroid polytope $P(\mathrm{D})$ is always a coarsening of Σ_{B_n} [ACEP20, Section 4.4]. Hence, under the standard correspondence between nef toric line bundles and polytopes, the polytope $P(\mathrm{D})$ defines a line bundle whose K-class we denote $[P(\mathrm{D})] \in K(X_{B_n})$. See [CLS11, Chapter 6] and [EFLS, Section 2.2] for details. The assignment $\mathrm{D} \mapsto [P(\mathrm{D})]$ is *valuative* in the following sense.

Definition 2.7. For a subset $S \subset \mathbb{R}^n$, let $\mathbf{1}_S \colon \mathbb{R}^n \to \mathbb{Z}$ be defined by $\mathbf{1}_S(x) = 1$ if $x \in S$ and $\mathbf{1}_S(x) = 0$ if otherwise. Define the *valuative group* of delta-matroids on $[n, \bar{n}]$ to be

$$\mathbb{I}(\mathsf{DMat}_n) = \text{the subgroup of } \mathbb{Z}^{(\mathbb{R}^n)} \text{ generated by } \{\mathbf{1}_{P(\mathrm{D})} : \mathrm{D} \text{ a delta-matroid on } [n, \bar{n}]\}.$$

A function f on delta-matroids valued in an abelian group is *valuative* if it factors through $\mathbb{I}(\mathsf{DMat}_n)$.

We record the following useful consequence of [EFLS, Theorem D].

Theorem 2.8. Let $\mathscr{D} = \{\mathrm{D} \text{ a delta-matroid on } [n, \bar{n}] : \mathrm{D} \text{ has a realization } L \text{ with } [\mathcal{O}_{\overline{T \cdot [L]}}] = y(\mathrm{D})\}$. Then, the delta-matroids in $\mathscr{D}$ generate both the K-ring $K(X_{B_n})$, considered as an abelian group, and the valuative group $\mathbb{I}(\mathsf{DMat}_n)$. That is, the set $\{[P(\mathrm{D})] : \mathrm{D} \in \mathscr{D}\}$ generates $K(X_{B_n})$, and the set $\{1_{P(\mathrm{D})} : \mathrm{D} \in \mathscr{D}\}$ generates $\mathbb{I}(\mathsf{DMat}_n)$.

Proof. We first note that the set $\mathscr{D}$ includes the family of delta-matroids known as *Schubert delta-matroids* [EFLS, Definition 2.6]. Indeed, Schubert delta-matroids are realizable [EFLS, Example 6.3], and their base polytopes, being isomorphic to an polymatroid polytope, are normal [Wel76, Chapter 18.6, Theorem 3]. Hence, by Proposition 2.3, the set $\mathscr{D}$ includes all Schubert delta-matroids. Now, Schubert delta-matroids generate both $K(X_{B_n})$ [EFLS, Theorem D] and $\mathbb{I}(\mathsf{DMat}_n)$ [EFLS, Proposition 2.7]. □

Lastly, the K-class $y(\mathrm{D})$ relates to the geometry of X_{B_n} in the following way. When D has a realization $[L] \in OGr(n; 2n+1)$, there exists a unique T-equivariant map $\varphi_L \colon X_{B_n} \to OGr(n; 2n+1)$ such that the identity point of the torus $T \subset X_{B_n}$ is mapped to $[L]$ [EFLS, Proposition 7.2]. Note that its image is the torus-orbit-closure $\overline{T \cdot [L]}$.

Proposition 2.9. The assignment $\mathrm{D} \mapsto y(\mathrm{D})$ is the unique valuative map such that $y(\mathrm{D}) = \varphi_{L*}[\mathcal{O}_{X_{B_n}}]$ whenever D has a realization L.

Proof. The assignment $\mathrm{D} \mapsto y(\mathrm{D})$ is valuative because taking the Hilbert series of the tangent cone at a chosen point is valuative. When D has a realization L and $P(\mathrm{D})$ is very ample, the map φ_L, considered as a map $X_{B_n} \to \overline{T \cdot [L]}$ of toric varieties, is induced by a map of tori with a connected kernel. Hence, in this case we have $\varphi_{L*}[\mathcal{O}_{X_n}] = [\mathcal{O}_{\overline{T \cdot [L]}}]$ by [CLS11, Theorem 9.2.5] and $[\mathcal{O}_{\overline{T \cdot [L]}}] = y(\mathrm{D})$ by Proposition 2.3. The uniqueness then follows from Theorem 2.8.

To see that $y(\mathrm{D}) = \varphi_{L*}[\mathcal{O}_{X_{B_n}}]$ whenever D has a realization L, even if $P(\mathrm{D})$ is not very ample, we compute the pushforward using Atiyah–Bott. First, for a maximal admissible $B \subset [n, \bar{n}]$, the construction of the map φ_L shows that the fiber $\varphi_L^{-1}(L_B)$ is

$$\varphi_L^{-1}(L_B) = \begin{cases} \left\{ \mathrm{pt}_w \in X_{B_n}^T : \begin{array}{l} w \in W \text{ such that the dual cone of} \\ \mathbb{R}_{\geq 0}\{P(\mathrm{D}) - \mathbf{e}_{B \cap [n]}\} \text{ contains } \sigma_w \end{array} \right\} & \text{if } B \text{ a feasible set of } \mathrm{D} \\ \varnothing & \text{otherwise.} \end{cases}$$

We note that because the normal fan of $P(\mathrm{D})$ is a coarsening of Σ_{B_n}, for B a feasible set of D, the cones $\{\sigma_w : \mathrm{pt}_w \in \varphi_L^{-1}(L_B)\}$ form a polyhedral subdivision of the dual cone of $\mathbb{R}_{\geq 0}\{P(\mathrm{D}) - \mathbf{e}_{B\cap[n]}\}$. Now, the desired result follows from combining [CG10, Theorem 5.11.7] and the generalized Brion's formula [Ish90, Theorem 2.3], [Bri88]. □

Remark 2.10. One could have defined a K-class on $OGr(n;2n+1)$ for an arbitrary delta-matroid D via the formula in Proposition 2.3 instead of Proposition-Definition 2.2. Abusing notation, denote this alternate K-class by $[\mathcal{O}_{\overline{T\cdot \mathrm{D}}}]$, even though D may not be realizable. Proposition 2.3 states that $y(\mathrm{D}) = [\mathcal{O}_{\overline{T\cdot \mathrm{D}}}]$ exactly when $P(\mathrm{D})$ is very ample (with respect to $\mathbb{Z}^n$). Unlike $\mathrm{D} \mapsto y(\mathrm{D})$, the assignment $\mathrm{D} \mapsto [\mathcal{O}_{\overline{T\cdot \mathrm{D}}}]$ enjoys the feature that $[\mathcal{O}_{\overline{T\cdot \mathrm{D}}}] = [\mathcal{O}_{\overline{T\cdot [L]}}]$ whenever D has a realization L, but it is not valuative by Proposition 2.9. Moreover, Theorem A fails when $[\mathcal{O}_{\overline{T\cdot \mathrm{D}}}]$ is used in place of $y(\mathrm{D})$, and we do not know a description of $\pi_{1*}\pi_n^*\big([\mathcal{O}_{\overline{T\cdot \mathrm{D}}}]\cdot[\mathcal{O}(1)]\big)$ in terms of known delta-matroid invariants. See Section 5 for examples and questions about $[\mathcal{O}_{\overline{T\cdot \mathrm{D}}}]$.

3. THE EXCEPTIONAL HIRZEBRUCH–RIEMANN–ROCH FORMULA

In this section, we prove Theorem B. We first construct ψ and prove that it is an isomorphism after inverting 2. Then, we discuss how ψ relates to the *isotropic tautological classes* of delta-matroids constructed in [EFLS], which we use to finish the proof of Theorem B.

3.1. **The isomorphism.** We follow the notation and conventions in [EFLS, Sections 2.1 and 3.1], recalling what is necessary. For a variety with a T-action, we will denote the Chow ring and equivariant Chow ring by $A^\bullet(X)$ and $A^\bullet_T(X)$ respectively. We use the language of moment graphs; see [FS10, Section 2.4] or [Mac07, Lecture 2].

We first define the moment graph Γ associated to the T-action on X_{B_n}. The vertex set $V(\Gamma)$ is the signed permutation group W, which indexes the torus-fixed points of X_{B_n}, and the edges $E(\Gamma)$ are given by $(w, w\tau)$ for a transposition $\tau \in \{(1,2),(2,3),\ldots,(n-1,n),(n,\bar{n})\}$, indexing T-invariant $\mathbb{P}^1$'s joining torus-fixed points of X_{B_n}. Denote $\tau_{i,i+1} := (i,i+1)$ and $\tau_n := (n,\bar{n})$. We have edge labels $c(w,w\tau)$ which are characters of T up to sign (i.e., elements of $\mathbb{Z}^n/\pm 1$) by taking $c(w,w\tau_n) = \pm\mathbf{e}_{w(n)} \in \mathbb{Z}^n/\pm 1$ and $c(w,w\tau_{i,i+1}) = \pm(\mathbf{e}_{w(i)} - \mathbf{e}_{w(i+1)}) \in \mathbb{Z}^n/\pm 1$, recalling the convention that $\mathbf{e}_{\bar{i}} = -\mathbf{e}_i$.

By the identification of the character lattice of T with $\mathbb{Z}^n$, we write $K_T(\mathrm{pt}) = \mathbb{Z}[T_1^{\pm 1},\ldots,T_n^{\pm 1}]$ and $A^\bullet_T(\mathrm{pt}) = \mathbb{Z}[t_1,\ldots,t_n]$. By equivariant localization we have

$$K_T(X_{B_n}) = \{(f_v)_{v\in V(\Gamma)} : f_i - f_j \equiv 0 \pmod{1 - \prod_{k=1}^n T_k^{c(ij)_k}} \text{ for all } (i,j)\in E(\Gamma)\} \subset \bigoplus_{v\in\Gamma} K_T(\mathrm{pt}),$$

$$A^\bullet_T(X_{B_n}) = \{(f_v)_{v\in V(\Gamma)} : f_i - f_j \equiv 0 \pmod{\sum_{k=1}^n c(ij)_k\cdot t_k} \text{ for all } (i,j)\in E(\Gamma)\} \subset \bigoplus_{v\in\Gamma} A^\bullet_T(\mathrm{pt}).$$

Note that both compatibility conditions are invariant under $c(ij)\mapsto -c(ij)$. These are algebras over the rings $\mathbb{Z}[T_1^{\pm 1},\ldots,T_n^{\pm 1}]$ and $\mathbb{Z}[t_1,\ldots,t_n]$ respectively, which are identified as subrings of $K_T(X_{B_n})$ and $A^\bullet_T(X_{B_n})$ via the constant collections of $(f_v)_{v\in V}$. Additionally, we have that

$$K(X_{B_n}) = K_T(X_{B_n})/(T_1-1,\ldots,T_n-1) \text{ and } A^\bullet(X_{B_n}) = A^\bullet_T(X_{B_n})/(t_1,\ldots,t_n).$$

Finally, there is are W-actions on $K_T(X_{B_n})$ by $(w\cdot f)_{w'}(T_1,\dots,T_n)=f_{w^{-1}w'}(T_{w(1)},\dots,T_{w(n)})$, and on $A_T(X_{B_n})$ by $(w\cdot f)_{w'}(t_1,\dots,t_n)=f_{w^{-1}w'}(t_{w(1)},\dots,t_{w(n)})$, where we set

$$T_{\bar{i}}=T_i^{-1} \text{ and } t_{\bar{i}}=-t_i.$$

This action descends to usual action of $W\subset \operatorname{Aut} X_{B_n}$ on $K(X_{B_n})$ and $A^\bullet(X_{B_n})$.

Theorem 3.1. There is an injective ring map

$$\psi_T\colon K_T(X_{B_n})\to A_T^\bullet(X_{B_n})[1/(1\pm t_i)]:=A_T^\bullet(X_{B_n})[\{\tfrac{1}{1-t_i},\tfrac{1}{1+t_i}\}_{1\le i\le n}]$$

obtained by

$$(\psi_T(f))_w(t_1,\dots,t_n)=f_w\left(\frac{1+t_1}{1-t_1},\dots,\frac{1+t_n}{1-t_n}\right).$$

This map descends to a non-equivariant map $\psi\colon K(X_{B_n})\to A^\bullet(X_{B_n})$, which is injective and becomes an isomorphism after tensoring with $\mathbb{Z}[\frac{1}{2}]$.

Finally, ψ_T and ψ are W-equivariant in the sense that they intertwine the W-actions:

$$\psi_T(w\cdot f)=w\cdot\psi_T(f) \text{ and } \psi(w\cdot f)=w\cdot\psi(f).$$

Proof. The map ψ_T is an injective ring homomorphism if it is well-defined, so we need to check that the compatibility conditions are preserved by ψ_T. Let $p(z)=\frac{1+z}{1-z}$.

- If $c(ij)=\pm\mathbf{e}_k$, then $f_i(T_1,\dots,T_n)=f_j(T_1,\dots,T_n)$ when $T_k=1$. Because $p(0)=1$, this implies that $f_i(p(t_1),\dots,p(t_n))=f_j(p(t_1),\dots,p(t_n))$ when $t_k=0$.
- If $c(ij)=\pm(\mathbf{e}_k-\mathbf{e}_\ell)$, then $f_i(T_1,\dots,T_n)=f_j(T_1,\dots,T_n)$ when $T_k=T_\ell$. This implies that $f_i(p(t_1),\dots,p(t_n))=f_j(p(t_1),\dots,p(t_n))$ when $t_i=t_j$.
- If $c(ij)=\pm(\mathbf{e}_k+\mathbf{e}_\ell)$, then $f_i(T_1,\dots,T_n)=f_j(T_1,\dots,T_n)$ when $T_k=T_\ell^{-1}$. Because $p(z)=p(-z)^{-1}$, this implies that $f_i(p(t_1),\dots,p(t_n))=f_j(p(t_1),\dots,p(t_n))$ when $t_k=-t_\ell$.

We now check that the map ψ_T descends non-equivariantly to a map $\psi\colon K(X_{B_n})\to A^\bullet(X_{B_n})$. Note that under the map $A_T^\bullet(X_{B_n})\to A^\bullet(X_{B_n})$ we have $1\pm t_i\mapsto 1$, so there is an induced map $A_T^\bullet(X_{B_n})[\frac{1}{1\pm t_i}]\to A^\bullet(X_{B_n})$. To obtain the map ψ, we have to show that under the composition $K_T(X_{B_n})\to A^\bullet(X_{B_n})[\frac{1}{1\pm t_i}]\to A^\bullet(X_{B_n})$, the ideal $(T_1-1,\dots,T_n-1)$ gets mapped to 0. Indeed, $\psi_T(T_i-1)=\frac{2t_i}{1-t_i}$, which gets mapped to 0 under the map $A_T^\bullet(X_{B_n})[\frac{1}{1\pm t_i}]\to A^\bullet(X_{B_n})$ because t_i maps to 0.

We now check that ψ is an isomorphism after inverting 2. Note that under the map $K_T(X_{B_n})\to A_T^\bullet(X_{B_n})[\frac{1}{1\pm t_i}][\frac{1}{2}]$, the element $1+T_i$ maps to the unit $\frac{2}{1-t_i}$, and hence, by the universal property of localization, we have a map $K_T(X_{B_n})[\frac{1}{1+T_i}][\frac{1}{2}]\to A_T^\bullet(X_{B_n})[\frac{1}{1\pm t_i}][\frac{1}{2}]$. We claim that this is an isomorphism.

Indeed, first note that it is clearly injective by definition of ψ_T, so we just have to check surjectivity. For $g\in A^\bullet(X_{B_n})[\frac{1}{1\pm t_i}][\frac{1}{2}]$, it is easy to see that $g_w(\frac{T_1-1}{T_1+1},\dots,\frac{T_n-1}{T_n+1})\in K_T(\mathrm{pt})[\frac{1}{1+T_i}][\frac{1}{2}]$, and arguing as before, we see that

$$w\mapsto g_w\left(\frac{T_1-1}{T_1+1},\dots,\frac{T_n-1}{T_n+1}\right)$$

gives a preimage of g in $K_T(X_{B_n})[\frac{1}{1+T_i}][\frac{1}{2}]$.

Now the ideal $(T_1 - 1, \ldots, T_n - 1) \subset K_T(X_{B_n})[\frac{1}{1+T_i}][\frac{1}{2}]$ maps to the ideal $(\frac{-2t_1}{1-t_1}, \ldots, \frac{-2t_n}{1-t_n}) = (t_1, \ldots, t_n) \subset A^\bullet(X_{B_n})[\frac{1}{1\pm t_i}][\frac{1}{2}]$. Hence we obtain that $\psi \otimes \mathbb{Z}[\frac{1}{2}]$ is the isomorphism

$$\begin{aligned} K(X_{B_n})[\frac{1}{2}] = K_T(X_{B_n})[\frac{1}{2}]/(T_1 - 1, \ldots, T_n - 1) &= K_T(X_{B_n})[\frac{1}{1+T_i}][\frac{1}{2}]/(T_1 - 1, \ldots, T_n - 1) \\ &\cong A_T^\bullet(X_{B_n})[\frac{1}{1 \pm t_i}][\frac{1}{2}]/(t_1, \ldots, t_n) \\ &= A_T^\bullet(X_{B_n})[\frac{1}{2}]/(t_1, \ldots, t_n) = A^\bullet(X_{B_n})[\frac{1}{2}]. \end{aligned}$$

Finally, we check W-equivariance. Let $\epsilon_i(w)$ equal 1 if $w(i) \in \{1, \ldots, n\}$ and -1 if $w(i) \in \{\overline{1}, \ldots, \overline{n}\}$. Then for $f \in K_T(X_{B_n})$ we verify W-equivariance of ψ_T by computing

$$(w \cdot \psi_T(f))_{w'} = f_{w^{-1}w'}\left(\frac{1 + t_{w(1)}}{1 - t_{w(1)}}, \ldots, \frac{1 + t_{w(n)}}{1 - t_{w(n)}}\right), \text{ and}$$

$$(\psi_T(w \cdot f))_{w'} = f_{w^{-1}w'}\left(\left(\frac{1 + \epsilon_1(w)t_{w(1)}}{1 - \epsilon_1(w)t_{w(1)}}\right)^{\epsilon_1(w)}, \ldots, \left(\frac{1 + \epsilon_n(w)t_{w(n)}}{1 - \epsilon_n(w)t_{w(n)}}\right)^{\epsilon_n(w)}\right)$$

which are equal as $p(z) = \frac{1+z}{1-z}$ has $p(z) = p(-z)^{-1}$. The W-equivariance then descends to ψ. □

Remark 3.2. Although we state the theorem above for X_{B_n}, we note that the only hypothesis on the moment graph Γ used in the proof up to the verification of W-equivariance is that all edge labels lie in the set $\{\pm\mathbf{e}_k : 1 \le k \le n\} \cup \{\pm(\mathbf{e}_k + \mathbf{e}_\ell) : 1 \le k < \ell \le n\} \cup \{\pm(\mathbf{e}_k - \mathbf{e}_\ell) : 1 \le k < \ell \le n\}$.

Remark 3.3. The map $\psi\colon K(X_{B_n}) \to A^\bullet(X_{B_n})$ differs from the previous Hirzebruch–Riemann–Roch-type isomorphisms for X_{B_n} established in [EFLS], but is related as follows. Let ϕ^B and ζ^B be the exceptional isomorphisms $K(X_{B_n}) \xrightarrow{\sim} A^\bullet(X_{B_n})$ as in [EFLS, Theorem C] and [EFLS, Proposition 3.7]. Comparing the formulas for their T-equivariant maps, one can show that ψ is the unique ring map such that

$$\psi([\mathcal{L}]) = \phi^B([\mathcal{L}]) \cdot \zeta^B([\mathcal{L}]) \quad \text{for any } T\text{-equivariant line bundle } \mathcal{L} \text{ on } X_{B_n}.$$

3.2. **Isotropic tautological classes.** We now discuss the "isotropic tautological class" $[\mathcal{I}_D] \in K(X_{B_n})$ of a delta-matroid D, which was introduced in [EFLS]. We show how this class is related to $[P(\mathrm{D})]$ via the ψ map, which will allow us to use the relationship between $[\mathcal{I}_\mathrm{D}]$ and interlace polynomials established in [EFLS, Theorem 7.15].

By pulling back the tautological sequence $0 \to \mathcal{S} \to \mathcal{O}_{Gr(n;2n+1)}^{\oplus 2n+1} \to \mathcal{Q} \to 0$ involving the tautological subbundle and quotient bundle on the Grassmannian, one has a short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_{OGr(n;2n+1)}^{\oplus 2n+1} \to \mathcal{Q} \to 0 \tag{1}$$

of vector bundles on $OGr(n; 2n+1)$. For a realization $[L] \in OGr(n; 2n+1)$ of a delta-matroid D, pulling back the sequence via φ_L yields T-equivariant vector bundles $\mathcal{I}_L$ and $\mathcal{Q}_L$ on X_{B_n}. In general, we have the following T-equivariant K-classes for a delta-matroid [EFLS, Proposition 7.4]. Denote $T_{\overline{i}} = T_i^{-1}$ for $i \in [n]$, and let $B_w(\mathrm{D})$ denote the *w-minimal feasible set* of D for $w \in W$, which is the feasible set corresponding to the vertex of $P(\mathrm{D})$ that minimizes the inner product with any vector $\mathbf{v}$ in the interior of σ_w.

Definition 3.4. For a delta-matroid D on $[n,\bar{n}]$, define $[\mathcal{I}_\mathrm{D}] \in K_T(X_{B_n})$ to be the *isotropic tautological class* of D, given by

$$[\mathcal{I}_\mathrm{D}]_w = \sum_{i\in B_w(\mathrm{D})} T_i \quad \text{for all } w \in W.$$

Define $[\mathcal{Q}_\mathrm{D}] \in K_T(X_{B_n})$ as $[\mathcal{O}_{X_{B_n}}^{\oplus 2n+1}] - [\mathcal{I}_\mathrm{D}]$, that is,

$$[\mathcal{Q}_\mathrm{D}]_w = 1 + \sum_{i\in[n,\bar{n}]\setminus B_w(\mathrm{D})} T_i.$$

We will use the following fundamental computation relating Chern classes of isotropic tautological classes and interlace polynomials. For $[\mathcal{E}] \in K(X_{B_n})$, let $c_i(\mathcal{E})$ denote its i-th Chern class, and denote by $c(\mathcal{E}, q) = \sum_{i\geq 0} c_i(\mathcal{E}) q^i$ its Chern polynomial. Recall that γ is the class of the anti-canonical divisor on X_{B_n}, which is the line bundle on X_{B_n} corresponding to the cross polytope.

Theorem 3.5. [EFLS, Theorem 7.15] Let D be a delta-matroid on $[n,\bar{n}]$. Then

$$\int_{X_{B_n}} c(\mathcal{I}_\mathrm{D}^\vee, v)\cdot \frac{1}{1-\gamma} = (1+v)^n \operatorname{Int}_\mathrm{D}\left(\frac{1-v}{1+v}\right).$$

Many constructions using isotropic tautological classes are valuative (cf. [BEST23, Proposition 5.6]), which is often useful when combined with Theorem 2.8.

Lemma 3.6. Any function that maps a delta-matroid D to a fixed polynomial expression in the exterior powers of $[\mathcal{I}_\mathrm{D}]$ or $[\mathcal{Q}_\mathrm{D}]$ or their duals is valuative, and similarly for a fixed polynomial expression in the Chern classes of $[\mathcal{I}_\mathrm{D}]$ or $[\mathcal{Q}_\mathrm{D}]$.

Proof. Let $\mathbb{Z}^{2^{[n,\bar{n}]}}$ be the free abelian group with basis given by subsets of $[n,\bar{n}]$. By [EHL, Proposition A.4] (see also [McM09, Theorem 4.6]), the function

$$\{\text{delta-matroids on } [n,\bar{n}]\} \to \bigoplus_{w\in W} \mathbb{Z}^{2^{[n,\bar{n}]}} \text{ given by } \mathrm{D} \mapsto \sum_{w\in W} \mathbf{e}_{B_w(\mathrm{D})}$$

is valuative. Any such polynomial expression depends only on $B_w(\mathrm{D})$ for each $w \in W$, and so it factors through this map and is therefore valuative. □

We also note the following property of Chern classes of $[\mathcal{I}_\mathrm{D}]$ and $[\mathcal{Q}_\mathrm{D}]$.

Proposition 3.7. Let D be a delta-matroid. Then $c(\mathcal{I}_\mathrm{D}) = c(\mathcal{Q}_\mathrm{D}^\vee)$ and $c(\mathcal{I}_\mathrm{D})c(\mathcal{I}_\mathrm{D}^\vee) = 1$.

Proof. We claim that one has the following short exact sequence of vector bundles

$$0 \to \mathcal{I} \to \mathcal{Q}^\vee \to \mathcal{O}_{OGr(n;2n+1)} \to 0.$$

The claim implies the proposition for realizable delta-matroids, and by valuativity (Theorem 2.8 and Lemma 3.6), for all delta-matroids. For the claim, let b be the map $\Bbbk^{2n+1} \to (\Bbbk^{2n+1})^\vee$ given by the bilinear pairing of the quadratic form q, that is, $\mathrm{b}(x)\colon y \mapsto q(x+y) - q(x) - q(y)$.

Note that if $L \subseteq \Bbbk^{2n+1}$ is isotropic, then $\mathrm{b}(L) \subseteq (\Bbbk^{2n+1}/L)^\vee \subseteq (\Bbbk^{2n+1})^\vee$, since $\mathrm{b}(\ell)(\ell') = q(\ell+\ell') - q(\ell) - q(\ell') = 0$ for all $\ell, \ell' \in L$. When $\operatorname{char} \Bbbk \neq 2$, the map b is an isomorphism, and when $\operatorname{char} \Bbbk = 2$, its kernel is $\operatorname{span}(\mathbf{e}_0)$, which is not isotropic. Hence, the map b gives an injection of vector bundles $0 \to \mathcal{I} \to \mathcal{Q}^\vee$, whose quotient line bundle is necessarily trivial because $\det \mathcal{I} \simeq \det \mathcal{Q}^\vee$ from (1).

Alternatively, one can prove the proposition via localization as follows. In $K_T(X_{B_n})$, we have that $[\mathcal{I}_{\mathrm{D}}] + 1 = [\mathcal{Q}_{\mathrm{D}}^\vee]$, which gives that $c(\mathcal{I}_{\mathrm{D}}) = c(\mathcal{Q}_{\mathrm{D}}^\vee)$, and therefore that $c(\mathcal{I}_{\mathrm{D}}^\vee) = c(\mathcal{Q}_{\mathrm{D}})$. Because $[\mathcal{I}_{\mathrm{D}}] + [\mathcal{Q}_{\mathrm{D}}] = [\mathcal{O}_{X_{B_n}}^{\oplus 2n+1}]$, we have that $c(\mathcal{I}_{\mathrm{D}})c(\mathcal{Q}_{\mathrm{D}}^\vee) = 1$, and substituting gives the result. □

In order to prove Theorem B, it remains to prove the Hirzebruch–Riemann–Roch-type formula. We prepare by doing the following computation, which will be used in the proof of Theorem A as well.

Proposition 3.8. Let D be a delta-matroid. Then $\psi([P(\mathrm{D})]) = c(\mathcal{I}_{\mathrm{D}}^\vee)$.

Proof. The class in $K_T(X_{B_n})$ defined by the line bundle corresponding to $\widehat{P(\mathrm{D})}$ under the usual correspondence between polytopes and nef toric line bundles on a toric variety has

$$[\widehat{P(\mathrm{D})}]_w = \prod_{i \in B_w(\mathrm{D})} T_{\bar{i}}.$$

Therefore, we see that

$$\psi^T([\widehat{P(\mathrm{D})}])_w = \prod_{a \in B_w(\mathrm{D}) \cap [n]} \frac{1-t_a}{1+t_a} \cdot \prod_{\bar{a} \in B_w(\mathrm{D}) \cap [\bar{n}]} \frac{1+t_a}{1-t_a}.$$

On the other hand, by the definition of $[\mathcal{I}_{\mathrm{D}}]$ and $[\mathcal{Q}_{\mathrm{D}}]$, we have that

$$c^T(\mathcal{I}_{\mathrm{D}})_w = \prod_{i \in B_w(\mathrm{D})} (1+t_i), \text{ and } c^T(\mathcal{Q}_{\mathrm{D}})_w = \prod_{i \in B_w(\mathrm{D})} (1-t_i).$$

We see that $\psi^T([\widehat{P(\mathrm{D})}]) = c^T(\mathcal{Q}_{\mathrm{D}})/c^T(\mathcal{I}_{\mathrm{D}})$. Because $c(\mathcal{I}_{\mathrm{D}}^\vee) = c(\mathcal{I}_{\mathrm{D}})^{-1} = c(\mathcal{Q}_{\mathrm{D}})$ by Proposition 3.7, we get that

$$\psi([\widehat{P(\mathrm{D})}]) = \psi([P(\mathrm{D})]^2) = c(\mathcal{I}_{\mathrm{D}}^\vee)^2.$$

In a graded ring, a class which has degree zero part equal to 1 has at most one square root with degree zero part equal to 1. Using this, we conclude that $\psi([P(\mathrm{D})]) = c(\mathcal{I}_{\mathrm{D}}^\vee)$. □

Proof of Theorem B. We have already constructed ψ, so it suffices to show that, for any $[\mathcal{E}] \in K(X_{B_n})$,

$$\chi(X_{B_n}, [\mathcal{E}]) = \frac{1}{2^n} \int_{X_{B_n}} \psi([\mathcal{E}]) \cdot \frac{1}{1-\gamma}.$$

By Theorem 2.8, $K(X_{B_n})$ is spanned by the classes $[P(\mathrm{D})]$ for D a delta-matroid, so it suffices to check this for $[\mathcal{E}] = [P(\mathrm{D})]$. Note that $\chi(X_{B_n}, [P(\mathrm{D})])$ is the number lattice points in $P(\mathrm{D})$, which is the number of feasible sets of D. It follows from Proposition 3.5 that $\frac{1}{2^n}\int_{X_{B_n}} c(\mathcal{I}_{\mathrm{D}}^\vee) \cdot \frac{1}{1-\gamma}$ is the number of feasible sets of D as well, so the result follows from Proposition 3.8. □

4. The push-pull computation

Our strategy to prove Theorem A is based on transferring the computation of $\pi_{1*}\pi_n^*(y(\mathrm{D})\cdot[\mathcal{O}(1)])$ to a computation on $OGr(n;2n+1)$. This idea first appeared in [FS12, Lemma 4.1] and was also used in [DES21]. This is implemented in Proposition 4.1. We then reduce the computation to a computation on X_{B_n}, following the strategy in [BEST23, Section 10.2].

Proposition 4.1. For $\epsilon \in K(OGr(n;2n+1))$, define a polynomial

$$R_\epsilon(v) = \sum_{i\geq 0} \chi(OGr(n;2n+1), \epsilon\cdot[\textstyle\bigwedge^i \mathcal{Q}^\vee])v^i.$$

Then $\pi_{1*}\pi_n^*\epsilon = R_\epsilon(u-1) \in K(\mathbb{P}^{2n})$, where $u = [\mathcal{O}_H] \in K(\mathbb{P}^{2n})$ is the class of the structure sheaf of a hyperplane $H \subset \mathbb{P}^{2n}$.

Proof. We prove the claim in a slighter more general setting: Let X be a variety with a short exact sequence of vector bundles $0 \to \mathcal{S} \to \mathcal{O}_X^{\oplus N} \to \mathcal{Q} \to 0$. Let $\mathbb{P}_X(\mathcal{S}) = \operatorname{Proj}\operatorname{Sym}^\bullet \mathcal{S}^\vee$ be the projective bundle with the projection $\pi\colon \mathbb{P}_X(\mathcal{S}) \to X$ and the inclusion $\mathbb{P}_X(\mathcal{S}) \hookrightarrow X\times\mathbb{P}^{N-1}$. Let $\rho\colon \mathbb{P}_X(\mathcal{S}) \to \mathbb{P}^{N-1}$ be the composition $\mathbb{P}_X(\mathcal{S}) \hookrightarrow X\times\mathbb{P}^{N-1} \to \mathbb{P}^{N-1}$. We claim that for $\epsilon \in K(X)$, one has

$$\sum_{i\geq 0} \chi\big(X, \epsilon\cdot[\textstyle\bigwedge^i \mathcal{Q}^\vee]\big)(u-1)^i = \rho_*\pi^*\epsilon,$$

where u is the class of the structure sheaf of a hyperplane in $\mathbb{P}^{N-1}$.

To prove the claim, since $K(\mathbb{P}^{N-1}) \simeq \mathbb{Z}[u]/(u^N)$, and since $\chi(\mathbb{P}^{N-1}, u^k)$ is equal to 1 if $0 \leq k \leq N-1$ and is equal to 0 if $k \geq N$, we first note that

$$\xi = \sum_{i\geq 0} \chi\big(\mathbb{P}^{N-1}, \xi\cdot u^{N-1-i}\cdot(1-u)\big)u^i \quad \text{for } \xi \in K(\mathbb{P}^{N-1}).$$

We consider the polynomial

$$\begin{aligned}\sum_{i\geq 0} \chi\big(\mathbb{P}^{N-1}, \rho_*\pi^*\epsilon\cdot u^{N-1-i}(1-u)\big)v^i &= \chi\left(\mathbb{P}^{N-1}, \rho_*\pi^*\epsilon\cdot v^N\cdot\frac{1-u}{v}\cdot\frac{1}{1-uv^{-1}}\right)\\ &= v^N\chi\left(\mathbb{P}^{N-1}, \rho_*\pi^*\epsilon\cdot\frac{1}{1+(1-u)^{-1}(v-1)}\right).\end{aligned}$$

Letting $\lambda = (1-u)^{-1} = [\mathcal{O}(1)] \in K(\mathbb{P}^{N-1})$ and substituting v with $v+1$, the right-hand-side becomes

$$(v+1)^N\chi\left(\mathbb{P}^{N-1}, \rho_*\pi^*\epsilon\cdot\frac{1}{1+\lambda v}\right) = (v+1)^N\chi\left(X, \epsilon\cdot\pi_*\rho^*\left(\frac{1}{1+\lambda v}\right)\right),$$

where the equality is due to the projection formula in K-theory. Thus, to finish we need show

$$(v+1)^N\pi_*\rho^*\left(\frac{1}{1+\lambda v}\right) = \sum_{i\geq 0}[\textstyle\bigwedge^i \mathcal{Q}^\vee]v^i.$$

But this follows by combining the following three facts from [Har77, III.8] and [Eis95, A.2]:

- We have $\pi_*\rho^*(\lambda^i) = [\operatorname{Sym}^i \mathcal{S}^\vee]$ for all $i \geq 0$.

- We have $\left(\sum_{i\geq 0}[\bigwedge^i \mathcal{S}^\vee]v^i\right)\left(\sum_{i\geq 0}[\bigwedge^i \mathcal{Q}^\vee]v^i\right) = (v+1)^N$ from the dual short exact sequence $0 \to \mathcal{Q}^\vee \to (\mathcal{O}_X^{\oplus N})^\vee \to \mathcal{S}^\vee \to 0$.
- We have $\left(\sum_{i\geq 0}(-1)^i[\operatorname{Sym}^i \mathcal{S}^\vee]v^i\right)\left(\sum_{i\geq 0}[\bigwedge^i \mathcal{S}^\vee]v^i\right) = 1$ from the exactness of the Koszul complex $\bigwedge^\bullet \mathcal{S}^\vee \otimes \operatorname{Sym}^\bullet \mathcal{S}^\vee \to \mathcal{O}_X \to 0$.

Lastly, the desired result follows from the general claim by setting $X = OGr(n; 2n+1)$ and $\mathcal{S} = \mathcal{I}$, since $OFl(1, n; 2n+1) = \mathbb{P}_{OGr(n;2n+1)}(\mathcal{I})$. □

Before proving Theorem A, we make one more preparatory computation.

Proposition 4.2. Let D be a delta-matroid. Then

$$\psi(\sum_{p\geq 0}[\wedge^p \mathcal{Q}_{\mathrm{D}}^\vee]v^p) = (v+1)^{n+1} \cdot c\left(\mathcal{I}_{\mathrm{D}}, \frac{v-1}{v+1}\right) \cdot c(\mathcal{I}_{\mathrm{D}}).$$

Proof. We compute equivariantly. We have that

$$\sum_{p\geq 0}[\wedge^p \mathcal{Q}_{\mathrm{D}}^\vee]_w v^p = (1+v) \prod_{i\in B_w(\mathrm{D})}(1+T_i v),$$

see, e.g., [EHL, Section 2]. Therefore, we get that

$$\begin{aligned}\psi^T(\sum_{p\geq 0}[\wedge^p \mathcal{Q}_{\mathrm{D}}^\vee])_w v^p &= (1+v) \prod_{i\in B_w(\mathrm{D})}\left(1+\frac{1+t_i}{1-t_i}v\right)\\ &= (1+v)^{n+1} \prod_{i\in B_w(\mathrm{D})}\left(1+\frac{t_i(v-1)}{v+1}\right) \cdot \prod_{i\in B_w(\mathrm{D})}\frac{1}{(1-t_i)}\\ &= (1+v)^{n+1}\cdot c^T\left(\mathcal{I}_{\mathrm{D}}, \frac{v-1}{v+1}\right)\cdot c^T(\mathcal{I}_{\mathrm{D}}^\vee)^{-1}.\end{aligned}$$

As $c(\mathcal{I}_{\mathrm{D}}^\vee)^{-1} = c(\mathcal{I}_{\mathrm{D}})$ by Proposition 3.7, the result follows. □

Proof of Theorem A. By Proposition 4.1, we need to show that

$$R_{y(\mathrm{D})\cdot[\mathcal{O}(1)]}(v) := \sum_{p\geq 0}\chi(OGr(n;2n+1), y(\mathrm{D})\cdot[\mathcal{O}(1)]\cdot[\wedge^p \mathcal{Q}^\vee])v^p = (v+1)\operatorname{Int}_{\mathrm{D}}(v).$$

The left-hand-side is valuative by Proposition 2.9, and the right-hand-side also by [ESS21, Theorem 3.6]. Thus, by Theorem 2.8, it suffices to verify this equality when D has a realization $[L] \in OGr(n; 2n+1)$ such that $y(\mathrm{D}) = [\mathcal{O}_{\overline{T\cdot[L]}}]$. As in the proof of Proposition 2.9, in this case we have a toric map $\varphi_L\colon X_{B_n} \to \overline{T\cdot[L]}$ such that $\varphi_{L*}[\mathcal{O}_{X_{B_n}}] = y(\mathrm{D})$, and by construction $\varphi_L^*[\mathcal{O}(1)] = [P(\mathrm{D})]$ and $\varphi_L^*[\wedge^p \mathcal{Q}^\vee] = [\wedge^p \mathcal{Q}_{\mathrm{D}}^\vee]$. Hence, by the projection formula, we have that

$$R_{y(\mathrm{D})\cdot[\mathcal{O}(1)]}(v) = \sum_{p\geq 0}\chi(X_{B_n}, [P(\mathrm{D})]\cdot[\wedge^p \mathcal{Q}_{\mathrm{D}}^\vee])v^p.$$

Applying Theorem B and Proposition 4.2, we get that

$$\begin{aligned} R_{y(\mathrm{D})\cdot[\mathcal{O}(1)]}(v) &= \frac{1}{2^n}\int_{X_{B_n}} \frac{1}{1-\gamma}\cdot c(\mathcal{I}_{\mathrm{D}}^\vee)\cdot(v+1)^{n+1}\cdot c\left(\mathcal{I}_{\mathrm{D}}, \frac{v-1}{v+1}\right)\cdot c(\mathcal{I}_{\mathrm{D}}) \\ &= \frac{(v+1)^{n+1}}{2^n}\int_{X_{B_n}} \frac{1}{1-\gamma}\cdot c\left(\mathcal{I}_{\mathrm{D}}, \frac{v-1}{v+1}\right) \\ &= (v+1)\,\mathrm{Int}_{\mathrm{D}}(v). \end{aligned}$$

In the second line we used Proposition 3.7, and in the third line we used Proposition 3.5. □

5. Structure sheaves of orbit closures

We noted in Remark 2.10 that, using the formula in Proposition 2.3, one may assign a K-class $[\mathcal{O}_{\overline{T\cdot \mathrm{D}}}]$ to a delta-matroid D, different from $y(\mathrm{D})$. It has the feature that $[\mathcal{O}_{\overline{T\cdot \mathrm{D}}}] = [\mathcal{O}_{\overline{T\cdot [L]}}]$ whenever D has a realization $[L] \in OGr(n; 2n+1)$. Here, we collect various examples and questions about this K-class. The Macaulay2 code used for the computation of these examples can be found at `https://github.com/chrisweur/KThryDeltaMat`. A database of small delta-matroids can be found at `https://eprints.bbk.ac.uk/id/eprint/19837/` [FMN18].

We start with the smallest example where $y(\mathrm{D}) \neq [\mathcal{O}_{\overline{T\cdot \mathrm{D}}}]$.

Example 5.1. Let $L \subset \Bbbk^7$ be the maximal isotropic subspaces given by the row span of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & a & b & 0 \\ 0 & 1 & 0 & -a & 0 & c & 0 \\ 0 & 0 & 1 & -b & -c & 0 & 0 \end{pmatrix}$$

for a, b, c generic elements of $\Bbbk$. Then the delta-matroid D represented by L has feasible sets

$$\{1,2,3\}, \{1,\bar{2},\bar{3}\}, \{\bar{1},2,\bar{3}\}, \{\bar{1},\bar{2},3\}.$$

The stabilizer of $[L]$ is $\{(1,1,1),(-1,-1,-1)\} \in T$, so the map $X_{B_3} \to \overline{T\cdot[L]}$ is a double cover. This implies that $y(\mathrm{D}) \neq [\mathcal{O}_{\overline{T\cdot[L]}}]$. Alternatively, one can verify that $P(\mathrm{D})$ is not very ample with respect to $\mathbb{Z}^3$ and using Proposition 2.3. We have $\pi_{1*}\pi_n^*([\mathcal{O}_{\overline{T\cdot[L]}}]\cdot[\mathcal{O}(1)]) = R_{[\mathcal{O}_{\overline{T\cdot[L]}}]\cdot[\mathcal{O}(1)]}(u-1)$ by Proposition 4.1. A computer computation shows that

$$R_{[\mathcal{O}_{\overline{T\cdot[L]}}]\cdot[\mathcal{O}(1)]}(v) = 4v^2 + 8v + 4 = (v+1)\,\mathrm{Int}_{\mathrm{D}}(v).$$

In other words, here Theorem A holds with $[\mathcal{O}_{\overline{T\cdot[L]}}]$ in place of $y(\mathrm{D})$ although $y(\mathrm{D}) \neq [\mathcal{O}_{\overline{T\cdot[L]}}]$.

Let us say that a delta-matroid has property (∗) if Theorem A holds with $[\mathcal{O}_{\overline{T\cdot \mathrm{D}}}]$ in place of $y(\mathrm{D})$, that is, by Proposition 4.1, if

$$R_{[\mathcal{O}_{\overline{T\cdot \mathrm{D}}}]\cdot[\mathcal{O}(1)]}(v) = (v+1)\,\mathrm{Int}_{\mathrm{D}}(v). \tag{$*$}$$

We now feature an example where (∗) fails.

Example 5.2. Let D be the delta-matroid with feasible sets

$\{\bar{1},\bar{2},\bar{3},\bar{4}\},\{1,\bar{2},\bar{3},\bar{4}\},\{\bar{1},2,\bar{3},\bar{4}\},\{\bar{1},\bar{2},3,\bar{4}\},\{\bar{1},\bar{2},\bar{3},4\}\{\bar{1},2,3,4\},\{1,\bar{2},3,4\},\{1,2,\bar{3},4\},\{1,2,3,\bar{4}\}.$

A computer computation shows that $(v+1)\operatorname{Int}_{\mathrm{D}}(v) = 9+16v+7v^2$, but

$$R_{[\mathcal{O}_{\overline{T\cdot \mathrm{D}}}]\cdot[\mathcal{O}(1)]}(v) = 9+16v+6v^2-v^3+v^4+v^5.$$

A computer search shows that Example 5.2 is the only delta-matroid up to $n=4$ that fails $(*)$. The delta-matroids in the above two examples differ in the following ways. The delta-matroid in Example 5.1

- is realizable,
- is *even* in the sense that the parity of $|B\cap[n]|$ is constant over all feasible sets B, and
- has the polytope $P(\mathrm{D})$ very ample with respect to the lattice (affinely) generated by its vertices.

The last property, when D has a realization $[L]$, is equivalent to stating that $\overline{T\cdot[L]}$ is a normal variety. All three properties fail for the delta-matroid in Example 5.2. We thus ask:

Question 5.3. When does Theorem A hold with $[\mathcal{O}_{\overline{T\cdot \mathrm{D}}}]$ in place of $y(\mathrm{D})$? More specifically, is $(*)$ satisfied when

- D is realizable?
- D is an even delta-matroid?
- the polytope $P(\mathrm{D})$ is very ample with respect to the lattice (affinely) generated by its vertices?

We expect $(*)$ to fail for some realizable delta-matroid, but do not know any examples. We conclude with the following realizable even delta-matroid example.

Example 5.4. Let G be a graph on vertices $[7]$ with edges $\{12,13,23,34,45,56,57,67\}$. Let $A(G)$ be its adjacency matrix, considered over $\mathbb{F}_2$ so that it is skew-symmetric with zero diagonal entries. Let D be the delta-matroid realized by the row span of the $7\times(7+7+1)$ matrix $[\,A\mid I_7\mid 0\,]$. That is, its feasible sets are

$$\left\{\begin{matrix}\text{maximal admissible subsets } B\subset[7,\bar{7}] \text{ such that the principal minor}\\ \text{of } A(G) \text{ corresponding to the subset } B\cap[7] \text{ is nonzero}\end{matrix}\right\}.$$

The polytope $P(\mathrm{D})$ is not very ample with respect to the lattice (affinely) generated by its vertices, demonstrated as follows. One verifies that $P(\mathrm{D})$ contains the origin, and the semigroup $\mathbb{Z}_{\geq 0}\{P(\mathrm{D})\cap\mathbb{Z}^7\}$ is generated by

$$\{\mathbf{e}_{12},\mathbf{e}_{13},\mathbf{e}_{23},\mathbf{e}_{34},\mathbf{e}_{45},\mathbf{e}_{56},\mathbf{e}_{57},\mathbf{e}_{67}\}.$$

In the intersection of the cone $\mathbb{R}_{\geq 0}\{P(\mathrm{D})\}$ and the lattice $\mathbb{Z}\{P(\mathrm{D})\cap\mathbb{Z}^7\}$, we have the point

$$(1,1,1,0,1,1,1) = \frac{1}{2}(\mathbf{e}_{12}+\mathbf{e}_{13}+\mathbf{e}_{23}) + \frac{1}{2}(\mathbf{e}_{56}+\mathbf{e}_{57}+\mathbf{e}_{67}) = \mathbf{e}_{13}+\mathbf{e}_{23}-\mathbf{e}_{34}+\mathbf{e}_{45}+\mathbf{e}_{67},$$

but this point is not in the semigroup $\mathbb{Z}_{\geq 0}\{P(\mathrm{D}) \cap \mathbb{Z}^7\}$. In particular, the torus-orbit-closure is not normal. Nonetheless, this even delta-matroid satisfies (∗): a computer computation shows that

$$R_{[\mathcal{O}_{\overline{T\cdot \mathrm{D}}}]\cdot[\mathcal{O}(1)]}(v) = 32 + 92v + 92v^2 + 36v^3 + 4v^4 = (v+1)\,\mathrm{Int}_{\mathrm{D}}(v).$$

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