SIGNED PERMUTOHEDRA, DELTA-MATROIDS, AND BEYOND

CHRISTOPHER EUR, ALEX FINK, MATT LARSON, HUNTER SPINK

ABSTRACT. We establish a connection between the algebraic geometry of the type B permutohedral toric variety and the combinatorics of delta-matroids. Using this connection, we compute the volume and lattice point counts of type B generalized permutohedra. Applying tropical Hodge theory to a new framework of "tautological classes of delta-matroids," modeled after certain vector bundles associated to realizable delta-matroids, we establish the log-concavity of a Tutte-like invariant for a broad family of delta-matroids that includes all realizable delta-matroids. Our results include new log-concavity statements for all (ordinary) matroids as special cases.

CONTENTS

1. INTRODUCTION

For a nonnegative integer n, let $[n] = \{1, \ldots, n\}$. For a subset $S \subseteq [n]$, let $\mathbf{e}_S = \sum_{i \in S} \mathbf{e}_i \in \mathbb{R}^n$ be the sum of the standard basis vectors indexed by S. If $n \geq 1$, the $\boldsymbol{A_{n-1}}$ **permutohedral fan** $\Sigma_{A_{n-1}}$ is the complete fan in $\mathbb{R}^n$ whose maximal cones are the chambers of the arrangement of hyperplanes

$$H_{\mathbf{e}_i - \mathbf{e}_j} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i - x_j = 0\} \quad \text{for all } 1 \leq i < j \leq n.$$

A polytope $P \subset \mathbb{R}^n$ is an $\boldsymbol{A_{n-1}}$ **generalized permutohedron** if its normal fan coarsens the fan $\Sigma_{A_{n-1}}$. The polyhedral properties of A_{n-1} generalized permutohedra and the algebraic geometry of the toric variety $X_{A_{n-1}}$ associated to $\Sigma_{A_{n-1}}$ (as a fan in $\mathbb{R}^n/\mathbb{R}(1, \ldots, 1)$) have been well studied as a way to illuminate the structure of several combinatorial objects [Pos09, AA], including graphs, posets, and, notably in recent years, matroids.

Definition 1.1. A **matroid** M on $[n]$ is a nonempty collection $\mathcal{B}$ of subsets of $[n]$, called the **bases** of M, such that the polytope

$$P(\mathrm{M}) = \text{the convex hull of } \{\mathbf{e}_B \mid B \in \mathcal{B}\} \subset [0,1]^n,$$

has all edges parallel translates of $\mathbf{e}_i - \mathbf{e}_j$ for various $i, j \in [n]$, or, equivalently, such that $P(\mathrm{M})$ is an A_{n-1} generalized permutohedron with all vertices lying in $\{0,1\}^n$.

Recently, an interpretation of matroids as elements in the Chow cohomology ring of $X_{A_{n-1}}$ has led to fruitful developments in matroid theory [HK12, AHK18, BST, LdMRS20]. Conversely, this interpretation allows matroid theory to inform the geometry of $X_{A_{n-1}}$ [Ham17, EHL]. Many of these developments have recently been unified, recovered, and extended under the new framework of "tautological classes of matroids" [BEST], modeled after certain torus-equivariant vector bundles on $X_{A_{n-1}}$.

Meanwhile, the fan $\Sigma_{A_{n-1}}$ generalizes to the Coxeter complex Σ_Φ of an arbitrary crystallographic root system Φ, the toric variety $X_{A_{n-1}}$ generalizes to the toric variety X_Φ of Σ_Φ, and the combinatorial objects such as graphs, posets, and matroids generalize appropriately to their Coxeter analogues (see [ACEP20, §4] and references therein). For instance, in the theory of Coxeter matroids [BGW03], matroids in the usual sense are exactly the type A minuscule Coxeter matroids. Several works [Pro90, DL94, Ste94, Kly95] have studied the Chow cohomology ring of X_Φ. Missing in these previous works is an interaction between Coxeter matroids and the Chow cohomology ring of X_Φ that generalizes the interaction between matroids and the Chow cohomology ring of $X_{A_{n-1}}$.

We establish here such an interaction when Φ is a root system of type B, noting that the type B minuscule Coxeter matroids are exactly delta-matroids (Definition 1.3). This interaction interfaces particularly well with the framework of "tautological classes of delta-matroids" we develop in Section 7, which are modeled after toric vector bundles associated to maximal isotropic subspaces that realize delta-matroids. Some barriers to establishing a uniform treatment for arbitrary Coxeter types can be found in Remark 3.6.

1.1. Main combinatorial consequences.

Definition 1.2. Let $n \geq 0$. The $\boldsymbol{B_n}$ **permutohedral fan** Σ_{B_n} is the complete fan in $\mathbb{R}^n$ whose maximal cones are the chambers of the arrangement of hyperplanes

$$\begin{aligned} H_{\mathbf{e}_i \pm \mathbf{e}_j} &= \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i \pm x_j = 0\} \quad \text{for all } i \neq j \in [n], \text{ and} \\ H_{\mathbf{e}_i} &= \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i = 0\} \quad \text{for all } i \in [n]. \end{aligned}$$

The fan Σ_{B_n} is the normal fan of the type B_n permutohedron Π_{B_n}, also called the signed permutohedron, which is the convex hull of $\{w \cdot (n, \dots, 1) \in \mathbb{R}^n : w \in \mathfrak{S}_n^B\}$, where $\mathfrak{S}_n^B$ is the signed permutation group (see §2.1). A polytope $P \subset \mathbb{R}^n$ is a $\boldsymbol{B_n}$ **generalized permutohedron** if its normal fan Σ_P coarsens Σ_{B_n}, or, equivalently, if each edge of P is parallel to $\mathbf{e}_i + \mathbf{e}_j$, $\mathbf{e}_i - \mathbf{e}_j$, or $\mathbf{e}_i$ for various $i, j \in [n]$.

A celebrated result of Postnikov [Pos09] gives a formula for the volumes and lattice point enumerators of A_{n-1} generalized permutohedra in terms of **transversals** of subsets $S_1, \dots, S_k$ of $[n]$, i.e., subsets $\tau \subseteq [n]$ such that there exist a bijection $j : \{1, \dots, k\} \to \tau$ with $j(i) \in S_i$ for all

$i \in \{1, \ldots, k\}$. We give a formula for the volumes and lattice point enumerators of B_n generalized permutohedra as follows.

Let $[\overline{n}] = \{\overline{1}, \ldots, \overline{n}\}$, and let $[n, \overline{n}] = [n] \sqcup [\overline{n}]$, which is endowed with the obvious involution $\overline{(\cdot)}$. For $S \subseteq [n, \overline{n}]$, we denote $\mathbf{e}_S = \sum_{i \in S} \mathbf{e}_i$, where $\mathbf{e}_{\overline{j}} := -\mathbf{e}_j$ for $j \in [n]$. Define the set AdS of **admissible subsets** of $[n, \overline{n}]$ to be

$$\mathsf{AdS} = \{S \subset [n, \overline{n}] \text{ such that } \{i, \bar{i}\} \not\subseteq S \text{ for all } i \in [n]\}, \text{ and define } \mathsf{AdS}_n = \{S \in \mathsf{AdS} : |S| = n\}$$

to be the set of maximal admissible subsets. A **signed transversal** of $S_1, \ldots, S_n$ is an admissible subset $\tau \in \mathsf{AdS}_n$ such that there exists a bijection $j : \{1, \ldots, n\} \to \tau$ with $j(i) \in S_i$ for all $i = 1, \ldots, n$. For an admissible subset $S \in \mathsf{AdS}$, let

$$\Delta^0_S = \text{the simplex which is the convex hull of } \{\mathbf{e}_i \mid i \in S\} \cup \{\mathbf{0}\} \text{ in } \mathbb{R}^n.$$

Theorem A. Let P be a lattice B_n generalized permutohedron (i.e., P has vertices in $\mathbb{Z}^n$).

(a) There exists a unique set of integers $\{c_S \in \mathbb{Z} \mid S \in \mathsf{AdS} \setminus \{\varnothing\}\}$ such that the signed Minkowski sum $\displaystyle\sum_{S \in \mathsf{AdS} \setminus \{\varnothing\}} c_S \Delta^0_S$ equals P. Hence we may write $P = P(\{c_S\})$.

(b) For any sequence $(S_1, \ldots, S_n)$ of nonempty admissible subsets of $[n, \overline{n}]$, one has that

$$\text{mixed volume of } \{\Delta^0_{S_1}, \ldots, \Delta^0_{S_n}\} = |\{\text{signed transversals of } S_1, \ldots, S_n\}|.$$

In particular, normalizing the volume of the standard simplex $\Delta^0_{[n]}$ to be 1, one has

$$\operatorname{Vol}\big(P(\{c_S\})\big) = \sum_{(S_1, \ldots, S_n)} |\{\text{signed transversals of } S_1, \ldots, S_n\}| \cdot c_{S_1} c_{S_2} \cdots c_{S_n}$$

where the sum is over all sequences $(S_1, \ldots, S_n)$ of nonempty admissible subsets.

(c) Let Ψ be the linear operator on polynomials that sends a polynomial $f(x_1, \ldots, x_m)$ to the polynomial obtained by replacing each monomial $x_1^{d_1} \cdots x_m^{d_m}$ in f with $\binom{x_1}{d_1} \cdots \binom{x_m}{d_m}$. Let $\square = [0, 1]^n$ be the standard unit cube in $\mathbb{R}^n$. Then, we have

$$\#\text{ lattice points of } \big(P(\{c_S\}) - \square\big) = \Psi\Big(\operatorname{Vol}\big(P(\{c_S\})\big)\Big),$$

where $P(\{c_S\}) - \square$ is the polytope $P(\{c'_S\})$. Here $c'_S = c_S - 1$ if $S = \{i\} \subseteq [n]$ and $c'_S = c_S$ otherwise.

The statements (a), (b), and (c) generalize to type B the classical type A results [ABD10, Proposition 2.3], [Pos09, Theorem 9.3], and [Pos09, Theorem 11.3], respectively. Hence, Theorem A fully answers [ACEP20, Question 9.3] for type B. The statement (a) was also shown in [Bas21] via a study of Tits algebras, and a different set of polytopes satisfying the property in (a) was obtained in [PPR] via a study of shard polytopes. Neither work gives a formula for the volume or lattice point enumerator. We will deduce Theorem A via our study of delta-matroids.

Definition 1.3. A **delta-matroid** D on ground set $[n, \overline{n}]$ is a nonempty collection $\mathcal{F} \subseteq \mathsf{AdS}_n$ of admissible subsets of $[n, \overline{n}]$ of cardinality n, called the **feasible sets** of D, such that the polytope

$$P(\mathrm{D}) = \text{the convex hull of } \{\mathbf{e}_{B \cap [n]} \mid B \in \mathcal{F}\} \subset [0, 1]^n$$

has all edges parallel translates of $\mathbf{e}_i + \mathbf{e}_j$, $\mathbf{e}_i - \mathbf{e}_j$, or $\mathbf{e}_i$ for various $i, j \in [n]$, or, equivalently, such that $P(\mathrm{D})$ is a B_n generalized permutohedron with all vertices lying in $\{0,1\}^n$. For $i \in [n]$, we say that i is a **loop**, resp. **coloop**, of D if no, resp. every, feasible set contains i.

We often identify a delta-matroid D with its polytope $P(\mathrm{D})$. Delta-matroids were introduced in [Bou87] by generalizing the basis exchange axiom for matroids. Just as matroids are combinatorial abstractions of graphs, delta-matroids are combinatorial abstractions of graphs embedded in surfaces, see [CMNR19a, CMNR19b]. For the equivalence of the definition of delta-matroids in those works and the one given here, see [BGW03, Ch. 4].

A matroid M on $[n]$ with set of bases $\mathcal{B}$ defines a delta-matroid D in two different ways: first, by its base polytope $P(\mathrm{M})$, and, second, by its independence polytope

$$IP(\mathrm{M}) = \text{the convex hull of } (\mathbf{e}_I \mid I \subseteq [n] \text{ such that } I \subseteq B \text{ for some } B \in \mathcal{B}) \subset [0,1]^n.$$

We will frequently use $P(\mathrm{M})$ and $IP(\mathrm{M})$ to refer to the delta-matroids obtained from M as above.

We introduce a new invariant of delta-matroids defined by a recursive relation similar to the one satisfied by Tutte polynomials of matroids. See Definition 5.1 for the deletion $\mathrm{D} \setminus i$, contraction D/i, and projection $\mathrm{D}(i)$ of a delta-matroid D.

Definition 1.4. For a delta-matroid D on $[n, \overline{n}]$ with feasible sets $\mathcal{F}$, the ***U*-polynomial** $U_{\mathrm{D}}(u,v)$ is the unique bivariate polynomial satisfying the properties:

- (Base case) If $n = 0$, then $U_{\mathrm{D}}(u,v) = 1$.
- (Recursive relation) If $n \geq 1$ and $i \in [n]$, then

$$U_{\mathrm{D}}(u,v) = \begin{cases} U_{\mathrm{D}\setminus i}(u,v) + U_{\mathrm{D}/i}(u,v) + uU_{\mathrm{D}(i)}(u,v) & \text{if } i \text{ is neither a loop nor a coloop} \\ (u+v+1) \cdot U_{\mathrm{D}\setminus i}(u,v) & \text{if } i \text{ is a loop or a coloop.} \end{cases}$$

Proposition 5.2 verifies that this recursive definition is well-defined. Specializing $U_{\mathrm{D}}(u,v)$ at $u = 0$, one obtains the **interlace polynomial** $\mathrm{Int}_{\mathrm{D}}(v)$, introduced in [ABS04] for graphs and generalized to delta-matroids in [BH14]. See [Mor17] for a survey on interlace polynomials. [1] The invariant U_{D} also gives rise to two invariants of (ordinary) matroids. Let T_{M} denote the Tutte polynomial of M. One computes, as done in Examples 5.5 and 5.6, that

$$U_{P(\mathrm{M})}(u,v) = \sum_{T \subseteq S \subseteq [n]} u^{|S-T|} v^{\mathrm{corank}_{\mathrm{M}}(S) + \mathrm{nullity}_{\mathrm{M}}(T)},$$

so in particular $\mathrm{Int}_{P(\mathrm{M})}(v) = T_{\mathrm{M}}(v+1, v+1)$, and

$$U_{IP(\mathrm{M})}(u,v) = (u+1)^{n - \mathrm{rank}(\mathrm{M})}\, T_{\mathrm{M}}\left(u+2, \frac{u+v+1}{u+1}\right).$$

We establish a log-concavity property for the U-polynomial for delta-matroids which have an **enveloping matroid** (Definition 6.6), a condition necessary for applying tools from the tropical Hodge theory developed in [ADH]. Such delta-matroids include $P(\mathrm{M})$ and $IP(\mathrm{M})$ when M is a

[1] In our terms the "interlace polynomial" defined in [ABS04] equals $\mathrm{Int}_{\mathrm{D}}(v-1)$. Our definition agrees with [Mor17, Definition 28] and the polynomial denoted q_1 in [BH14].

matroid (Proposition 6.10), and include realizable delta-matroids (Proposition 6.8), in particular the adjacency delta-matroids of graphs (Example 6.4) and delta-matroids from graphs embedded on surfaces (Example 6.5). We say that the coefficients of a homogeneous polynomial f of degree d form a **log-concave unbroken array** if for any $1 \leq i < j \leq n$ and any monomial $x^{\mathbf{m}}$ of degree $d' \leq d$, the coefficients of $\{x_i^k x_j^{d-d'-k} x^{\mathbf{m}}\}$ form a nonnegative log-concave sequence with no internal zeros.

Theorem B. Let D be a delta-matroid which has an enveloping matroid. Then the polynomials

$$(y+q)^n\, U_{\mathrm{D}}\left(\frac{x}{y+q}, \frac{y-q}{y+q}\right) \quad \text{and} \tag{1.1}$$

$$(y+w)^n\, U_{\mathrm{D}}\left(\frac{2z+x}{y+w}, \frac{y-z}{y+w}\right) \tag{1.2}$$

have a log-concave unbroken array of coefficients. In fact, they are denormalized Lorentzian polynomials in the sense of [BH20, BLP].

Setting $x = 0$ and $q = 1$ in (1.1) implies that the transformation $(y+1)^n \operatorname{Int}_{\mathrm{D}}(\frac{y-1}{y+1})$ of the interlace polynomial has nonnegative log-concave coefficients with no internal zeros, and hence has unimodal coefficients. We note that the interlace polynomial of a realizable delta-matroid can have non-unimodal coefficients (Example 8.3); see Remark 8.2 for a history of conjectures about unimodality for the interlace polynomial. Theorem B also yields new log-concavity results for (ordinary) matroids. For instance, that $U_{P(\mathrm{M})}(u, 0)$ has log-concave coefficients implies that the sequence

$$a_i = |\{T \subseteq S \subseteq [n] \colon T \text{ independent in } \mathrm{M} \text{ and } S \text{ spanning in } \mathrm{M},\ |S| - |T| = i\}|$$

is log-concave. See Corollary 8.1 for more implications of Theorem B. See Theorems 7.15 and 7.14 for the algebro-geometric results underlying the formulas (1.1) and (1.2) respectively, and see §8 for the derivation of log-concavity from these formulas using tropical Hodge theory.

Conjecture 1.5. The hypothesis that D has an enveloping matroid can be removed in Theorem B.

1.2. **Underlying geometry.** We obtain Theorems A and B by establishing a new connection between the algebraic geometry of the B_n permutohedral fan Σ_{B_n} and the combinatorics of delta-matroids. The fan Σ_{B_n}, as a rational fan over $\mathbb{Z}^n$, defines a smooth projective toric variety X_{B_n} which we call the $\boldsymbol{B_n}$**-permutohedral variety**. We follow the conventions in [Ful93, CLS11] for toric varieties and polyhedra, and we work over an algebraically closed field $\Bbbk$. The toric variety X_{B_n} is equipped with two well-studied rings, the Chow cohomology ring $A^\bullet(X_{B_n})$ and the Grothendieck ring of vector bundles $K(X_{B_n})$.

We construct an isomorphism between the rings $K(X_{B_n})$ and $A^\bullet(X_{B_n})$, different from the classical Hirzebruch–Riemann–Roch theorem. Recall that the Hirzebruch–Riemann–Roch theorem states that for an arbitrary smooth projective variety X, the Chern character map $ch : K(X) \otimes \mathbb{Q} \xrightarrow{\sim} A^\bullet(X) \otimes \mathbb{Q}$ is an isomorphism such that

$$\chi([\mathcal{E}]) = \int_X ch([\mathcal{E}]) \cdot \operatorname{Td}(X) \quad \text{for all } [\mathcal{E}] \in K(X),$$

where $\chi : K(X) \to \mathbb{Z}$ is the sheaf Euler characteristic map, $\int_X$ is the degree map, and $\operatorname{Td}(X) \in A^\bullet(X) \otimes \mathbb{Q}$ is the Todd class of X.

To state our exceptional Hirzebruch–Riemann–Roch-type theorem, we need the following definitions. Note that the product fan $(\Sigma_{B_1})^n$, which is the fan induced by the arrangement of coordinate hyperplanes in $\mathbb{R}^n$, is a coarsening of Σ_{B_n}. Hence, since the toric variety of Σ_{B_1} is $\mathbb{P}^1$, we have a birational map $X_{B_n} \to (\mathbb{P}^1)^n$ of toric varieties. Let $\boxplus\mathcal{O}(1)$ be the vector bundle on X_{B_n} obtained as the direct sum of the pullbacks of $\mathcal{O}_{\mathbb{P}^1}(1)$ from each $\mathbb{P}^1$ in the product $(\mathbb{P}^1)^n$.

Theorem C. There exists a ring isomorphism $\phi^B : K(X_{B_n}) \to A^\bullet(X_{B_n})$ such that

$$\chi([\mathcal{E}]) = \int_{X_{B_n}} \phi^B([\mathcal{E}]) \cdot c(\boxplus\mathcal{O}(1)) \quad \text{for all } [\mathcal{E}] \in K(X_{B_n}),$$

where $c(\boxplus\mathcal{O}(1)) = c_0(\boxplus\mathcal{O}(1)) + \cdots + c_n(\boxplus\mathcal{O}(1))$ denotes total Chern class of $\boxplus\mathcal{O}(1)$.

We define the map ϕ^B and prove Theorem C in §3. We note that the map ϕ^B in Theorem C differs from ch and is an isomorphism integrally, and the class $c(\boxplus\mathcal{O}(1))$ differs from the Todd class of X_{B_n}. The isomorphism ϕ^B here is closely related to the type A exceptional Hirzebruch–Riemann–Roch isomorphisms that appeared in [BEST] and [EHL] (see §3.3).

The combinatorial utility of Theorem C is mediated by our Theorem D that describes a basis of the ring $K(X_{B_n})$ in terms of Schubert delta-matroids (Proposition-Definition 2.6), which correspond to the Bruhat cells of a type B generalized flag variety (Example 6.3). Recall that there is a standard correspondence between polytopes and base-point-free line bundles on toric varieties [CLS11, §6.2].

Theorem D. The classes of line bundles on X_{B_n} corresponding to the polytopes of Schubert delta-matroids without coloops form a basis for $K(X_{B_n})$.

The first step of the proof of Theorem D is that $K(X_{B_n})$ is isomorphic to a combinatorially defined ring, the **polytope algebra** $\overline{\mathbb{I}}(\Sigma_{B_n})$ of indicator functions of lattice B_n generalized permutohedra modulo translation (see Definition 2.4). This is a special case of the folklore statement that $K(X_\Sigma)$ is isomorphic to a polytope algebra for an arbitrary smooth projective fan Σ, proven precisely in [EHL, Appendix A]. The isomorphism sends the class $[\mathbf{1}(P)]$ of a B_n generalized permutohedron P to the K-class of the corresponding line bundle. The proof of Theorem D proceeds by showing, using polyhedral properties special to Boolean cubes, that $\overline{\mathbb{I}}(\Sigma_{B_n})$ is generated by classes of delta-matroid polytopes, and indeed by classes of coloop-free Schubert delta-matroid polytopes, which are shown to satisfy no linear relations.

By combining Theorem C with Theorem D, we construct in Corollary 4.5 a graded basis for $A^\bullet(X_{B_n})$ indexed by coloop-free Schubert delta-matroids. By considering the basis elements in $A^1(X_{B_n})$, we deduce statement (a) of Theorem A. The rest of Theorem A is deduced from Theorem C in §4.2. Theorem B is proved by constructing torus-equivariant nef vector bundles on X_{B_n} which are related to delta-matroids; see §7.2 and §7.3. The proof of Theorem B invokes Theorem C in §7.4 to compute certain intersection numbers. Their log-concavity properties are established using tropical Hodge theory in §8.

Acknowledgments. We thank Steven Noble for pointing out Example 6.11. The first author is partially supported by the US National Science Foundation (DMS-2001854). The third author is supported by an NDSEG fellowship.

2. POLYTOPE ALGEBRAS OF DELTA-MATROIDS

In this section, we prove Theorem D, which describes $K(X_{B_n})$ in terms of delta-matroids.

2.1. **The fan Σ_{B_n} and the signed permutation group $\mathfrak{S}_n^B$.** Let n be a nonnegative integer. Recall that the B_n permutohedral fan Σ_{B_n} was defined to be the complete fan in $\mathbb{R}^n$ whose maximal cones are the chambers of the type B arrangement of hyperplanes, the union of all hyperplanes of the form $\{x_i \pm x_j = 0\}$ and $\{x_i = 0\}$.

Definition 2.1. The Weyl reflection group corresponding to the real hyperplane arrangement defining Σ_{B_n} is the **signed permutation group** $\mathfrak{S}_n^B$, which is the subgroup

$$\mathfrak{S}_n^B = \{w \in \mathfrak{S}_{[n,\overline{n}]} \mid w(\bar{i}) = \overline{w(i)} \text{ for all } i \in [n,\overline{n}]\} \subset \mathfrak{S}_{[n,\overline{n}]},$$

where $\mathfrak{S}_{[n,\overline{n}]}$ denotes the symmetric group on $[n,\overline{n}]$.

A permutation σ of $[n]$ can be extended to a signed permutation of $[n,\overline{n}]$ by setting $\sigma(\bar{i}) = \overline{\sigma(i)}$. In this way, the permutation group $\mathfrak{S}_n$ is naturally a parabolic subgroup of $\mathfrak{S}_n^B$, viewed as the stabilizer of $[n] \subset [n,\overline{n}]$. Then we have an expression of $\mathfrak{S}_n^B$ as a semidirect product

$$\mathfrak{S}_n^B = \mathfrak{S}_n \ltimes \{\pm 1\}^n,$$

where $\{\pm 1\}^n \trianglelefteq \mathfrak{S}_n^B$ is the **sign group** such that the ith copy of $\{\pm 1\}$ is the subgroup generated by the transposition $(i, \bar{i})$. We denote the map to the set of left cosets of $\mathfrak{S}_n$ by

$$(\epsilon_1, \ldots, \epsilon_n)\colon \mathfrak{S}_n^B \to \{\pm 1\}^n,$$

which can also be described by

$$\epsilon_i(w) = \begin{cases} 1 & i \in w([n]) \\ -1 & i \notin w([n]). \end{cases}$$

Recall that we have defined $\mathbf{e}_{\bar{i}} = -\mathbf{e}_i \in \mathbb{R}^n$ for $i \in [n]$. We next fix notation for cones of Σ_{B_n}.

Proposition 2.2. The maximal cones of Σ_{B_n} are given by

$$C_w = \operatorname{cone}\{\mathbf{e}_{w(1)}, \ldots, \mathbf{e}_{w(1)} + \cdots + \mathbf{e}_{w(n)}\}$$

for each $w \in \mathfrak{S}_n^B$. The cone C_w is the unique maximal cone containing $w \cdot (n, \ldots, 1)$. The dual cones are given by

$$C_w^\vee = \operatorname{cone}\{\mathbf{e}_{w(1)}, \mathbf{e}_{w(2)} - \mathbf{e}_{w(1)}, \ldots, \mathbf{e}_{w(n)} - \mathbf{e}_{w(n-1)}\}.$$

We describe here the various (left) actions of $\mathfrak{S}_n^B$ we will consider.

- $\mathfrak{S}_n^B$ acts on $\mathbb{R}^n$ by $w \cdot \mathbf{e}_i = \mathbf{e}_{w(i)}$. This is the geometric definition of the Weyl group as the set of isometries preserving the type B hyperplane arrangement.
- $\mathfrak{S}_n^B$ acts on the set of maximal cones of Σ_{B_n} through its action on $\mathbb{R}^n$ by $w \cdot C_{w'} = C_{ww'}$.
- $\mathfrak{S}_n^B$ acts on the set of delta-matroids D through the action on the ground set $[n,\overline{n}]$.

- $\mathfrak{S}_n^B$ acts on the set of delta-matroid polytopes $P(\mathrm{D})$ through its action on the set of delta-matroids. This is *not* induced by the above $\mathfrak{S}_n^B$-action on $\mathbb{R}^n$ (which does not preserve the cube $[0,1]^n$ containing all delta-matroid polytopes), but rather the $\mathfrak{S}_n^B$-action on $\mathbb{R}^n$ conjugated by translation by $(-\frac{1}{2}, \dots, -\frac{1}{2})$. Hence $\mathfrak{S}_n$ acts in the usual way by permuting coordinates, but the ith copy of $\{\pm 1\}$ in the sign group acts by reflection in the $x_i = \frac{1}{2}$ hyperplane.

2.2. **The polytope algebra.** We collect some facts about McMullen's polytope algebra; see [EHL, Appendix A] for a survey and references. For a polyhedron $P \subseteq \mathbb{R}^n$, possibly unbounded, let $\mathbf{1}(P) : \mathbb{R}^n \to \mathbb{Z}$ be its indicator function, defined so that $\mathbf{1}(P)(x)$ equals 1 if $x \in P$ and 0 if not. Let $\mathscr{P}$ be a collection of polyhedra in $\mathbb{R}^n$.

Definition 2.3. The **indicator group** $\mathbb{I}(\mathscr{P})$ is the group of functions $\mathbb{R}^n \to \mathbb{Z}$ generated by the indicator functions $\mathbf{1}(P)$ for $P \in \mathscr{P}$. A function $f : \mathscr{P} \to G$ valued in an abelian group G is called **strongly valuative** if it factors through the map $\mathbf{1} : \mathscr{P} \to \mathbb{I}(\mathscr{P})$.

Let $\mathbb{Z}^n + \mathscr{P} = \{m + P : m \in \mathbb{Z}^n,\ P \in \mathscr{P}\}$ be the set of lattice translates of polyhedra in $\mathscr{P}$.

Definition 2.4. The **translation-invariant indicator group** $\overline{\mathbb{I}}(\mathscr{P})$ is the quotient

$$\overline{\mathbb{I}}(\mathscr{P}) = \mathbb{I}(\mathbb{Z}^n + \mathscr{P})/(\mathbf{1}(m+P) - \mathbf{1}(P) : m \in \mathbb{Z}^n, P \in \mathscr{P}).$$

We write $[f]$ for the class of a function $f \in \mathbb{I}(\mathbb{Z}^n + \mathscr{P})$ in this quotient. For a polyhedron $P \in \mathscr{P}$, we often write $[P]$ for the class $[\mathbf{1}(P)]$.

Suppose now that $\mathscr{P}$ is the set $\mathscr{P}_{\mathbb{Z},\Sigma}$ of lattice **deformations** of a smooth projective fan Σ in $\mathbb{R}^n$, that is, $\mathscr{P}_{\mathbb{Z},\Sigma} = \{P \subset \mathbb{R}^n$ a lattice polytope whose normal fan coarsens $\Sigma\}$. In this case, the group $\overline{\mathbb{I}}(\mathscr{P}_{\mathbb{Z},\Sigma})$ is isomorphic to the subalgebra of McMullen's polytope algebra spanned by polytopes in $\mathscr{P}_{\mathbb{Z},\Sigma}$ [EHL, Proposition A.6] (see also [McM09]). In particular, $\overline{\mathbb{I}}(\mathscr{P}_{\mathbb{Z},\Sigma})$ acquires the structure of a unital commutative ring [McM89, Lemma 6], with the product induced by $[P] \cdot [Q] = [P + Q]$.

The polytope algebra $\overline{\mathbb{I}}(\mathscr{P}_{\mathbb{Z},\Sigma})$ relates to the geometry of the smooth projective toric variety X_Σ of the fan Σ as follows. The standard correspondence between polyhedra and divisors on toric varieties [CLS11, §6.2] (see also [ACEP20, §2.4]) gives a bijection between polytopes $P \in \mathscr{P}_{\mathbb{Z},\Sigma}$ and base-point-free torus-invariant divisors D_P on X_Σ. Let $\mathcal{O}_{X_\Sigma}(D_P)$ denote the corresponding line bundle. We then have the following folklore isomorphism.

Theorem 2.5. [EHL, Theorem A.9] (cf. [Mor93, Theorem 8]). The assignment $[P] \mapsto [\mathcal{O}_{X_\Sigma}(D_P)]$ defines an isomorphism of rings $\overline{\mathbb{I}}(\mathscr{P}_{\mathbb{Z},\Sigma}) \xrightarrow{\sim} K(X_\Sigma)$.

We now specialize to the B_n permutohedral fan. Let

$$\mathsf{GP}_{\mathbb{Z},B_n} = \mathscr{P}_{\mathbb{Z},\Sigma_{B_n}}$$

be the set of B_n generalized permutohedra that are lattice polytopes. Then

$$\mathsf{DMat}_n = \text{the set of all delta-matroids on } [n,\overline{n}]$$

is identified with the subset of $\mathsf{GP}_{\mathbb{Z},B_n}$ consisting of polytopes with vertices in $\{0,1\}^n$.

2.3. **Schubert delta-matroids.** We now describe a special family of delta-matroids that we will use to provide bases for $\mathbb{I}(\mathsf{GP}_{\mathbb{Z},B_n})$ and $\overline{\mathbb{I}}(\mathsf{GP}_{\mathbb{Z},B_n})$. Through the identification of $w \in \mathfrak{S}_n^B/\mathfrak{S}_n$ with $w \cdot [n] \in \mathsf{AdS}_n$, the Bruhat order provides a partial order on AdS_n, namely the (hyperoctahedral) Gale order of [BGW03, §3.1.2], given as follows. Endow $[n, \overline{n}]$ with the total order

$$\overline{n} < \cdots < \overline{1} < 1 < \cdots < n. \tag{2.1}$$

Then, the Gale order on AdS_n is the corresponding **dominance order**, which is described in two equivalent ways:

- Given $S, S' \in \mathsf{AdS}_n$, we have $S \leq S'$ if and only if $|S \cap U| \leq |S' \cap U|$ for every upper segment U of the order (2.1).
- In terms of elementwise inequalities, if $S = \{i_1, \ldots, i_n\}$ and $S' = \{j_1, \ldots, j_n\}$ with $i_1 < \cdots < i_n$ and $j_1 < \cdots < j_n$, then $S \leq S'$ if and only if $i_k \leq j_k$ for all k.

Proposition-Definition 2.6. [BGW03, §6.1.1] Each lower interval $[[\overline{n}], S]$ in the Gale order is the set of feasible sets of a delta-matroid Ω_S. We call the Ω_S for $S \in \mathsf{AdS}_n$ the **standard Schubert delta-matroids**. A **Schubert delta-matroid** is a $\mathfrak{S}_n^B$-image of a standard Schubert delta-matroid.

For $S \in \mathsf{AdS}_n$, the standard Schubert delta-matroid polytope $P(\Omega_S)$ is the independence polytope of a type A Schubert matroid in the following way. The **standard Schubert matroid** Ω_T^A of a subset $T \subseteq [n]$ is the matroid on $[n]$ whose set of bases is

$$\Omega_T^A = \{B \subseteq [n] : |B| = |T| \text{ and } B \leq T \text{ in the dominance order}\}$$

where the dominance order is taken with respect to the ground set ordering $1 < \cdots < n$.

Lemma 2.7. For $S, S' \in \mathsf{AdS}_n$, then the following are equivalent.

(1) $S \leq S'$ in the Gale order;
(2) $|S \cap \{i, \ldots, n\}| \leq |S' \cap \{i, \ldots, n\}|$ for all $1 \leq i \leq n$; and
(3) There exists $B \subset [n]$ with $|B| = |S' \cap [n]|$ such that $S \cap [n] \subset B \leq S' \cap [n]$, where the inequality is taken in the dominance order.

Proof. All equivalences are easy to verify directly, so we omit the proof. □

A **Schubert matroid** is a $\mathfrak{S}_n$-image of a standard Schubert matroid. From the equivalence of the first and third parts of Lemma 2.7, we see that $P(\Omega_S) = IP(\Omega^A_{S\cap[n]})$, and so the subset

$$\mathsf{SchDMat}_n = \text{the set of all Schubert delta-matroids on } [n, \overline{n}]$$

of DMat_n is identified with the set of $\mathfrak{S}_n^B$-images of independence polytopes of Schubert matroids on $[n]$. The name "Schubert (delta-)matroid" reflects a relationship with Schubert cells explained in Example 6.3.

2.4. **Intersecting with unit cubes.** We record here some key properties concerning how lattice B_n generalized permutohedra intersect with unit cubes. We will use them to prove Theorem D and some related isomorphisms in the next subsection.

The natural level of generality of our first proposition, Proposition 2.10, is not only lattice B_n generalized permutohedra but also their unbounded analogues. A polyhedron $P \subseteq \mathbb{R}^n$ is **lattice** (over $\mathbb{Z}^n$) if the affine span $\operatorname{aff}(F)$ of any face F of P contains a coset of a subgroup of $\mathbb{Z}^n$ of rank $\dim F$. If P is bounded, i.e., P is a polytope, this is equivalent to the vertices of P being lattice points, because the differences between vertices of F generate the subgroup sought for any face F.

Lemma 2.8. Let $P \subset \mathbb{R}^n$ be a closed convex set and $u : \mathbb{R}^n \to \mathbb{R}$ a linear functional. If $P^+ = P \cap \{x \in \mathbb{R}^n : x_1 \geq 0\}$ is nonempty, then u is bounded below on P^+ if and only if there exists $r \geq 0$ such that $u - rx_1$ is bounded below on P.

Proof. If u is bounded below on P^+ and attains its minimum at any point $x \in P^+$ with $x_1 > 0$, then $r = 0$ suffices; otherwise take $r = \lim_{y \to 0^+} \frac{1}{y} \min\{u(x) : x_1 = y\}$. The converse is clear because $u \geq u - rx_1$ on P^+. ☐

Lemma 2.9. Let σ be a cone of Σ_{B_n}, and let u lie in the relative interior of σ. Then both the set of cones of Σ_{B_n} which meet $\operatorname{cone}\{u, \mathbf{e}_1\}$ and the order in which $u + \lambda \mathbf{e}_1$ meets these cones as $\lambda \geq 0$ increases are functions of σ, independent of u.

In lieu of a proof of Lemma 2.9 we describe the cones arising. This is easier in the language of total preorders. Arbitrary cones of Σ_{B_n} are in bijection with total preorders $\leq$ on $[n, \overline{n}]$ such that for $i, j \in [n, \overline{n}]$, $i \leq j$ if (and only if) $\overline{j} \leq \overline{i}$, via the map

$$\leq \mapsto C_{\leq} = \operatorname{cone}\Big\{ \sum_{j \leq i} \mathbf{e}_j : i \in [n, \overline{n}] \Big\}.$$

In Lemma 2.9, if $\sigma = C_{\leq}$, then the cones whose relative interiors meet $\operatorname{cone}\{u, \mathbf{e}_1\}$ are the $C_{\preceq}$ for all $\preceq$ such that $\leq$ and $\preceq$ have the same restriction to $[n, \overline{n}] \setminus \{1, \overline{1}\}$, and for all $i \in [n, \overline{n}]$, if $1 \leq i$ then $1 \preceq i$.

Proposition 2.10. Let P be a lattice polyhedron, possibly unbounded, whose normal fan coarsens a subfan of Σ_{B_n}. If $m \in \mathbb{Z}^n$ and $P \cap (m + [0,1]^n)$ is nonempty, then $P \cap (m + [0,1]^n) \in \mathsf{GP}_{\mathbb{Z}, B_n}$.

The counterpart for type A generalized permutohedra follows from [Sch03, (44.70)] on intersections with coordinate half-spaces, which implies that Theorem 2.15 below also holds for type A.

Proof. By translating we may assume that $m = 0$. The cube $\square = [0,1]^n$ is an intersection of coordinate half-spaces. So we reduce to considering the intersection of P with a coordinate half-space H^+, say $\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 \geq 0\}$, and showing that if $P \cap H^+$ is nonempty, then it is a lattice polyhedron and has normal fan coarsening a subfan of Σ_{B_n}. This proves the proposition together with the observation that $P \cap \square$ is bounded because $\square$ is.

First, we show that $P \cap H^+$ is lattice. Note that for any face G of $P \cap H^+$, there is a face F of P such that either

(1) $G = F \cap H^+$ and $\dim G = \dim F$, or
(2) $G = F \cap H$ and $\dim G = \dim F - 1$.

In the former case, $\operatorname{aff}(G) = \operatorname{aff}(F)$. In the latter case, fix a cone of Σ_{B_n} maximal among those normal to F. This cone has the form

$$\operatorname{cone}\{\mathbf{e}_{w(1)} + \mathbf{e}_{w(2)} + \cdots + \mathbf{e}_{w(i_k)} : k = 1, \ldots, m\}$$

for some $w \in \mathfrak{S}_n^B$ and $\{i_1, \dots, i_m\} \subseteq [n]$ by Proposition 2.2. Thus

$$\begin{aligned} \operatorname{aff}(F) &= \{x \in \mathbb{R}^n : x_{w(1)} + \cdots + x_{w(i_k)} = a_{i_k} \text{ for all } k = 1, \dots, m\}, \\ &= \{x \in \mathbb{R}^n : x_{w(i_{k-1})+1} + \cdots + x_{w(i_k)} = a_{i_k} - a_{i_{k-1}} \text{ for all } k = 1, \dots, m\} \end{aligned} \tag{2.2}$$

where the a_i are integers because P is lattice. The lattice points in $\operatorname{aff}(G) = \operatorname{aff}(F) \cap H$ are those with $x_1 = 0$, which form a coset of a subgroup of corank 1 among the lattice points in $\operatorname{aff}(F)$ because x_1 appears in at most one equation in (2.2). We have thus shown $P \cap H^+$ is lattice.

Now we prove that the normal fan of $P \cap H^+$ coarsens a subfan of Σ_{B_n}. Write $\operatorname{face}_u Q$ for the face of a polytope Q on which a linear functional $u : \mathbb{R}^n \to \mathbb{R}$ attains its minimum; set $\operatorname{face}_u Q = \varnothing$ by convention if no minimum is attained. The assumption on P is that for each cone σ of Σ_{B_n} with relative interior σ°, it holds that $\operatorname{face}_u P = \operatorname{face}_v P$ for all $u, v \in \sigma^\circ$. Our claim is that the same is true of $P \cap H^+$.

Fix a cone σ of Σ_{B_n} and $u, v \in \sigma^\circ$. By Lemma 2.8, $\operatorname{face}_u(P \cap H^+) = \varnothing$ if and only if $u - rx_1$ lies outside the normal fan of P for all $r \geq 0$, where x_1 is the first coordinate functional, and likewise for v. By Lemma 2.9, whether this happens depends only on σ, not on u or v. So it remains to handle the case $\operatorname{face}_u(P \cap H^+) \neq \varnothing$. If $\operatorname{face}_u P$ is not disjoint from H^+, we are done, since in this case

$$\operatorname{face}_u(P \cap H^+) = (\operatorname{face}_u P) \cap H^+ = (\operatorname{face}_v P) \cap H^+ = \operatorname{face}_v(P \cap H^+).$$

If they are disjoint, let $r \in \mathbb{R}$ be minimal such that $F := \operatorname{face}_{u - rx_1} P$ intersects H^+, where x_1 is the first coordinate functional; some such r exists by our earlier invocation of Lemma 2.8. Note that $r > 0$, so u is a positive combination of x_1 and $u - rx_1$. Since $\operatorname{face}_{x_1}(P \cap H^+) = P \cap H$ and $\operatorname{face}_{u - rx_1}(P \cap H^+) = F \cap H^+$ intersect in their common face $F \cap H$, this implies $\operatorname{face}_u(P \cap H^+) = F \cap H$. Again by Lemma 2.9, the faces of the form $\operatorname{face}_{u - rx_1} P$, and their order they appear in as r varies, depend only on σ, so we have $\operatorname{face}_v(P \cap H^+) = F \cap H$ also. □

Let

$$C = -C_{\mathrm{id}}^\vee = \operatorname{cone}\{-\mathbf{e}_1, \mathbf{e}_1 - \mathbf{e}_2, \dots, \mathbf{e}_{n-1} - \mathbf{e}_n\} = \{x \in \mathbb{R}^n : \sum_{i=k}^n x_i \leq 0 \text{ for } k \in [n]\}.$$

This is the type B_n negative root cone for the choice of positive roots corresponding to our Gale order [BGW03, §3.2.2].

Lemma 2.11. Let $m \in \{0,1\}^n$ and let $S \in \mathsf{AdS}_n$ be the size n admissible set such that m is the indicator vector of $S \cap [n]$. Then $P(\Omega_S) = (m + C) \cap [0,1]^n$.

Proof. The half-space description of $m + C$ is

$$m + C = \{x \in \mathbb{R}^n : \sum_{i=k}^n x_i \leq \sum_{i=k}^n m_i \text{ for } k \in [n]\}. \tag{2.3}$$

By the equivalence of the first and second parts of Lemma 2.7, we see that $x \in (m + C) \cap \{0,1\}^n$ if and only if for the admissible set $S' \in \mathsf{AdS}_n$ such that x is the indicator vector of $S' \cap [n]$, we have $S' \leq S$ in the Gale order. Therefore $(m + C) \cap [0,1]^n$ and Ω_S contain the same set of lattice points. Since C is the dual of a cone of Σ_{B_n}, Proposition 2.10 applies and shows that $(m + C) \cap [0,1]^n$ is a lattice polytope. But Ω_S is also a lattice polytope, so they are equal. □

Proposition 2.12. Let $m \in \mathbb{Z}^n$. If the intersection $(m+C) \cap [0,1]^n$ is nonempty, then it is a standard Schubert delta-matroid polytope.

Proof. Assume that $(m+C) \cap [0,1]^n$ is nonempty. We construct a sequence $m^0 = m, m^1, \dots$ of integer vectors so that

$$(m^j + C) \cap [0,1]^n = (m+C) \cap [0,1]^n. \tag{2.4}$$

One of the m^j will lie in $\{0,1\}^n$, whereupon the proposition follows from Lemma 2.11.

Denote the generators of C, the negative simple roots, by $\alpha_1 = -\mathbf{e}_1$ and $\alpha_i = \mathbf{e}_{i-1} - \mathbf{e}_i$ for $i = 2, \dots, n$. An arbitrary lattice point of $m^j + C$ has the form $x = m^j + \sum_{i=1}^n a_i \alpha_i$ for nonnegative integers a_i. If $m_i^j > 1$ then we let $m^{j+1} = m^j + (m_i^j - 1)\alpha_i$. In this case $x_i \leq 1$ only if $a_i > m_i^j - 1$, so $m^j + C$ and $m^j + (m_i^j - 1)\alpha_i + C$ have the same intersection with $[0,1]^n$ and (2.4) holds. Similarly, if $m_i^j < 0$, then we let $m^{j+1} = m^j + (-m_i^j)\alpha_{i+1}$, and (2.4) holds because $x_i \geq 0$ only if $a_{i+1} > -m_i$ (note that $i < n$ in this case, which follows from $(m+C) \cap [0,1]^n$ being nonempty).

The sequence $(\sum_{i=1}^n i m_i^j)_{j \geq 0}$ is decreasing by construction, and bounded below by 0, because if $\sum_{i=1}^n i m_i^j < 0$ the functional $\sum_{i=1}^n i x_i$ takes negative values on $m^j + C$ and nonnegative values on $[0,1]^n$, implying $(m^j + C) \cap [0,1]^n = \varnothing$. So it is finite, i.e., the case $m^j \in \{0,1\}$ happens after finitely many steps. □

Corollary 2.13. The set $\mathsf{SchDMat}_n$ is closed under nonempty intersections with faces of $[0,1]^n$.

Proof. By the $\mathfrak{S}_n^B$ symmetry and iteration, it's enough to prove that if $P = P(\mathrm{D})$ for D a standard Schubert delta-matroid and F is a facet of $[0,1]^n$, then $P \cap F \in \mathsf{SchDMat}_n$. Write $P = (m+C) \cap [0,1]^n$ as in Proposition 2.12, and $F = H \cap [0,1]^n$ for a hyperplane $H = \{x \in \mathbb{R}^n : x_i = s\}$ where $i \in [n]$ and $s \in \{0,1\}$. Then $P \cap F = (m+C) \cap H \cap [0,1]^n$. Let $\pi : H \to \mathbb{R}^{n-1}$ be the map omitting the ith coordinate. Using (2.3) and its counterpart for B_{n-1}, one can check that $(m+C) \cap H$ is identified by π with a translate of the cone $-C_{\mathrm{id}}^\vee$ which is dual to a cone in $\Sigma_{B_{n-1}}$. Therefore π takes $P \cap F$ to a type B_{n-1} standard Schubert delta-matroid polytope. This implies that $P \cap F$ is a Schubert delta-matroid polytope, as follows. In the case $H = \{x \in \mathbb{R}^n : x_n = 0\}$, if $\pi(P \cap F) = P(\Omega_S)$ for S a maximal admissible subset of $[n-1]$, then $P \cap F = P(\Omega_{S \cup \{\overline{n}\}})$ by Lemma 2.11. The other possible choices of H are $\mathfrak{S}_n^B$ images of this one, so in general $P \cap F$ is a $\mathfrak{S}_n^B$ image of $P(\Omega_{S \cup \{\overline{n}\}})$. □

Corollary 2.14. Let $\square'$ be a face of $[0,1]^n$, and σ be a cone of Σ_{B_n}. For $m \in \mathbb{Z}^n$, if the intersection $(m + \sigma^\vee) \cap \square'$ is nonempty, then it is in $\mathsf{SchDMat}_n$.

Proof. If σ is a maximal cone of Σ_{B_n}, then $\sigma^\vee$ is a Weyl image of the cone $C = -C_{\mathrm{id}}^\vee$ above, and the result follows from Proposition 2.12 and Corollary 2.13.

For an arbitrary cone σ, we reduce to the preceding case. The cone σ is a face of a maximal cone τ of Σ_{B_n}, so $\sigma^\vee$ is a tangent cone of $\tau^\vee$, that is, $\sigma^\vee = -F + \tau^\vee$ for a face $F \subset \tau^\vee$. Now for $m' \in -F \cap \mathbb{Z}^n$, we have

$$\sigma^\vee \supseteq (-F \cap (m' + F)) + \tau^\vee = m' + \tau^\vee.$$

If m' is chosen deep enough in the interior of $-F$, the defining halfspaces of $m + m' + \tau^\vee$ will all contain $\square'$, so $m + \sigma^\vee$ and $m + m' + \tau^\vee$ will have the same intersection with $\square'$. □

2.5. **Bases from Schubert delta-matroids.** We are now ready to prove the following intermediate step for the proof of Theorem D.

Theorem 2.15. One has

$$\mathbb{I}(\mathbb{Z}^n + \mathsf{SchDMat}_n) = \mathbb{I}(\mathbb{Z}^n + \mathsf{DMat}_n) = \mathbb{I}(\mathsf{GP}_{\mathbb{Z},B_n}).$$

Proof. Let $P \subset \mathbb{R}^n$ be a lattice B_n generalized permutohedron. We will write $\mathbf{1}(P)$ as a sum of indicator functions of lattice translates of Schubert delta-matroid polytopes. This will prove that $\mathbb{I}(\mathsf{GP}_{\mathbb{Z},B_n}) \subset \mathbb{I}(\mathbb{Z}^n + \mathsf{SchDMat}_n)$, and the left-to-right inclusions in the theorem are clear.

Recall the signed permutohedron Π_{B_n}. By the Brianchon–Gram theorem applied to $P + \varepsilon\Pi_{B_n}$ in the pointwise limit $\varepsilon \to 0^+$, we have

$$\mathbf{1}(P) = \sum_{\sigma \in \Sigma_{B_n}} (-1)^{\operatorname{codim} \sigma}\, \mathbf{1}(P + \sigma^\vee).$$

Note that $P + \sigma^\vee$ is a lattice translate of $\sigma^\vee$.

Tile $\mathbb{R}^n$ by lattice translates of Boolean cubes $[0,1]^n$. Let $\mathcal{C}$ be the set of all such cubes that meet P, together with their common internal faces, so that we have an inclusion-exclusion relation

$$\mathbf{1}\Big(\bigcup_{F \in \mathcal{C}} F\Big) = \sum_{F \in \mathcal{C}} (-1)^{\operatorname{codim}(F)}\, \mathbf{1}(F).$$

Then

$$\mathbf{1}(P) = \sum_{F \in \mathcal{C}} (-1)^{\operatorname{codim}(F)}\, \mathbf{1}(P \cap F) = \sum_{F \in \mathcal{C}} \sum_{\sigma \in \Sigma_{B_n}} (-1)^{\operatorname{codim}(F) + \operatorname{codim}(\sigma)}\, \mathbf{1}((P + \sigma^\vee) \cap F).$$

By Corollary 2.14, the right hand side is in $\mathbb{I}(\mathbb{Z}^n + \mathsf{SchDMat}_n)$. □

We remark that the second equality of the theorem could have been proved using the tiling by Boolean cubes and Proposition 2.10 without invoking the Brianchon–Gram theorem.

Corollary 2.16. One has

$$\overline{\mathbb{I}}(\mathsf{SchDMat}_n) = \overline{\mathbb{I}}(\mathsf{DMat}_n) = \overline{\mathbb{I}}(\mathsf{GP}_{\mathbb{Z},B_n}).$$

Proof. What is left to prove after Theorem 2.15 is that the three groups of relations are equal. These are generated by $\mathbf{1}(m + P) - \mathbf{1}(P)$ where $m \in \mathbb{Z}^n$ and $P \in \mathsf{SchDMat}_n$, DMat_n, and $\mathsf{GP}_{\mathbb{Z},B_n}$ respectively. If $P \in \mathsf{GP}_{\mathbb{Z},B_n}$, then another use of Theorem 2.15 gives us a finite expression

$$\begin{aligned}\mathbf{1}(m + P) - \mathbf{1}(P) &= \sum_{Q \in \mathsf{SchDMat}_n, v \in \mathbb{Z}^n} a_{Q,v}\, (\mathbf{1}(m + v + Q) - \mathbf{1}(v + Q)) \\ &= \sum_{Q \in \mathsf{SchDMat}_n, v \in \mathbb{Z}^n} a_{Q,v}\, \big((\mathbf{1}(m + v + Q) - \mathbf{1}(Q)) - (\mathbf{1}(v + Q) - \mathbf{1}(Q))\big).\end{aligned}$$

So the relations for $\overline{\mathbb{I}}(\mathsf{GP}_{\mathbb{Z},B_n})$ are also relations for $\overline{\mathbb{I}}(\mathsf{SchDMat}_n)$, and the other containments are obvious. □

We prepare for the proof of Theorem D by proving the analogous fact for $\mathbb{I}(\mathsf{DMat}_n)$.

Proposition 2.17. The set $\{\mathbf{1}(P) : P \in \mathsf{SchDMat}_n\}$ is a basis for $\mathbb{I}(\mathsf{DMat}_n)$.

Proof. The first equality in Theorem 2.15 implies that every $\mathbf{1}(P)$ for P a delta-matroid polytope can be expressed as a linear combination of indicator functions of Schubert delta-matroid polytopes. Here we note that a lattice translate of a Schubert delta-matroid polytope $P(\mathrm{D})$, provided it is contained in the unit cube, is again a Schubert delta-matroid polytope because it is a $\mathfrak{S}_n^B$-image of $P(\mathrm{D})$.

For linear independence, suppose we have a nontrivial relation

$$\sum_{i=1}^{k} a_i \mathbf{1}(P_i) = 0 \qquad \text{with } k \geq 1 \text{ and } a_1, \dots, a_k \neq 0$$

where $P_1, \dots, P_k$ are Schubert delta-matroids. By Proposition 2.12, there exists $w \in \mathfrak{S}_n^B$ and $m \in \mathbb{Z}^n$ such that $P_1 = [0,1]^n \cap (m + w \cdot C)$. Without loss of generality, we may assume that P_1 does not contain P_i for all $i > 1$. In particular, no P_i for $i > 1$ is contained in $m + w \cdot C$. Now, [ESS21, Theorem 2.3] implies that the assignment

$$P \mapsto \begin{cases} 1 & \text{if } P \subset m + w \cdot C \text{ and } P \cap m \neq \varnothing \\ 0 & \text{otherwise} \end{cases}$$

defines a strongly valuative function on $\mathsf{GP}_{\mathbb{Z}^n, B_n}$. Applying this function to both sides of the relation $\sum_{i=1}^{k} a_i \mathbf{1}(P_i) = 0$ then yields $a_1 = 0$, a contradiction. □

We are ready to prove Theorem D. Converted to a statement about polyhedra by using Theorem 2.5, the theorem asserts that a basis of $\overline{\mathbb{I}}(\mathsf{GP}_{\mathbb{Z}, B_n})$ is

$$\mathsf{SchDMat}_n^{\mathsf{clf}} := \{\mathrm{D} \in \mathsf{SchDMat}_n : \mathrm{D} \text{ has no coloops}\}.$$

The superscript $^{\mathsf{clf}}$ stands for "coloop-free." We verify that, among the polytopes of the delta-matroids in $\mathsf{SchDMat}_n^{\mathsf{clf}}$, there is exactly one translate of any Schubert delta-matroid polytope. For any $\mathrm{D} \in \mathsf{SchDMat}$, changing any coloops D may have to loops gives a translate in $\mathsf{SchDMat}_n^{\mathsf{clf}}$. If for two delta-matroids D and D′ we have $P(\mathrm{D}') = m + P(\mathrm{D})$ for some $m \in \mathbb{Z}^n$, then $m \in \{-1, 0, 1\}^n$; if for some i we have $m_i = 1$, then $P(\mathrm{D}') \subseteq \{x \in \mathbb{R}^n : x_i = 1\}$ and $P(\mathrm{D}) \subseteq \{x \in \mathbb{R}^n : x_i = 0\}$, and if $m_1 = -1$ then these containments hold vice versa, so not both D and D′ are coloop-free.

Our method for proving Theorem D can also be used to deduce the counterpart of the theorem in type A, i.e., that coloop-free Schubert matroids are a basis for the translation-invariant polytope algebra of lattice type A generalized permutohedra. Another proof of the type A theorem can be assembled from [BEST, Theorem D] and the analogous theorem for the cohomology ring in type A appearing in [Ham17].

Proof of Theorem D. Theorem 2.15 shows that $\{[P] : P \in \mathsf{SchDMat}_n^{\mathsf{clf}}\}$ generates $\overline{\mathbb{I}}(\mathsf{SchDMat}_n)$. So we must prove linear independence.

We first show translates of coloop-free Schubert delta-matroids are linearly independent in $\mathbb{I}(\mathbb{Z}^n + \mathsf{SchDMat}_n)$. Suppose we are given a finite relation

$$\sum_{P \in \mathsf{SchDMat}_n^{\mathsf{clf}}, m \in \mathbb{Z}^n} a_{P,m} \, \mathbf{1}(m + P) = 0.$$

Let $V \subseteq \mathbb{Z}^n$ be the set of vectors v such that, for some (P, m) with $a_{P,m} \neq 0$, $m + P$ intersects the translate $[0,1)^n + v$ of the half-open cube. Our objective is to prove V empty. Suppose otherwise, and let $v \in V$ be lexicographically minimum. Restricting our relation to the closed cube $v + [0,1]^n$ gives

$$\sum_{P \in \mathsf{SchDMat}_n^{\mathsf{clf}}, m \in \mathbb{Z}^n} a_{P,m} \, \mathbf{1}((m+P) \cap (v + [0,1]^n)) = 0.$$

If $(m + P) \cap (v + [0,1]^n)$ is nonempty, then it has the form $v + Q$ for some $Q \in \mathsf{SchDMat}_n$ by Corollary 2.13. Letting

$$J(Q) = \{(P, m) : (m + P) \cap (v + [0,1]^n) = v + Q\},$$

we collect identical translates:

$$\sum_{Q \in \mathsf{SchDMat}_n} \left(\sum_{(P,m) \in J(Q)} a_{P,m} \right) \mathbf{1}(v + Q) = 0.$$

By Proposition 2.17, every inner sum is zero. For any $Q \in \mathsf{SchDMat}_n^{\mathsf{clf}}$, minimality of v implies that the only possibly nonzero summand in this inner sum is the one indexed by $(P, m) = (Q, v)$, so $a_{Q,v} = 0$. But this contradicts $v \in V$.

Now, a linear dependence in $\bar{\mathbb{I}}(\mathsf{GP}_{\mathbb{Z},B_n})$,

$$\sum_{P \in \mathsf{SchDMat}_n^{\mathsf{clf}}} a_P \, [\mathbf{1}(P)] = 0,$$

lifts to $\mathbb{I}(\mathsf{GP}_{\mathbb{Z},B_n})$ as a relation

$$\sum_{P \in \mathsf{SchDMat}_n^{\mathsf{clf}}} a_P \, \mathbf{1}(P) + \sum_{Q, m \in \mathbb{Z}^n \setminus \{0\}} b_{Q,m} \, (\mathbf{1}(m + Q) - \mathbf{1}(Q)) = 0$$

over some family of lattice B_n generalized permutohedra Q, where finitely many $b_{Q,m}$ are nonzero. Applying Theorem 2.15 to these Q, this can be rewritten

$$\sum_{P \in \mathsf{SchDMat}_n^{\mathsf{clf}}} a_P \, \mathbf{1}(P) + \sum_{P \in \mathsf{SchDMat}_n, m \neq 0} c_{P,m} \, (\mathbf{1}(m + P) - \mathbf{1}(P)) = 0.$$

Every $P \in \mathsf{SchDMat}_n$ has a lattice translate $P' \in \mathsf{SchDMat}_n^{\mathsf{clf}}$, and we can use the relation $\mathbf{1}(m + Q) - \mathbf{1}(Q) = (\mathbf{1}(m + Q) - \mathbf{1}(Q')) - (\mathbf{1}(Q) - \mathbf{1}(Q'))$ for any polytopes Q, Q' to rewrite the second sum:

$$\sum_{P \in \mathsf{SchDMat}_n^{\mathsf{clf}}} a_P \, \mathbf{1}(P) + \sum_{P' \in \mathsf{SchDMat}_n^{\mathsf{clf}}, m \neq 0} d_{P',m} \, (\mathbf{1}(m + P') - \mathbf{1}(P')) = 0.$$

The earlier lifted linear independence statement implies that each polytope in the above sum has a zero coefficient, i.e., $d_{P,m} = 0$ for all $m \neq 0$ and $a_P - \sum_{m \neq 0} d_{P,m} = 0$. Therefore $a_P = 0$ for all $P \in \mathsf{SchDMat}_n^{\mathsf{clf}}$. $\square$

3. THE EXCEPTIONAL HIRZEBRUCH–RIEMANN–ROCH-TYPE THEOREM

We prove Theorem C, relating the Grothendieck ring of vector bundles $K(X_{B_n})$ to the Chow cohomology $A^\bullet(X_{B_n})$, in two parts. In §3.2, we establish the isomorphism $\phi^B : K(X_{B_n}) \to A^\bullet(X_{B_n})$ via localization methods in torus-equivariant geometry. Then, in §3.3, we establish the formula involving the sheaf Euler characteristic by relating the isomorphism ϕ^B to a similar isomorphism for stellahedral varieties established in [EHL].

3.1. **K-rings and Chow rings of X_{B_n}.** Let $K_T(X_{B_n})$ be the T-equivariant K-ring of X_Σ, which is the Grothendieck ring of T-equivariant vector bundles on X_{B_n}, and let $A^\bullet_T(X_{B_n})$ be the T-equivariant Chow ring in the sense of [EG98]. We describe the equivariant and non-equivariant K and Chow rings of X_{B_n}. We will make use of a description of $K_T(X_{B_n})$ and $A^\bullet_T(X_{B_n})$ coming from equivariant localization. See [EHL, Section 2] for a review of equivariant localization.

We first set up some notation. To describe the adjacent maximal cones in Σ_{B_n}, we use the following special involutions in $\mathfrak{S}_n^B$:

- $\tau_{i,i+1} = (i, i+1)(\bar{i}, \overline{i+1})$ for $1 \le i \le n-1$, and
- $\tau_n = (n, \overline{n})$.

Then C_w is adjacent to $C_{w'}$ exactly if $w = w'\tau_{i,i+1}$ for some i, in which case the common facet normal $\hat{n}(ww')$ is $\pm(\mathbf{e}_{w(i)} - \mathbf{e}_{w(i+1)})$, or $w = w'\tau_n$, in which case $\hat{n}(ww') = \pm\mathbf{e}_{w(n)}$. Recall that $K_T(\mathrm{pt}) = \mathbb{Z}[T_1^{\pm1}, \dots, T_n^{\pm1}]$ and $A^\bullet_T(\mathrm{pt}) = \mathbb{Z}[t_1, \dots, t_n]$. Let $T_{\bar{i}} = T_i^{-1}$ and $t_{\bar{i}} = -t_i$ for $i \in [n]$.

Theorem 3.1. [VV03, Pay06] The following hold.

(1) The injective localization map $K_T(X_{B_n}) \to K_T(X_{B_n}^T) = \bigoplus_{w\in\mathfrak{S}_n^B} K_T(\mathrm{pt})$ identifies $K_T(X_{B_n})$ with the set of collections of elements $(f_w)_{w\in\mathfrak{S}_n^B} \in \bigoplus_{w\in\mathfrak{S}_n^B} \mathbb{Z}[T_1^{\pm1}, \dots, T_n^{\pm1}]$ such that
   - if $w\tau_{i,i+1} = w'$ for $1 \le i \le n-1$, then $f_w \equiv f_{w'} \bmod 1 - T_{w(i)}T_{w(i+1)}^{-1}$, and
   - if $w\tau_n = w'$ then $f_w \equiv f_{w'} \bmod 1 - T_{w(n)}$.

   The diagonal embedding of $\mathbb{Z}[T_1^{\pm1}, \dots, T_n^{\pm1}]$ into $\bigoplus_{w\in\mathfrak{S}_n^B} K_T(\mathrm{pt})$ identifies $\mathbb{Z}[T_1^{\pm1}, \dots, T_n^{\pm1}]$ with a subring of $K_T(X_{B_n})$, and the K-ring $K(X_{B_n})$ is given by
$$K(X_{B_n}) = K_T(X_{B_n})/(T_1 - 1, \dots, T_n - 1).$$

(2) The injective localization map $A^\bullet_T(X_{B_n}) \to A^\bullet_T(X_{B_n}^T) = \bigoplus_{w\in\mathfrak{S}_n^B} A^\bullet_T(\mathrm{pt})$ identifies $A^\bullet_T(X_{B_n})$ with the set of collections of elements $(f_w)_{w\in\mathfrak{S}_n^B} \in \bigoplus_{w\in\mathfrak{S}_n^B} \mathbb{Z}[t_1, \dots, t_n]$ such that
   - if $w\tau_{i,i+1} = w'$ for $1 \le i \le n-1$, then $f_w \equiv f_{w'} \bmod t_{w(i)} - t_{w(i+1)}$, and
   - if $w\tau_n = w'$ then $f_w \equiv f_{w'} \bmod t_{w(n)}$.

   The diagonal embedding of $\mathbb{Z}[t_1, \dots, t_n]$ into $\bigoplus_{w\in\mathfrak{S}_n^B} A^\bullet_T(\mathrm{pt})$ identifies $\mathbb{Z}[t_1, \dots, t_n]$ with a subring of $A^\bullet_T(X_{B_n})$, and the Chow ring $A^\bullet(X_{B_n})$ is given by
$$A^\bullet(X_{B_n}) = A^\bullet_T(X_{B_n})/(t_1, \dots, t_n).$$

Let us now describe $\mathfrak{S}_n^B$-actions on $K_T(X_{B_n})$ and $A^\bullet_T(X_{B_n})$. To do so, we prepare with some generalities on maps between torus-equivariant K-rings for actions of potentially different tori. For $i = 1, 2$, let T_i be a torus and X_i a smooth projective T_i-variety. Suppose we have a map of tori

$\varphi : T_1 \to T_2$ and a map $\overline{\varphi} : X_1 \to X_2$ with the commuting diagram

$$\begin{array}{ccc} T_1 \times X_1 & \xrightarrow{\varphi \times \overline{\varphi}} & T_2 \times X_2 \\ \downarrow & & \downarrow \\ X_1 & \xrightarrow{\quad \varphi \quad} & X_2, \end{array}$$

where the two vertical maps are the torus actions. Then, by treating X_2 as a T_1-variety via φ, we have the induced maps

$$K_{T_2}(X_2) \to K_{T_1}(X_2) \overset{\overline{\varphi}^*}{\to} K_{T_1}(X_1) \tag{3.1}$$

where the first map is the "forgetful map" and the second map is the pullback map of maps between T_1-varieties. We similarly have induced maps of equivariant Chow rings.

In our situation, we will have $T_1 = T_2 = T$ and $X_1 = X_2 = X_{B_n}$ in the following way. An element $w \in \mathfrak{S}_n^B$ acts on $\mathbb{R}^n$ by $\mathbf{e}_i \mapsto \mathbf{e}_{w(i)}$. We consider $\mathbb{R}^n$ as the real vector space $\operatorname{Cochar}(T) \otimes \mathbb{R}$ that contains the fan Σ_{B_n}. This $\mathfrak{S}_n^B$-action defines an automorphism $\varphi_w \colon T \to T$ given by $T_i \mapsto T_{w(i)}$. Since the $\mathfrak{S}_n^B$-action maps Σ_{B_n} isomorphically onto itself, the map φ_w extends to an automorphism $\overline{\varphi}_w \colon X_{B_n} \to X_{B_n}$. The map $\overline{\varphi}_w$ is not a T-equivariant map, but it fits into the commuting diagram

$$\begin{array}{ccc} T \times X_{B_n} & \xrightarrow{\varphi_w \times \overline{\varphi}_w} & T \times X_{B_n} \\ \downarrow & & \downarrow \\ X_{B_n} & \xrightarrow{\quad \overline{\varphi}_w \quad} & X_{B_n}. \end{array}$$

Hence, we have the maps

$$\psi_w : K_T(X_{B_n}) \to K_T(X_{B_n}) \overset{\overline{\varphi}_w^*}{\to} K_T(X_{B_n})$$

as in (3.1), and similarly for $A_T^\bullet(X_{B_n})$. The assignments $w \mapsto \psi_{w^{-1}}$ give a $\mathfrak{S}_n^B$-action descending to the usual $\mathfrak{S}_n^B$-action on $K(X_{B_n})$ and $A^\bullet(X_{B_n})$. In terms of the localization description of $K_T(X_{B_n})$ and $A_T^\bullet(X_{B_n})$ in Theorem 3.1, the action has the following explicit description:

(1) An element $w \in \mathfrak{S}_n^B$ acts on $f \in K_T(X_{B_n})$ by $(w \cdot f)_{w'} = f_{w^{-1}w'}(T_{w(1)}, \dots, T_{w(n)})$.
(2) An element $w \in \mathfrak{S}_n^B$ acts on $f \in A_T^\bullet(X_{B_n})$ by $(w \cdot f)_{w'} = f_{w^{-1}w'}(t_{w(1)}, \dots, t_{w(n)})$.

3.2. **The exceptional isomorphism.**

Theorem 3.2. There is an injective ring map

$$\phi_T^B \colon K_T(X_{B_n}) \to A_T^\bullet(X_{B_n})[1/(1 \pm t_i)] := A_T^\bullet(X_{B_n})[\{\tfrac{1}{1-t_i}, \tfrac{1}{1+t_i}\}_{1 \le i \le n}]$$

obtained by

$$(\phi_T^B(f))_w(t_1, \dots, t_n) = f_w(h_{\epsilon_1(w)}(t_1), \dots, h_{\epsilon_n(w)}(t_n))$$

where

$$h_\epsilon(t) = (1 + \epsilon t)^\epsilon := \begin{cases} 1 + t & \epsilon = +1 \\ \frac{1}{1-t} & \epsilon = -1 \end{cases}.$$

This equivariant map ϕ_T^B descends to a non-equivariant isomorphism $\phi^B\colon K(X_{B_n}) \xrightarrow{\sim} A^\bullet(X_{B_n})$. Finally, ϕ^B and ϕ_T^B are $\mathfrak{S}_n^B$-equivariant in the sense that they intertwine the above $\mathfrak{S}_n^B$-actions:

$$\phi_T^B(w \cdot f) = w \cdot \phi_T^B(f), \text{ and } \phi^B(w \cdot f) = w \cdot \phi^B(f).$$

Proof. We first check that ϕ_T^B is $\mathfrak{S}_n^B$-equivariant. For $f \in K_T(X_{B_n})$, we have that

$$(\phi_T^B(w \cdot f))_{w'} = f_{w^{-1}w'}(h_{\epsilon_1(w')}(T_{w(1)}), \ldots, h_{\epsilon_n(w')}(T_{w(n)})), \text{ and}$$
$$(w \cdot \phi_T^B(f))_{w'} = f_{w^{-1}w'}((1+\epsilon_1(w')t_{w(1)})^{\epsilon_1(w')}, \ldots, (1+\epsilon_n(w')t_{w(n)})^{\epsilon_n(w')}),$$

which are equal. We now check the congruence conditions. First, we check for $w' = w\tau_{i,i+1}$ that

$$(\phi_T^B(f))_w \equiv (\phi_T^B(f))_{w'} \mod t_{w(i)} - t_{w(i+1)}.$$

By $\mathfrak{S}_n^B$-equivariance, this is equivalent to

$$(\phi_T^B(w^{-1} \cdot f))_{\mathrm{id}} \equiv (\phi_T^B(w^{-1} \cdot f))_{\tau_{i,i+1}} \mod t_i - t_{i+1},$$

which by definition of ϕ_T^B, and the fact that $\epsilon_j(\mathrm{id}) = \epsilon_j(\tau_{i,i+1}) = 1$ for all j, is equivalent to

$$(w^{-1} \cdot f)_{\mathrm{id}}(t_1+1, \ldots, t_n+1) \equiv (w^{-1} \cdot f)_{\tau_{i,i+1}}(t_1+1, \ldots, t_n+1) \mod t_i - t_{i+1}.$$

Since $w^{-1} \cdot f \in K_T(X_{B_n})$, we have $((w^{-1} \cdot f)_{\mathrm{id}}(T_1, \ldots, T_n) \equiv (w^{-1} \cdot f)_{\tau_{i,i+1}}(T_1, \ldots, T_n)) \mod T_i = T_{i+1}$, and the result follows from replacing T_j with $t_j + 1$ for all j. Now, we check for $w' = w\tau_n$ that

$$(\phi_T^B(f))_w \equiv (\phi_T^B(f))_{w'} \mod t_{w(n)}.$$

Indeed, this similarly follows from the fact that $w \cdot f \in K_T(X_{B_n})$ and the compatibility

$$(w^{-1} \cdot f)_{\mathrm{id}}(T_1, \ldots, T_n) \equiv (w^{-1} \cdot f)_{\tau_n}(T_1, \ldots, T_n) \mod T_n - 1.$$

As we now know that ϕ_T^B is well-defined, from the defining formula it is trivial to check that it is indeed a ring map, and is injective as well.

We now check that the map ϕ_T^B descends non-equivariantly to a map $\phi^B\colon K(X_{B_n}) \to A^\bullet(X_{B_n})$. Note that under the map $A_T^\bullet(X_{B_n}) \to A^\bullet(X_{B_n})$ we have $1 \pm t_i \mapsto 1$, so there is an induced map $A_T^\bullet(X_{B_n})[\frac{1}{1\pm t_i}] \to A^\bullet(X_{B_n})$. To obtain the map ϕ^B, we have to show that under the composite $K_T(X_{B_n}) \to A_T^\bullet(X_{B_n})[\frac{1}{1\pm t_i}] \to A^\bullet(X_{B_n})$, the ideal $(T_1 - 1, \ldots, T_n - 1)$ gets mapped to 0. Indeed, $\phi_T^B(T_i - 1) = t_i \cdot r_i$ where $(r_i)_w$ is 1 if $\epsilon_i(w) = 1$ and $\frac{1}{1-t_i}$ if $\epsilon_i(w) = -1$. Therefore $\phi_T^B(T_i - 1)$ is zero under the map $A_T^\bullet(X_{B_n})[\frac{1}{1\pm t_i}] \to A^\bullet(X_{B_n})$ because t_i maps to 0.

The $\mathfrak{S}_n^B$-equivariance of ϕ^B follows immediately from the $\mathfrak{S}_n^B$-equivariance of ϕ_T^B, so it remains to check that ϕ^B is an isomorphism. For this, we identify the image of ϕ_T^B. Note that $\phi^B(K_T(X_{B_n}))$ lies in the subring $R \subset A_T^\bullet(X_{B_n})[\frac{1}{1\pm t_i}]$ consisting of those g where g_w lies in the ring $K_T(\mathrm{pt})[\frac{1}{1+\epsilon_1(w)t_1}, \ldots, \frac{1}{1+\epsilon_n(w)t_n}]$ for all w. Define

$$h_\epsilon^{-1}(T) = \epsilon(T^\epsilon - 1) := \begin{cases} T - 1 & \epsilon = +1 \\ 1 - T^{-1} & \epsilon = -1. \end{cases}$$

It is easy to see that for $g \in R$ we have $g_w(h_{\epsilon_1(w)}^{-1}(t_1), \ldots, h_{\epsilon_n(w)}^{-1}(t_n)) \in K_T(\mathrm{pt})$ for all w, and, arguing as before, we see that

$$w \mapsto g_w(h_{\epsilon_1(w)}^{-1}(t_1), \ldots, h_{\epsilon_n(w)}^{-1}(t_n))$$

gives a preimage of g under ϕ_T^B. Hence $\phi_T^B\colon K_T(X_{B_n}) \to R$ is an isomorphism. Now, note that the r_i constructed above has the property that $r_i \in R^\times$, so the ideal $(T_1 - 1, \ldots, T_n - 1) \subset K_T(X_{B_n})$ maps under ϕ_T^B to the ideal $(t_1, \ldots, t_n) \subset R$. Hence because

$$A_T(X_{B_n}) \subset R \subset A_T(X_{B_n})\left[\frac{1}{1 \pm t_i}\right]$$

and $\frac{1}{1\pm t_i}$ gets sent to 1 after quotienting by $(t_1, \ldots, t_n)$, we conclude that ϕ^B induces an isomorphism

$$K(X_{B_n}) \cong R/(t_1, \ldots, t_n) = A_T(X_{B_n})\left[\frac{1}{1 \pm t_i}\right]/(t_1, \ldots, t_n) = A^\bullet(X_{B_n}). \qquad \square$$

3.3. **Stellahedral geometry.** We show that the isomorphism ϕ^B of Theorem 3.2 satisfies

$$\chi([\mathcal{E}]) = \int_{X_{B_n}} \phi^B([\mathcal{E}]) \cdot c(\boxplus \mathcal{O}(1))$$

for any $[\mathcal{E}] \in K(X_{B_n})$, thereby completing the proof of Theorem C. While one can prove this via the Atiyah-Bott localization formula, as in [BEST], we present a more geometric proof that explains how our result relates to a previous exceptional Hirzebruch–Riemann–Roch-type theorem given in [EHL] for stellahedral varieties. Note that $(\Sigma_{B_1})^n$ is a fan in $\mathbb{R}^n$ whose cones are

$$\mathrm{Cone}(\mathbf{e}_i : i \in S) \quad \text{for } S \text{ an admissible subset of } [n, \overline{n}].$$

Definition 3.3. The **stellahedral fan** Σ_{St_n} is a fan in $\mathbb{R}^n$ obtained from $(\Sigma_{B_1})^n$ by iteratively performing stellar subdivisions on all faces of the nonpositive orthant $\mathrm{Cone}(\mathbf{e}_i : i \in [\bar{n}])$ starting with the maximal face.

Note that the B_n permutohedral fan Σ_{B_n} is obtained by performing such iterated stellar subdivisions on all the orthants. In other words, the fan Σ_{B_n} is the common refinement of the 2^n different "copies" of the stellahedral fan: For each admissible subset $\tau \in \mathsf{AdS}_n$, we have the "copy" of the stellahedral fan obtained from $(\Sigma_{B_1})^n$ by performing the iterated stellar subdivision on the orthant $\mathrm{Cone}(\mathbf{e}_i : i \in \tau)$. See Figure 1 for an illustration when $n = 2$.

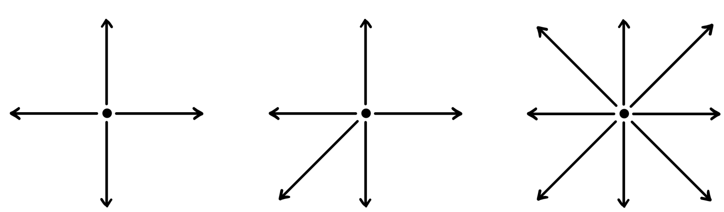

FIGURE 1. The fans $(\Sigma_{B_1})^2$ (left), Σ_{St_2} (middle), and Σ_{B_2} (right)

The **stellahedral variety** X_{St_n} is the toric variety associated to the fan Σ_{St_n}. Since the fans Σ_{B_n}, Σ_{St_n}, and $(\Sigma_{B_1})^n$ form a sequential coarsening, we have a natural sequence of maps $X_{B_n} \to X_{St_n} \to (\mathbb{P}^1)^n$ of toric varieties. Recall that $\boxplus\mathcal{O}(1)$ denotes the vector bundle on X_{B_n} that is the direct sum of the pullbacks of $\mathcal{O}_{\mathbb{P}^1}(1)$ from each $\mathbb{P}^1$ factor in $(\mathbb{P}^1)^n$. We reuse the notation $\boxplus\mathcal{O}(1)$ for the similar vector bundle pulled back only to X_{St_n}.

Stellahedral varieties play a central role in the proof the top-heavy conjecture and the nonnegativity of Kazhdan-Lusztig polynomials of matroids [BHM+22, BHM+]. The connection between stellahedral varieties and matroids was further developed in [EHL]. In our case, we will need the following exceptional Hirzebruch–Riemann–Roch-type theorem for stellahedral varieties.

Theorem 3.4. [EHL, Theorem 1.9 & Theorem 6.1] There is an isomorphism $\phi_T\colon K_T(X_{St_n}) \to A^\bullet_T(X_{St_n})[1/(1-t_i)]$ defined by

$$f_x(T_1,\dots,T_n) \mapsto f_x(\tfrac{1}{1-t_1},\dots,\tfrac{1}{1-t_n})$$

where $f_x(T_1,\dots,T_n) \in \mathbb{Z}[T_1^{\pm},\dots,T_n^{\pm}]$ is the localization value of a K-class $f \in K_T(X_{St_n})$ at a T-fixed point x of X_{St_n}. It descends to an isomorphism $\phi\colon K(X_{St_n}) \to A^\bullet(X_{St_n})$ which satisfies

$$\chi([\mathcal{E}]) = \int_{X_{St_n}} \phi([\mathcal{E}]) \cdot c(\boxplus \mathcal{O}(1)) \quad \text{for any } [\mathcal{E}] \in K(X_{St_n}).$$

The isomorphism ϕ^B of Theorem 3.2 is an extension of this isomorphism ϕ as follows.

Lemma 3.5. Let $p\colon X_{B_n} \to X_{St_n}$ be the toric morphism described above. The following diagram commutes:

$$\begin{array}{ccc} K(X_{St_n}) & \xrightarrow{\phi} & A^\bullet(X_{St_n}) \\ \downarrow{\scriptstyle p^*} & & \downarrow{\scriptstyle p^*} \\ K(X_{B_n}) & \xrightarrow{\phi^B} & A^\bullet(X_{B_n}). \end{array}$$

Proof. For a matroid M on $[n]$, its independence polytope $IP(\mathrm{M})$ is a deformation of Σ_{St_n} and hence defines a class $[IP(\mathrm{M})]$ in the polytope algebra $\overline{\mathbb{I}}(\mathscr{P}_{\mathbb{Z},\Sigma_{St_n}})$ [EHL, Example 3.15]. Moreover, the set $\{[IP(\mathrm{M})] : \mathrm{M}$ a matroid on $[n]\}$ spans $\overline{\mathbb{I}}(\mathscr{P}_{\mathbb{Z},\Sigma_{St_n}})$ [EHL, Proposition 7.4], which is isomorphic to $K(X_{St_n})$ via Theorem 2.5. Hence, it suffices to show the commutativity of the diagram on the elements $[IP(\mathrm{M})]$ for M matroids on $[n]$. Now, for $i \in [n]$ and any maximal cone σ of Σ_{B_n} containing $\mathbf{e}_i$, the T-equivariant localization value of $[IP(\mathrm{M})]$ at σ is a Laurent polynomial in the variables T_j for $j \neq i$, because the vertex of $IP(\mathrm{M})$ minimizing the standard pairing with a vector in the interior of σ has zero as its ith coordinate. By the descriptions of the maps ϕ_T and ϕ_T^B, this implies that $p^*\phi_T([IP(\mathrm{M})]) = \phi_T^B([IP(\mathrm{M})])$ for any matroid M on $[n]$. □

We can now finish the proof of Theorem C.

Proof of Theorem C. We have shown that ϕ^B is an isomorphism in Theorem 3.2. It remains to show the Hirzebruch–Riemann–Roch-type formula

$$\chi([\mathcal{E}]) = \int_{X_{St_n}} \phi^B([\mathcal{E}]) \cdot c(\boxplus \mathcal{O}(1)) \quad \text{for any } [\mathcal{E}] \in K(X_{B_n}).$$

Theorem D implies that $K(X_{B_n})$ is generated as an abelian group by Weyl images of independence polytopes of matroids. Hence, it suffices to check the Hirzebruch–Riemann–Roch-type formula for Weyl images of independence polytopes of matroids. Moreover, by Weyl-equivariance of ϕ^B, it suffices to check this for independence polytopes of matroids. Then this follows from the projection formula, Theorem 3.4, and Lemma 3.5. □

Remark 3.6. There are two obstructions to establishing analogues of Theorems C and D for arbitrary root systems. First, Propositions 2.10 and 2.12 about intersections with the unit cube, which were essential to our proof of Theorem D, no longer hold when the unit cube is replaced by (minuscule) weight polytopes of types other than A and B, for instance in type D. See [ESS21, Remark 3.15]. Second, the useful feature of Σ_{B_n} in the construction of the map ϕ_T^B in Theorem 3.2 and in the proof of Theorem C is that Σ_{B_n} can be viewed as a common refinement of 2^n "copies" of the stellahedral fan Σ_{St_n}. For arbitrary crystallographic root systems Φ, we do not know whether $K(X_\Phi)$ and $A^\bullet(X_\Phi)$ are isomorphic.

In Section 7.4, we will make use of the following "dual" version of ϕ^B. For a variety X, define the ring involution $D_K\colon K(X) \to K(X)$ by $[\mathcal{E}] \mapsto [\mathcal{E}^\vee]$ and the ring involution $D_A\colon A^\bullet(X) \to A^\bullet(X)$ by multiplication by $(-1)^d$ in degree d. Define the "dual" isomorphism $\zeta^B\colon K(X_{B_n}) \to A^\bullet(X_{B_n})$ by $D_A \circ \phi^B \circ D_K$. Similarly define ζ_T^B. The isomorphism ζ^B satisfies the following Hirzebruch–Riemann–Roch-type formula. To state it, let $\gamma \in A^1(X_{B_n})$ be the divisor class on X_{B_n} corresponding to the n-dimensional **cross polytope**, which is the B_n generalized permutohedron $\Diamond = \operatorname{Conv}(\mathbf{e}_i \mid i \in [n,\overline{n}]) \subset \mathbb{R}^n$.

Proposition 3.7. For any $[\mathcal{E}] \in K(X_{B_n})$, one has

$$\chi([\mathcal{E}]) = \int_{X_{B_n}} \zeta^B([\mathcal{E}]) \cdot c(\boxplus \mathcal{O}(-1)) \cdot (1 + \gamma + \cdots + \gamma^n).$$

Proof. A primitive vector in a ray of Σ_{B_n} is $\mathbf{e}_S$ for some nonempty admissible subset S of $[n,\overline{n}]$. We note that the minimum of the standard pairing $\langle x, \mathbf{e}_S\rangle$ for $x \in \Diamond$ is -1. Under the standard correspondence between polytopes and base-point-free divisors on toric varieties that we have been using, this means that γ is the sum of all boundary divisors on X_{B_n}. In other words, by [CLS11, Theorem 8.1.6], the line bundle $\mathcal{O}(-\gamma)$ is the canonical bundle of X_{B_n}. Applying Serre duality along with $\phi^B = D_A \circ \zeta^B \circ D_K$ to Theorem C, we have that

$$\begin{aligned}
\chi([\mathcal{E}]) &= (-1)^n \chi\big([\mathcal{O}(-\gamma)] \cdot D_K([\mathcal{E}])\big) \\
&= (-1)^n \int_{X_{B_n}} \phi^B\big([\mathcal{O}(-\gamma)] \cdot D_K([\mathcal{E}])\big) \cdot c(\boxplus \mathcal{O}(1)) \\
&= (-1)^n \int_{X_{B_n}} D_A\big(\zeta^B([\mathcal{O}(\gamma)] \cdot [\mathcal{E}]) \cdot c(\boxplus \mathcal{O}(-1))\big) \\
&= \int_{X_{B_n}} \zeta^B([\mathcal{O}(\gamma)]) \cdot \zeta^B([\mathcal{E}]) \cdot c(\boxplus \mathcal{O}(-1)).
\end{aligned}$$

It suffices now to show that $\zeta^B([\mathcal{O}(\gamma)]) = 1 + \gamma + \cdots + \gamma^n$. For this, we compute using torus-equivariant localization. For $w \in \mathfrak{S}_n^B$ such that $\operatorname{face}_v \Diamond = -\mathbf{e}_i$ for any $v \in C_w^\circ$, we have that $[\mathcal{O}(\gamma)]_w = T_i$. For such w, we must have that $i \in w([n])$, so this maps to $1/(1-t_i)$ under ζ_T^B. If $\operatorname{face}_v \Diamond = \mathbf{e}_i$, $[\mathcal{O}(\gamma)]_w = T_i^{-1}$, and we must have $i \notin w([n])$, so this maps to $1/(1+t_i)$ under ζ_T^B. We thus see that $\zeta^B([\mathcal{O}(\gamma)]) = c(\mathcal{O}(-\gamma))^{-1} = 1 + \gamma + \cdots + \gamma^n$, as desired. ☐

We now introduce a set of equivariant K-classes that is inspired by [BEST, Definition 10.4]. Say that a class $[\mathcal{E}] \in K_T(X_{B_n})$ has "nice Chern roots" if on the maximal cone corresponding to $w \in \mathfrak{S}_n^B$ we have $[\mathcal{E}]_w = a_{w,0} + \sum_{i \in w([n])} a_{w,i} T_i^{-1} - \sum_{i \notin w([n])} a_{w,i} T_i$.

We first define some notation. For $[\mathcal{E}] \in K_T(X_{B_n})$, let $c^T([\mathcal{E}], u) = c_0^T([\mathcal{E}]) + c_1^T([\mathcal{E}])u + \cdots \in A_T^\bullet(X_{B_n})[u]$ be the equivariant Chern polynomial. The equivariant Segre power series $s^T([\mathcal{E}], u) = s_0^T([\mathcal{E}]) + s_1^T([\mathcal{E}])u + \cdots \in A_T^\bullet(X_{B_n})[[u]]$ is defined by $s^T([\mathcal{E}], u) := c^T([\mathcal{E}], u)^{-1}$. Recall that the map that assigns a vector bundle $\mathcal{E}$ to its rank extends to a map $\mathrm{rk}\colon K(X_{B_n}) \to \mathbb{Z}$, and with u a formal variable we have that

$$\sum_{j=0}^{\infty} \textstyle\bigwedge^j [\mathcal{E}]_w u^j = \displaystyle\prod_{i=1}^{k_w} (1 + T^{m_{w,i}} u)^{a_{w,i}}, \text{ and } \sum_{j=0}^{\infty} \mathrm{Sym}^j [\mathcal{E}]_w u^j = \prod_{i=1}^{k_w} \left(\frac{1}{1 - T^{m_{w,i}} u} \right)^{a_{w,i}}$$

if we write $[\mathcal{E}]_w = \sum_{i=1}^{k_w} a_{w,i} T^{m_{w,i}}$.

Proposition 3.8. If $[\mathcal{E}]$ has nice Chern roots, then

$$\sum_{i \geq 0} \zeta_T^B(\textstyle\bigwedge^i [\mathcal{E}]) u^i = (u+1)^{\mathrm{rk}(\mathcal{E})}\, c^T \left([\mathcal{E}], \dfrac{u}{u+1} \right),$$

$$\sum_{i \geq 0} \phi_T^B(\textstyle\bigwedge^i [\mathcal{E}]) u^i = (u+1)^{\mathrm{rk}(\mathcal{E})}\, s^T([\mathcal{E}]^\vee)\, c^T \left([\mathcal{E}]^\vee, \dfrac{1}{u+1} \right),$$

$$\sum_{j \geq 0} \zeta_T^B(\mathrm{Sym}^j [\mathcal{E}]) u^j = \frac{1}{(1-u)^{\mathrm{rk}(\mathcal{E})}}\, s^T \left([\mathcal{E}], \frac{u}{u-1} \right), \text{ and}$$

$$\sum_{j \geq 0} \phi_T^B(\mathrm{Sym}^j [\mathcal{E}]) u^j = \frac{c^T([\mathcal{E}]^\vee)}{(1-u)^{\mathrm{rk}(\mathcal{E})}}\, s^T \left([\mathcal{E}]^\vee, \frac{1}{1-u} \right).$$

Proof. We prove the formulas involving ϕ^B; the formulas involving ζ^B are similar. Consider a maximal cone corresponding to $w \in \mathfrak{S}_n^B$, and write

$$[\mathcal{E}]_w = a_{w,0} + \sum_{i \in w([n])} a_{w,i} T_i^{-1} - \sum_{i \notin w([n])} a_{w,i} T_i.$$

Then

$$\begin{aligned}
\sum_{i \geq 0} \phi_T^B(\textstyle\bigwedge^i [\mathcal{E}])_w u^i &= (u+1)^{a_{w,0}} \displaystyle\prod_{i \in w([n])} (1 + \phi_T^B(T_i^{-1})u)^{a_{w,i}} \prod_{i \notin w([n])} (1 + \phi_T^B(T_i)u)^{a_{w,i}} \\
&= (u+1)^{a_{w,0}} \prod_{i \in w([n])} (1 + (1+t_i)^{-1}u)^{a_{w,i}} \prod_{i \notin w([n])} (1 + (1-t_i)^{-1}u)^{a_{w,i}} \\
&= (u+1)^{\mathrm{rk}(\mathcal{E})} \prod_{i \in w([n])} (1+t_i)^{-a_{w,i}} \left(1 + \frac{t_i}{u+1} \right) \prod_{i \notin w([n])} (1-t_i)^{-a_{w,i}} \left(1 - \frac{t_i}{u+1} \right) \\
&= (u+1)^{\mathrm{rk}(\mathcal{E})} s^T([\mathcal{E}]^\vee)_w\, c^T \left([\mathcal{E}]^\vee, \frac{1}{u+1} \right)_w,
\end{aligned}$$

and

$$\begin{aligned}\sum_{j\geq 0}\phi_T^B(\mathrm{Sym}^j[\mathcal{E}])_w u^j &= \frac{1}{(1-u)^{a_{w,0}}}\prod_{i\in w([n])}\left(\frac{1}{1-\phi_T^B(T_i^{-1})u}\right)^{a_{w,i}}\prod_{i\notin w([n])}\left(\frac{1}{1-\phi_T^B(T_i)u}\right)^{a_{w,i}}\\ &= \frac{1}{(1-u)^{a_{w,0}}}\prod_{i\in w([n])}\frac{1}{(1-(1+t_i)^{-1}u)^{a_{w,i}}}\prod_{i\notin w([n])}\frac{1}{(1-(1-t_i)^{-1}u)^{a_{w,i}}}\\ &= \frac{1}{(1-u)^{\mathrm{rk}(\mathcal{E})}}\prod_{i\in w([n])}\frac{1+t_i}{1+t_i/(1-u)}\prod_{i\notin w([n])}\frac{1-t_i}{1-t_i/(1-u)}\\ &= \frac{c^T([\mathcal{E}]^\vee)}{(1-u)^{\mathrm{rk}(\mathcal{E})}}\, s^T\left([\mathcal{E}]^\vee, \frac{1}{1-u}\right). \end{aligned}$$ □

4. The Chow cohomology ring of X_{B_n}

In this section, we first combine Theorems C and D to obtain a basis for the Chow cohomology ring $A^\bullet(X_{B_n})$. We then prove Theorem A by using the Hirzebruch–Riemann–Roch-type formula that ϕ^B satisfies.

4.1. **A Schubert basis.** We now describe the structure of the Chow cohomology ring $A^\bullet(X_{B_n})$ in terms of the "augmented Bergman classes" of matroids. Let M be a matroid of rank r on $[n]$. The **augmented Bergman fan** of M is a subfan Σ_{M} of the stellahedral fan Σ_{St_n} obtained by gluing together the order complex of lattice of flats and the independence complex of M; for a precise definition see [BHM+22, Definition 2.4]. Assigning weight 1 to each of its maximal cones defines a Minkowski weight $[\Sigma_{\mathrm{M}}]$, called the **augmented Bergman class** of M, which can be considered as an element in $A^{n-r}(X_{St_n})$. Augmented Bergman classes are nef Chow classes, and they span extremal rays of the cone of nef classes in $A^{n-r}(X_{St_n})$ [BHM+22, Proposition 2.8].

We will consider the pullbacks of augmented Bergman classes to X_{B_n} under the morphism $p\colon X_{B_n}\to X_{St_n}$ described in Section 3.3. These pullbacks continue to span extremal rays of the cone of nef classes in $A^\bullet(X_{B_n})$. For a matroid M, let $\mathrm{M}^\perp$ be the dual matroid. We will also refer to these pulled back classes as augmented Bergman classes. Only two properties of augmented Bergman classes will be essential to the rest of the paper:

(1) For any matroid M, the class $[\Sigma_{\mathrm{M}}]$ is nonzero.
(2) When M has rank $n-1$, the class $[\Sigma_{\mathrm{M}}]$ is the first Chern class of the line bundle corresponding to the simplex $IP(\mathrm{M}^\perp)$.

We now introduce some terminology. Say that a delta-matroid D with feasible sets $\mathcal{F}$ is **standard cornered** if whenever $B\in\mathcal{F}$ and $i\in B\cap[n]$, then $B\setminus\{i\}\cup\{\bar{i}\}\in\mathcal{F}$. For example, delta-matroids of the form $IP(\mathrm{M})$ are standard cornered. In fact this is the only example.

Lemma 4.1. Any standard cornered delta-matroid is of the form $IP(\mathrm{M})$ for a matroid M.

Proof. We show the matroid independent set axioms for $\mathcal{I}=\{B\cap[n] : B\in\mathcal{F}\}$. By assumption, $\mathcal{I}$ is a nonempty family of sets closed under taking subsets, so we must prove the independent set augmentation axiom. Let $A,B\in\mathcal{F}$ with $|A\cap[n]|<|B\cap[n]|$. Let F be the smallest face of $[0,1]^n$ containing $a=\mathbf{e}_{A\cap[n]}$ and $b=\mathbf{e}_{B\cap[n]}$. We have that $P(\mathrm{D})\cap F$ is a delta-matroid polytope. Let C be the vertex cone of a in $P(\mathrm{D})\cap F$ (with the apex of C at the origin). Then C contains $b-a$ and is generated by type B_n roots. Because $b - a$ has strictly positive sum of coordinates, C must have a generator with strictly positive sum of coordinates, either $\mathbf{e}_i$ or $\mathbf{e}_i + \mathbf{e}_j$ for some $i, j \in [n]$. So either $a + \mathbf{e}_i$ or $a + \mathbf{e}_i + \mathbf{e}_j$ lies in $P(\mathrm{D}) \cap F$; because D is standard cornered, the latter case implies the former one. By choice of F, the element i lies in $B \setminus A$, and hence $(A \cap [n]) \cup \{i\} \in \mathcal{I}$ meets the conditions of the independent set augmentation axiom. □

Say that a delta-matroid C is **cornered** if there is $w \in \mathfrak{S}_n^B$ such that $w \cdot \mathrm{C}$ is standard cornered. We now develop some properties of cornered delta-matroids.

Lemma 4.2. Let M be a matroid of rank r on $[n]$. Then the degree i part of $\phi^B([IP(\mathrm{M})])$ vanishes for $i > r$, is equal to $[\Sigma_{\mathrm{M}^\perp}]$ in degree r, and is 1 in degree 0.

Proof. That $\phi\colon K(X_{St_n}) \to A^\bullet(X_{St_n})$ has this property follows from [EHL, Lemma 5.9]. Then the result follows from Lemma 3.5. □

Lemma 4.3. Let $\mathrm{M}_1, \mathrm{M}_2$ be matroids on $[n]$, and suppose that $w_1 \cdot [IP(\mathrm{M}_1)] = w_2 \cdot [IP(\mathrm{M}_2)]$ for some $w_1, w_2 \in \mathfrak{S}_n^B$. Then the rank of M_1 is equal to the rank of M_2, and $w_1 \cdot [\Sigma_{\mathrm{M}_1^\perp}] = w_2 \cdot [\Sigma_{\mathrm{M}_2^\perp}]$.

Proof. By $\mathfrak{S}_n^B$-equivariance of ϕ^B, we must have that $w_1 \cdot [\Sigma_{\mathrm{M}_1^\perp}] = w_2 \cdot [\Sigma_{\mathrm{M}_2^\perp}]$. Lemma 4.2 identifies the rank of M as the degree of the top nonzero piece of $\phi^B([IP(\mathrm{M})])$. □

In particular, if $\mathrm{C} = w \cdot IP(\mathrm{M})$ is a cornered delta-matroid, then we define the **cornered rank** $\mathrm{rk}_{\mathrm{cor}}(\mathrm{C})$ as the rank of M, which is independent of the choice of M and w, and we define

$$[\Sigma_{\mathrm{C}}] := w \cdot [\Sigma_{\mathrm{M}^\perp}].$$

Note with this definition, for M a matroid we have $[\Sigma_{IP(\mathrm{M}^\perp)}] = [\Sigma_{\mathrm{M}}]$. With this notation, the following is an immediate consequence of Lemma 4.2.

Lemma 4.4. Let C be a cornered delta-matroid. Then the degree i part of $\phi^B([\mathrm{C}])$ vanishes for $i > \mathrm{rk}_{\mathrm{cor}}(\mathrm{C})$, is equal to $[\Sigma_{\mathrm{C}}]$ in degree $\mathrm{rk}_{\mathrm{cor}}(\mathrm{C})$, and is 1 in degree 0.

Now we construct our basis for $A^\bullet(X_{B_n})$, noting that Schubert delta-matroids are cornered.

Corollary 4.5. For any $0 \le r \le n$,

$$\{[\Sigma_{\mathrm{C}}] : \mathrm{C} \in \mathsf{SchDMat}_n^{\mathsf{clf}} \text{ and } \mathrm{rk}_{\mathrm{cor}}(\mathrm{C}) = r\}$$

is a basis for $A^r(X_{B_n})$.

Proof. Endow $K(X_{B_n})$ with a grading by declaring the rth graded piece to be generated by the elements $\{[P(\mathrm{C})]\colon \mathrm{C} \in \mathsf{SchDMat}_n^{\mathsf{clf}}, \mathrm{rk}_{\mathrm{cor}}(\mathrm{C}) = r\}$; this is well-defined by Theorem D. Combining Theorem D with Theorem C, we have that $\{\phi^B([P(\mathrm{C})]) : \mathrm{C} \in \mathsf{SchDMat}_n^{\mathsf{clf}}\}$ is a basis of $A^\bullet(X_{B_n})$. By Lemma 4.4, ϕ^B is lower-triangular with respect to the gradings on $K(X_{B_n})$ and $A^\bullet(X_{B_n})$ and the degree r part of $\phi^B([\mathrm{C}])$ is $[\Sigma_{\mathrm{C}}]$, so we conclude. □

Setting $r = 1$ in the corollary yields Theorem A(a) as follows.

Proof of Theorem A(a). The polytope of a delta-matroid in $\mathsf{SchDMat}_n^{\mathsf{clf}}$ of rank 1 is a translate of a simplex Δ_S^0 for $S \in \mathsf{AdS} \setminus \{\varnothing\}$, and vice versa. Namely, $P(\Omega_{[\overline{n}]\setminus\{\bar{i}\}\cup\{i\}}) = \Delta^0_{\{1,\dots,i\}}$, and if $\mathrm{D} = w \cdot \Omega_{[\overline{n}]\setminus\{\bar{i}\}\cup\{i\}}$, then $P(\mathrm{D}) = w \cdot P(\Omega_{[\overline{n}]\setminus\{\bar{i}\}\cup\{i\}})$ differs from $\Delta^0_{w\cdot\{1,\dots,i\}} = w \cdot \Delta^0_{\{1,\dots,i\}}$ only by the translations that distinguish the $\mathfrak{S}_n^B$-action on delta-matroid polytopes from the $\mathfrak{S}_n^B$-action on $\mathbb{R}^n$ in Section 2.1. No two simplices Δ_S^0 are translations of each other except for the pairs of line segments $\{\Delta^0_{\{i\}}, \Delta^0_{\{\bar{i}\}}\}$. Hence, setting $r = 1$ in Corollary 4.5, we have that the set

$$\{\text{the divisor class associated to } \Delta_S^0 : S \in \mathsf{AdS} \setminus \{\varnothing\} \text{ and } S \neq \{\bar{i}\} \text{ for } i \in [n]\}$$

is a basis of $A^1(X_{B_n})$. Thus, up to translation by a vector in $\mathbb{Z}^n$, every B_n generalized permutohedron is a signed Minkowski sum of the simplices Δ_S^0 in the displayed set. Since $\Delta^0_{\{\bar{i}\}} = \Delta^0_{\{i\}} - \mathbf{e}_i$, reinserting the segments $\Delta^0_{\{\bar{i}\}}$ into the set accounts for the translations. □

Remark 4.6. The h-vector of the Coxeter complex Σ_Φ of a root system Φ, or equivalently, the sequence of dimensions of the graded pieces of $A^\bullet(X_\Phi)$, is equal to the vector of Φ-Eulerian numbers [Bjö84, Bre94], which are defined in terms of the descents of elements in the Coxeter group associated to Φ. Concretely, in type B the set of descents of an element $w \in \mathfrak{S}_n^B$ is

$$\mathrm{des}(w) = \{i \in [n] : w(i-1) > w(i)\},$$

where we define $w(0) = 0$ to fit into the total order as $\overline{n} < \cdots < \overline{1} < 0 < 1 < \cdots < n$. The rth B_n Eulerian number is then

$$h_r(B_n) := |\{w \in \mathfrak{S}_n^B : \mathrm{des}(w) = r\}|.$$

In particular, Corollary 4.5 implies that the B_n Eulerian numbers count the coloop-free Schubert delta-matroids of cornered rank r. An analogous statement for type A was shown in [Ham17]. In neither type A nor type B do we know of a natural bijection between the set of Weyl group elements of fixed number of descents and the corresponding set of coloop-free Schubert (delta-)matroids.

4.2. **Volumes and lattice point enumerators.** We now compute volumes and lattice point counts of B_n generalized permutohedra by using Theorem C. We will use the following observation throughout. For an admissible subset $S \in \mathsf{AdS}$, let h_S be the divisor class on X_{B_n} associated to the simplex Δ_S^0. Because simplices are Weyl images of the independence polytopes of standard Schubert matroids of cornered rank 1, Lemma 4.4 implies that $\phi^B([\Delta_S^0]) = 1 + h_S$.

Proof of Theorem A(b). For a sequence $(S_1, \dots, S_n)$ of n admissible subsets, standard results in toric geometry [Ful93, §5.4] imply that the mixed volume of the corresponding simplices is the intersection product $\int_{X_{B_n}} h_{S_1} \cdots h_{S_n}$, which equals

$$\int_{X_{B_n}} (1 + h_{S_1}) \cdots (1 + h_{S_n}) = \int_{X_{B_n}} \phi^B([\Delta^0_{S_1}] \cdots [\Delta^0_{S_n}]) = \int_{X_{B_n}} \phi^B([\Delta^0_{S_1} + \cdots + \Delta^0_{S_n}]).$$

Let P be the Minkowski sum $\Delta^0_{S_1} + \cdots + \Delta^0_{S_n}$. By construction, the polytope P is "saturated towards the origin" in the following sense: For any subset $S \subseteq [n]$, denote $\mathrm{Orth}_S = \mathbb{R}^S_{\geq 0} \times \mathbb{R}^{[n]\setminus S}_{\leq 0}$. If $u \in P \cap \mathrm{Orth}_S$, then any $v \in \mathrm{Orth}_S$ such that $u - v \in \mathrm{Orth}_S$ is also in P. We tile $\mathbb{R}^n$ by lattice translates of the unit cube $\square = [0,1]^n$, and express

$$[P] = \Big(\sum_{m\in\mathbb{Z}^n} [P\cap(m+\square)]\Big)$$
$$+\text{ a linear combination of } \{[P\cap(m+F)] : m\in\mathbb{Z}^n,\ F \text{ a proper face of } \square\}$$

Every intersection $P\cap(m+\square)$ or $P\cap(m+F)$ in the expression is a translate of a delta-matroid polytope by Proposition 2.10. Because P is saturated towards the origin, these delta-matroid polytopes are cornered by Lemma 4.1. For such a delta-matroid C, by Lemma 4.4 we have $\int_{X_{B_n}} \phi^B([P(\mathrm{C})]) = 0$ when $P(\mathrm{C}) \neq \square$. When $P(\mathrm{C}) = \square$ we have

$$\int_{X_{B_n}} \phi^B([\square]) = \int_{X_{B_n}} \phi^B([\Delta^0_{\{1\}}]\cdots[\Delta^0_{\{n\}}]) = \int_{X_{B_n}} h_{\{1\}}\cdots h_{\{n\}} = 1.$$

We have thus reduced to counting the number of intersections where $P\cap(m+\square) = m+\square$. This happens only when $m+\square$ contains the origin, since each simplex is contained in the cross-polytope $\lozenge$, so $P\subset n\lozenge$, and every integral translate of $\square$ contained in $n\lozenge$ contains the origin. In other words, we are counting the set of cardinality-n admissible subsets $\tau\in \mathsf{AdS}_n$ such that $\mathrm{e}_\tau \in P$. This set, by construction of P, is in bijection with the set of signed transversals of $(S_1,\ldots,S_n)$. ☐

Proof of Theorem A(c). Denote by $\mathsf{AdS}^{\notin[n]}$ the subset $\{S\in\mathsf{AdS} : |S|>1 \text{ or } S=\{\bar{i}\}\subset[\overline{n}]\}$ of admissible subsets of $[n,\overline{n}]$. Note that the divisor class on X_{B_n} corresponding to the cube $\square=[0,1]^n$ is $h_{\{1\}}+\cdots+h_{\{n\}}$. By standard results in toric geometry [Ful93, §3.5], the quantity

$$\Big(\#\text{ lattice points of } \big(P(\{c_S\})-\square\big)\Big)$$

is computed by the Euler characteristic

$$\chi\Big([\sum_{S\in\mathsf{AdS}^{\notin[n]}} c_S\Delta^0_S + \sum_{i\in[n]}(c_i-1)\Delta^0_{\{i\}}]\Big).$$

Noting that $c(\boxplus\mathcal{O}(1)) = \prod_{i\in E}(1+h_{\{i\}})$, we apply Theorem C to obtain

$$\begin{aligned}
&\chi\Big([\sum_{S\in\mathsf{AdS}^{\notin[n]}} c_S\Delta^0_S + \sum_{i\in[n]}(c_i-1)\Delta^0_{\{i\}}]\Big)\\
&= \int_{X_{B_n}} \prod_{S\in\mathsf{AdS}^{\notin[n]}} \phi^B([\Delta^0_S])^{c_S}\cdot \prod_{i\in[n]}\phi^B([\Delta^0_{\{i\}}])^{c_i-1}\cdot c(\boxplus\mathcal{O}(1))\\
&= \int_{X_{B_n}} \prod_{S\in\mathsf{AdS}^{\notin[n]}} (1+h_S)^{c_S}\cdot\prod_{i\in[n]}(1+h_{\{i\}})^{c_i}\\
&= \int_{X_{B_n}} \prod_{S\in\mathsf{AdS}\setminus\{\varnothing\}} \left(\sum_{k=0}^{n}\binom{c_S}{k} h_S^{c_S}\right)\\
&= \Psi\Big(\operatorname{Vol}\big(\sum_{S\in\mathsf{AdS}\setminus\{\varnothing\}} c_S\Delta^0_S\big)\Big),
\end{aligned}$$

as desired. ☐

Finally, we note that the mixed volume computation above can be generalized to arbitrary cornered delta-matroids as follows.

Theorem 4.7. Let $\mathrm{C}_1, \dots, \mathrm{C}_k$ be cornered delta-matroids with $\sum \mathrm{rk}_{\mathrm{cor}}(\mathrm{C}_i) = n$, and write $\mathrm{C}_i = w_i \cdot IP(\mathrm{M}_i)$. Then we have

$$\int_{X_{B_n}} [\Sigma_{\mathrm{C}_1}] \cdots [\Sigma_{\mathrm{C}_k}] = \left| \left\{ \tau \in \mathsf{AdS}_n \;\middle|\; \begin{array}{c} \tau \text{ a signed transversal of } (w_1 \cdot B_1, \dots, w_1 \cdot B_1, \dots, w_k \cdot B_k, \dots, w_k \cdot B_k) \\ \text{where } B_i \text{ is a basis of } \mathrm{M}_i \text{ and } w_i \cdot B_i \text{ is repeated } \mathrm{rk}_{\mathrm{cor}}(\mathrm{C}_i) \text{ times} \end{array} \right\} \right|.$$

Proof. The argument is similar to the proof of Theorem A(b), so we sketch only the main steps. By Theorem C and Lemma 4.4, we have

$$\int_{X_{B_n}} [\Sigma_{\mathrm{C}_1}] \cdots [\Sigma_{\mathrm{C}_k}] = |\{m \in \mathbb{Z}^n : \mathrm{C}_1 + \cdots + \mathrm{C}_k \supseteq (m + \square)\}|$$

where $\square = [0,1]^n$. Write $w_i IP(\mathrm{M}_i)$ for the image of the polytope $IP(\mathrm{M}_i)$ under the isometry associated to w_i for the standard geometric action of $\mathfrak{S}_n^B$ on $\mathbb{R}^n$. Then $P(\mathrm{C}_1) + \cdots + P(\mathrm{C}_k)$ is an integral translate of $P = w_1 IP(\mathrm{M}_1) + \cdots + w_k IP(\mathrm{M}_k)$, so we may equivalently compute $|\{m \in \mathbb{Z}^n : P \supseteq (m + \square)\}|$. Because $w_i IP(\mathrm{M}_i) \subset \mathrm{rk}_{\mathrm{cor}}(\mathrm{C}_i)\lozenge$ for the cross-polytope $\lozenge$, we have $P \subseteq (\sum \mathrm{rk}_{\mathrm{cor}}(\mathrm{C}_i))\lozenge = n\lozenge$. Hence for $P \supseteq (m + \square)$, we must have that $n\lozenge \supset m + \square$ so $m + \square$ contains the origin. Hence, we are counting the number of $\tau \in \mathsf{AdS}_n$ such that $\mathbf{e}_\tau \in P$. The desired formula follows. $\square$

Corollary 4.8. For a matroid M of rank r and admissible subsets $S_1, \dots, S_r \in \mathsf{AdS}$, we have

$$\int_{X_{B_n}} [\Sigma_{\mathrm{M}}] \cdot h_{S_1} \cdots h_{S_r} = \left| \left\{ \tau \in \mathsf{AdS}_n \;\middle|\; \begin{array}{c} \tau \text{ a signed transversal of } (S_1, \dots, S_r, B, \dots, B) \\ \text{for some basis } B \text{ of } \mathrm{M}^\perp \end{array} \right\} \right|.$$

5. Tutte-like invariants of delta-matroids

We first recall some combinatorial operations on delta-matroids. In the context of multi-matroids these operations can be found in [Bou97].

Definition 5.1. Let D be a delta-matroid on $[n, \bar{n}]$, and let $i \in [n]$. We define three delta-matroids on $[n, \bar{n}] \setminus \{i, \bar{i}\}$ obtained from D as follows:

(1) If i is not a loop, the **contraction** D/i is the delta-matroid with feasible sets $B \setminus i$ for B a feasible set of D containing i.
(2) If i is not a coloop, the **deletion** $\mathrm{D} \setminus i$ is the delta-matroid with feasible sets $B \setminus \bar{i}$ for B a feasible set of D containing $\bar{i}$.
(3) We define the **projection** $\mathrm{D}(i)$ as the delta-matroid with feasible sets $B \setminus \{i, \bar{i}\}$ for B a feasible set of D.

If i is a loop (resp. coloop), we define $\mathrm{D}/i = \mathrm{D} \setminus i$ (resp. $\mathrm{D} \setminus i = \mathrm{D}/i$), so that $\mathrm{D}/i = \mathrm{D} \setminus i = \mathrm{D}(i)$.

If i is not a loop (resp. a coloop), then $P(\mathrm{D}/i)$ (resp. $P(\mathrm{D} \setminus i)$) is obtained by intersecting $P(\mathrm{D})$ with the hyperplane $x_i = 0$ (resp. $x_i = 1$). We obtain $P(\mathrm{D}(i))$ by taking the orthogonal projection of $P(\mathrm{D})$ onto $x_i = 0$. Therefore projections commute with each other and commute with deletion and contraction. For $I \subseteq [n]$, we write $\mathrm{D}(I)$ for the delta-matroid obtained by successively projecting along each $i \in I$.

In the introduction, we defined the U-polynomial $U_{\mathrm{D}}(u,v)$ and its specialization, the interlace polynomial $\mathrm{Int}_{\mathrm{D}}(v) = U_{\mathrm{D}}(0,v)$, via a recursion involving deletion, contraction, and projection, similar to the deletion-contraction recursion for the Tutte polynomial of a matroid. Like the Tutte polynomial of a matroid, the U-polynomial and the interlace polynomial also admit a non-recursive formula in the following way. For a delta-matroid D with feasible sets $\mathcal{F}$ and $S \in \mathsf{AdS}_n$, let

$$d_{\mathrm{D}}(S) = \frac{1}{2}\min_{B\in\mathcal{F}} |B \triangle S|, \text{ the lattice distance between } \mathbf{e}_{S\cap[n]} \text{ and } P(\mathrm{D}).$$

Proposition 5.2. For a delta-matroid D on $[n,\overline{n}]$, define polynomials $\mathrm{Int}'_{\mathrm{D}}(v)$ and $U'_{\mathrm{D}}(u,v)$ by

$$\mathrm{Int}'_{\mathrm{D}}(v) = \sum_{S\in\mathsf{AdS}_n} v^{d_{\mathrm{D}}(S)}, \text{ and } U'_{\mathrm{D}}(u,v) = \sum_{I\subseteq[n]} u^{|I|}\,\mathrm{Int}'_{\mathrm{D}(I)}(v).$$

Then $U'_{\mathrm{D}}(u,v)$ satisfies the recursion for $U_{\mathrm{D}}(u,v)$ in Definition 1.4. In particular, $U'_{\mathrm{D}} = U_{\mathrm{D}}$ and $\mathrm{Int}'_{\mathrm{D}} = \mathrm{Int}_{\mathrm{D}}$, and the recursive definition of U_{D} is independent of the element $i \in [n]$ chosen.

Proof. We first show that $\mathrm{Int}'_{\mathrm{D}}(v)$ satisfies the recursive property in Definition 1.4 with $u = 0$. [BH14, Theorem 30] states that if $i \in [n]$ is neither a loop nor coloop, then $\mathrm{Int}'_{\mathrm{D}}(v) = \mathrm{Int}'_{\mathrm{D}/i}(v) + \mathrm{Int}'_{\mathrm{D}\setminus i}(v)$, and that if every element is a loop or a coloop, then $\mathrm{Int}'_{\mathrm{D}}(v) = (1+v)^n$. If i is a loop or a coloop of D, then it continues to be so in D/J and $\mathrm{D}\setminus J$ for $J \subseteq [n]$ not containing i. Thus, we conclude that $\mathrm{Int}'_{\mathrm{D}}$ satisfies the desired recursive relation, and hence that $\mathrm{Int}'_{\mathrm{D}} = \mathrm{Int}_{\mathrm{D}}$.

For the U-polynomial, we have that

$$U'_{\mathrm{D}}(u,v) = uU'_{\mathrm{D}(i)}(u,v) + \sum_{J\not\ni i} u^{|J|}\,\mathrm{Int}_{\mathrm{D}(J)}(v).$$

If i neither a loop nor coloop of D, then i is neither a loop nor coloop of $\mathrm{D}(J)$ for any J not containing i. The defining recursion of the interlace polynomial gives that

$$\sum_{J\not\ni i} u^{|J|}\,\mathrm{Int}_{\mathrm{D}(J)}(u,v) = \sum_{J\not\ni i} u^{|J|}\left(\mathrm{Int}_{\mathrm{D}(J)/i}(u,v) + \mathrm{Int}_{\mathrm{D}(J)\setminus i}(u,v)\right) = U'_{\mathrm{D}/i}(u,v) + U'_{\mathrm{D}\setminus i}(u,v).$$

Combining these yields $U'_{\mathrm{D}}(u,v) = U'_{\mathrm{D}/i}(u,v) + U'_{\mathrm{D}\setminus i}(u,v) + uU'_{\mathrm{D}(i)}(u,v)$ if i is not a loop or coloop of D. If i is a loop or a coloop of D, then it continues to be so in $\mathrm{D}(J)$ for $J \subseteq [n]$ not containing i. Hence, if i is a loop or a coloop, we have

$$U'_{\mathrm{D}}(u,v) = \sum_{J\not\ni i} u^{|J|+1}\,\mathrm{Int}_{\mathrm{D}(J\cup i)}(v) + u^{|J|}\,\mathrm{Int}_{\mathrm{D}(J)}(v) = \sum_{J\not\ni i} u^{|J|}\Big(u\,\mathrm{Int}_{\mathrm{D}(J\cup i)}(v) + (v+1)\,\mathrm{Int}_{\mathrm{D}(J\cup i)}(v)\Big),$$

and hence $U'_{\mathrm{D}}(u,v) = (u+v+1)U'_{\mathrm{D}\setminus i}(u,v)$. □

Given two delta-matroids D_1, D_2 on disjoint ground sets, let $\mathrm{D}_1 \times \mathrm{D}_2$ be the delta-matroid on the union of the ground sets whose feasible sets are $B_1 \sqcup B_2$ for B_i feasible in D_i. When one observes that $d_{\mathrm{D}_1}(S_1)d_{\mathrm{D}_2}(S_2) = d_{\mathrm{D}_1\times\mathrm{D}_2}(S_1 \sqcup S_2)$ and that projections commute with products, Proposition 5.2 implies the following.

Corollary 5.3. For two delta-matroids D_1 and D_2 on disjoint ground sets, we have

$$U_{\mathrm{D}_1\times\mathrm{D}_2}(u,v) = U_{\mathrm{D}_1}(u,v)U_{\mathrm{D}_2}(u,v).$$

We also note the following property of U_{D} for future use.

Lemma 5.4. We have that

$$\sum_{I\subseteq[n]} a^{|I|}U_{\mathrm{D}(I)}(u,v) = U_{\mathrm{D}}(u+a,v).$$

Proof. We claim that, if i is not a loop or coloop, then

$$\sum_{I\subseteq[n]} a^{|I|}U_{\mathrm{D}(I)}(u,v) = U_{\mathrm{D}/i}(u+a,v) + U_{D\setminus i}(u+a,v) + (u+a)U_{D(i)}(u+a,v).$$

We induct on the size of the ground set. Note that

$$\sum_{i\in I\subseteq[n]} a^{|I|}U_{\mathrm{D}(I)}(u,v) = a\cdot \sum_{J\in[n]\setminus i} a^{|J|}U_{\mathrm{D}(i\cup J)} = a\cdot U_{\mathrm{D}(i)}(u+a,v), \text{ and}$$

$$\begin{aligned}\sum_{i\notin I\subseteq[n]} a^{|I|}U_{\mathrm{D}(I)}(u,v) &= \sum_{i\notin J\subset[n]} a^{|J|}(U_{\mathrm{D}/i(J)}(u,v) + U_{\mathrm{D}\setminus i(J)}(u,v) + uU_{\mathrm{D}(i\cup J)}(u,v)) \\ &= U_{\mathrm{D}/i}(u+a,v) + U_{\mathrm{D}\setminus i}(u+a,v) + uU_{\mathrm{D}(i)}(u+a,v).\end{aligned}$$

Summing these gives the claim. When i is a loop or coloop, it follows from the multiplicativity of the U-polynomial (Corollary 5.3) that the left-hand side satisfies the expected product formula. This shows that the left-hand side satisfies the defining recursion of the right-hand side. □

In the next two examples we compute the U-polynomials of delta-matroids from matroids.

Example 5.5. We compute $U_{IP(\mathrm{M})}$ for a matroid M on $[n]$ of rank r. An element $i \in [n]$ is a loop of D if i is a loop of M, and i is never a coloop of M. Then $\mathrm{D}(i)$ and D/i are both $IP(\mathrm{M}/i)$, and $\mathrm{D}\setminus i$ is $IP(\mathrm{M}\setminus i)$. Hence, $U_{IP(\mathrm{M})}$ is a Tutte–Grothendieck invariant, which implies that

$$U_{IP(\mathrm{M})}(u,v) = (u+1)^{n-r}T_{\mathrm{M}}\left(u+2, \frac{u+v+1}{u+1}\right).$$

Example 5.6. We compute $U_{P(\mathrm{M})}$ for a matroid M on $[n]$. Let $\mathrm{corank}_{\mathrm{M}}(S) = \mathrm{rk}_{\mathrm{M}}([n]) - \mathrm{rank}_{\mathrm{M}}(S)$ be the corank and $\mathrm{nullity}_{\mathrm{M}}(S) = |S| - \mathrm{rk}_{\mathrm{M}}(S)$ the nullity of a subset S in M. Then we claim that

$$U_{P(\mathrm{M})}(u,v) = \sum_{T\subseteq S\subseteq[n]} u^{|S-T|}v^{\mathrm{corank}_{\mathrm{M}}(S)+\mathrm{nullity}_{\mathrm{M}}(T)}.$$

Let $I\subseteq[n]$, and fix some $S\subseteq[n]\setminus I$. Then $d_{P(\mathrm{M})(I)}(S) = \min_{S\subseteq S'\subseteq S\cup I} d_{P(\mathrm{M})}(S')$, and

$$\begin{aligned}d_{P(\mathrm{M})}(S') &= \mathrm{corank}_{\mathrm{M}}(S') + \mathrm{nullity}_{\mathrm{M}}(S') \\ &= (\mathrm{corank}_{\mathrm{M}|_{S\cup I}/S}(S') + \mathrm{corank}_{\mathrm{M}}(S\cup I)) + (\mathrm{nullity}_{\mathrm{M}|_{S\cup I}/S}(S') + \mathrm{nullity}_{\mathrm{M}}(S)).\end{aligned}$$

The summand $\mathrm{corank}_{\mathrm{M}|_{S\cup I}}(S') + \mathrm{nullity}_{\mathrm{M}|_{S\cup I}}(S')$ achieves its minimum value 0 when S' is a basis of the minor $M|_{S\cup I}/S$. The other summand is the constant $\mathrm{corank}_{\mathrm{M}}(S\cup I)+\mathrm{nullity}_{\mathrm{M}}(S)$. The claim then follows from Proposition 5.2.

It would be interesting to compute the U-polynomial of other families of delta-matroids such as those arising from graphs and ribbon graphs (see Examples 6.4 and 6.5). Theorem B applies to these delta-matroids, and therefore gives log-concavity results.

6. Representability and enveloping matroids

We now discuss representability of delta-matroids and prepare for the construction of vector bundles associated to realizations of delta-matroids in Section 7.

6.1. **Torus-orbit closures.** We will discuss representability of delta-matroids using polytopes and torus-orbit closures. Let us prepare with generalities on torus-orbit closures in projective spaces and associated polytopes.

Let H be a torus with the character lattice $\mathrm{Char}(H)$. For a finite dimensional representation V of H, and a point $x \in \mathbb{P}(V)$, we define the **moment polytope** $P(\overline{H \cdot x})$ of its orbit closure $\overline{H \cdot x}$ as follows. Let $V \simeq \bigoplus_{i=0}^{N} V_i$ be the canonical decomposition into H-eigenspaces, where H acts on each V_i with character $a_i \in \mathrm{Char}(H)$. For a representative $v \in V$ of $x \in \mathbb{P}(V)$, let $\mathscr{A}$ be the set

$$\mathscr{A} = \Big\{a_i : v_i \neq 0 \text{ in the expression } v = \sum_{i=0}^{N} v_i, \text{ where } v_i \in V_i \text{ for all } i = 0, \ldots, N\Big\}$$

which is independent of the choice of v. We define

$$P(\overline{H \cdot x}) = \text{the convex hull of } \mathscr{A} \subset \mathrm{Char}(H) \otimes \mathbb{R}.$$

Over $\mathbb{C}$, this agrees with the classical notion of moment polytopes; see for instance [Ful93, §4.2] and [Sot03, §8]. Let us record the following basic facts.

Proposition 6.1. With notation as above:

(1) The (k-dimensional) H-orbits of $\overline{H \cdot x}$ are in bijection with the (k-dimensional) faces of $P(\overline{H \cdot x})$ (for all $0 \leq k \leq \dim H$). The character lattice of the quotient of H by the stabilizer of the orbit corresponding a face F is the sublattice $\mathbb{Z}\{F \cap \mathscr{A}\}$ of $\mathrm{Char}(H)$. (Here $F \cap \mathscr{A}$ is translated appropriately to contain the origin.)

(2) If $\iota : H' \hookrightarrow H$ is an inclusion of a subtorus H' with the corresponding linear projection $\iota^{\#} : \mathrm{Char}(H)_{\mathbb{R}} \to \mathrm{Char}(H')_{\mathbb{R}}$, then $P(\overline{H' \cdot x})$ equals the projection $\iota^{\#} P(\overline{H \cdot x})$.

Proof. The orbit closure $\overline{H \cdot x}$ is isomorphic to the H-variety

$$X_{\mathscr{A}} = \text{the closure of the image of } H \to \mathbb{P}^{|\mathscr{A}|-1} \text{ defined by } h \mapsto (h^a)_{a \in \mathscr{A}}.$$

The first statement is then [CLS11, Corollary 3.A.6]. The second statement follows by construction because the H-eigenspace V_i with weight $a_i \in \mathrm{Char}(H)$ is an H'-eigenspace with weight $\iota^{\#} a_i \in \mathrm{Char}(H')$. □

6.2. **Representable delta-matroids.** For a delta-matroid D with feasible sets $\mathcal{F}$, let

$$\widehat{P(\mathrm{D})} = 2P(\mathrm{D}) - \mathbf{e}_{[n]} = \text{the convex hull of } \{\mathbf{e}_B : B \in \mathcal{F}\} \subset [-1, 1]^n.$$

When $P(\mathrm{D}) = P(\mathrm{M})$ or $P(\mathrm{D}) = IP(\mathrm{M})$, we set $\widehat{P(\mathrm{M})} := \widehat{P(\mathrm{D})}$ and $\widehat{IP(\mathrm{M})} := \widehat{P(\mathrm{D})}$ respectively. We now describe representability of D in terms of the polytope $\widehat{P(\mathrm{D})}$ and torus-orbit closures in a type B Grassmannian.

The **standard** $(2n+1)$**-dimensional quadratic space** is $\Bbbk^{2n+1}$, whose coordinates are labelled $\{1,\dots,n,\overline{1},\dots,\overline{n},0\}$, and which is equipped with the quadratic form

$$q(x_1,\dots,x_n,x_{\bar{1}},\dots,x_{\bar{n}},x_0) = x_1x_{\bar{1}} + \cdots + x_nx_{\bar{n}} + x_0^2.$$

A maximal isotropic subspace $L \subset \Bbbk^{2n+1}$ is an n-dimensional subspace for which the restriction $q|_L$ is identically zero. The **maximal orthogonal Grassmannian**, denoted $OGr(n;2n+1)$, is a variety whose $\Bbbk$-valued points are in bijection with maximal isotropic subspaces of the standard $(2n+1)$-dimensional quadratic space $\Bbbk^{2n+1}$. By definition, $OGr(n;2n+1)$ is a closed subvariety of the Grassmannian $Gr(n;2n+1)$ with the Plücker embedding $Gr(n;2n+1) \hookrightarrow \mathbb{P}^{\binom{2n+1}{n}-1}$. The torus $\mathbb{G}_{\mathrm{m}}^{2n+1}$ acts on $Gr(n;2n+1)$ by its standard action on $\Bbbk^{2n+1}$. The torus $T = \mathbb{G}_m^n$ embeds into $\mathbb{G}_{\mathrm{m}}^{2n+1}$ by $(t_1,\dots,t_n) \mapsto (t_1,\dots,t_n,t_1^{-1},\dots,t_n^{-1},1)$, and the induced action of T on $Gr(n;2n+1)$ preserves $OGr(n;2n+1)$. We thus treat $OGr(n;2n+1)$ as a T-variety with the T-equivariant Plücker embedding in $\mathbb{P}^{\binom{2n+1}{n}-1}$.

Proposition 6.2. For $L \in OGr(n;2n+1)$ over $\Bbbk$, the set of admissible subsets

$$\mathcal{F} = \{S \in \mathsf{AdS}_n : \text{the composition } L \hookrightarrow \Bbbk^{2n+1} \twoheadrightarrow \Bbbk^S \text{ is an isomorphism}\}$$

is the set of feasible sets of a delta-matroid D, and the moment polytope $P(\overline{T\cdot[L]})$ of the orbit closure of $[L]$ as a point in $\mathbb{P}^{\binom{2n+1}{n}-1}$ is equal to $\widehat{P(\mathrm{D})}$.

In this case, we say that L is a B_n **representation** of D. We say that D is B_n **representable** if it has a B_n representation. Over $\mathbb{C}$, the proposition is [GS87, Section 7, Theorem 1]. A type C analogue of this statement for the Lagrangian Grassmannian, without the assertion about moment polytopes, appears in [BGW03, Theorem 3.4.3].

Proof. Index the coordinates of $\mathbb{P}^{\binom{2n+1}{n}-1}$ by size n subsets of $[n,\overline{n}] \cup \{0\}$. One verifies that:

- The T-fixed points of $OGr(n;2n+1)$ correspond to admissible subsets $B \in \mathsf{AdS}_n$ of size n, where B gives a point in $\mathbb{P}^{\binom{2n+1}{n}-1}$ whose Plücker coordinates are all zero except at B.
- The T-invariant closed curves of $OGr(n;2n+1)$ correspond to pairs of T-fixed points such that, writing B and B' for the corresponding admissible subsets, one has $\mathbf{e}_B - \mathbf{e}_{B'}$ parallel to $\mathbf{e}_i$, $\mathbf{e}_i + \mathbf{e}_j$, or $\mathbf{e}_i - \mathbf{e}_j$ for some $i,j \in [n]$.

The proposition now follows from Proposition 6.1(1). □

Example 6.3. Schubert delta-matroids are B_n representable, and their representations explain their name as follows. The closed cells X_v of the Schubert stratification of $OGr(n;2n+1)$ are indexed by $v \in \mathfrak{S}_n^B/\mathfrak{S}_n$, and the containment relation among the X_v is given by the reversed Bruhat order. If x is a general point of X_v, then the delta-matroid represented by the corresponding isotropic subspace is the standard Schubert delta-matroid $\Omega_{v\cdot[\overline{n}]}$. In particular, they are certain generalized Bruhat interval polytopes corresponding to Schubert cells [TW15]. This is analogous to the relationship between Schubert matroids on $[n]$ of rank r and the Schubert stratification of $Gr(r;n)$.

A maximal isotropic subspace L in $\Bbbk^{2n}$ with the quadratic form $q(x_1,\dots,x_n,x_{\overline{1}},\dots,x_{\overline{n}}) = x_1x_{\overline{1}} + \cdots + x_nx_{\overline{n}}$ yields a maximal isotropic subspace $L \oplus \{0\}$ in $\Bbbk^{2n+1}$, and hence a B_n representation of a delta-matroid D. In such case, we say that L is a D_n**-representation** of D. Such a delta-matroid enjoys the additional property that it is an **even delta-matroid**, meaning that the parity of $|B \cap [n]|$ for any feasible set B is the same [BGW03, Theorem 3.10.2].

In the literature, there are two prominent constructions of delta-matroids from graphs. Both constructions yield even delta-matroids with D_n-representations.

Example 6.4. Let G be a simple graph on vertex set $[n]$, and let A_G be its adjacency matrix with entries considered as elements of $\mathbb{F}_2$. As the matrix A_G is skew-symmetric, the row-span of the $n \times 2n$ matrix $[I_n|A_G]$ is an isotropic subspace of $\mathbb{F}_2^{2n}$, and hence defines an even delta-matroid $\mathrm{D}(G)$. The interlace polynomial was originally defined and studied as a graph invariant. See [Duc92, ABS04, AvdH04].

Example 6.5. A graph Γ embedded in a surface, also known as a ribbon graph, with edges labeled by $[n]$ defines a delta-matroid $\mathrm{D}(\Gamma)$ whose feasible sets are the "quasi-trees" of Γ. See [CMNR19a] for a history and proofs, and [CMNR19b] for further connection between delta-matroids and ribbon graphs generalizing the connection between matroids and graphs. [BGW03, Theorem 4.3.5] shows that such a delta-matroid has a D_n-representation (see also [BBGS00]).

6.3. **Enveloping matroids.** The notion of an *enveloping matroid* of a delta-matroid will play a crucial role when we construct "tautological classes of delta-matroids" in §7 and when we apply tools from tropical Hodge theory to prove Theorem B in §8.

Definition 6.6. Let M be a matroid on $[n, \bar{n}]$, and let D be a delta-matroid on $[n, \bar{n}]$. Then M is an **enveloping matroid** [2] of D if the image of $P(\mathrm{M})$ under the map $\mathrm{env}\colon \mathbb{R}^{2n} \to \mathbb{R}^n$ given by $\mathrm{env}(x_1, \ldots, x_n, x_{\bar{1}}, \ldots, x_{\bar{n}}) = (x_1 - x_{\bar{1}}, \ldots, x_n - x_{\bar{n}})$ is $\widehat{P(\mathrm{D})}$.

To avoid confusion with our notation that $\mathbf{e}_{\bar{i}} = -\mathbf{e}_i \in \mathbb{R}^n$, we use $\mathbf{u}_1, \ldots, \mathbf{u}_n, \mathbf{u}_{\bar{1}}, \ldots, \mathbf{u}_{\bar{n}}$ to refer to the standard basis of $\mathbb{R}^{2n}$. For $S \subset [n, \bar{n}]$, let $\mathbf{u}_S = \sum_{i \in S} \mathbf{u}_i$. If $S \in \mathsf{AdS}$, then $\mathrm{env}(\mathbf{u}_S) = \mathbf{e}_S$.

Existence of enveloping matroids behaves well with respect to operations on delta-matroids as follows. Let M be an enveloping matroid of a delta-matroid D on $[n, \overline{n}]$.

- For $w \in \mathfrak{S}_n^B$, the $\mathfrak{S}_n^B$-action on $[n, \bar{n}]$ makes $w \cdot \mathrm{M}$ an enveloping matroid of $w \cdot \mathrm{D}$.
- For $i \in [n]$, the matroid minor $\mathrm{M}/i \setminus \bar{i}$ (resp. $\mathrm{M} \setminus i/\bar{i}$) is an enveloping matroid for D/i (resp. $\mathrm{D} \setminus i$).
- If M' is an enveloping matroid of another delta-matroid D' on ground set disjoint from that of D, then $\mathrm{M} \times \mathrm{M}'$ is an enveloping matroid for $\mathrm{D} \times \mathrm{D}'$.
- The **dual delta-matroid** $\mathrm{D}^\perp$ is the delta-matroid with feasible sets $\{\overline{B} \colon B \text{ a feasible set of } \mathrm{D}\}$. Then the dual matroid $\mathrm{M}^\perp$ is an enveloping matroid for $\mathrm{D}^\perp$.

For future use in §8, we record an observation that loops and coloops of D and M are compatible.

Lemma 6.7. Let D be a delta-matroid with an enveloping matroid M, and let $i \in [n]$. Then i is a loop (resp. coloop) in D if and only if i is a loop and $\bar{i}$ a coloop (resp. i is a coloop and $\bar{i}$ a loop) in M. In particular, if D is loop-free and coloop-free, then so is M.

[2] [BGW03, Exercise 3.12.6] defines an enveloping matroid of D to be a matroid M whose set of bases that are admissible coincide with the set of feasible sets of D. Our definition is a stricter condition because we further require that $\mathrm{env}(\mathbf{e}_B)$ lie in $\widehat{P(\mathrm{D})}$ for any, not necessarily admissible, basis B of M.

Proof. Let us prove the statement for when i is a loop, i.e., the polytope $\widehat{P(D)} \subset \mathbb{R}^n$ is contained in the hyperplane $x_i = -1$. If a basis B of M contains i or does not contain $\bar{i}$, then $\mathrm{env}(\mathbf{u}_B)$ lies in $x_i \geq 0$. Hence i is a loop and $\bar{i}$ a coloop of M. The other direction is similar. □

We show that B_n representable delta-matroids have enveloping matroids. In particular, delta-matroids arising from graphs and graphs embedded on surfaces (Examples 6.4 and 6.5) have enveloping matroids.

Proposition 6.8. Let $L \subset \Bbbk^{2n+1}$ be a B_n representation of a delta-matroid D, and let L' denote the image of L under the projection to $\Bbbk^{2n}$ forgetting the x_0-coordinate. Then the matroid that L' represents is an enveloping matroid of D. In particular, every B_n representable delta-matroid has an enveloping matroid.

Proof. Let M be the matroid that L represents. As a point in $OGr(n; 2n+1) \subset Gr(n; 2n+1) \subset \mathbb{P}^{\binom{2n+1}{n}-1}$, the moment polytope of $\overline{\mathbb{G}_{\mathrm{m}}^{2n+1} \cdot [L]}$ is $P(\mathrm{M})$, whereas the moment polytope of $\overline{T \cdot [L]}$ is $\widehat{P(\mathrm{D})}$ by Proposition 6.2. Then Proposition 6.1(2) implies that the image of $P(\mathrm{M})$ under the composition $\mathrm{env} \circ \pi_0$ is $\widehat{P(\mathrm{D})}$, where $\pi_0 : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n}$ is the projection forgetting the 0th coordinate. Note that L' is a representation of $\mathrm{M} \setminus 0$, and $P(\mathrm{M} \setminus 0)$ is contained in $\mathrm{env} \circ \pi_0(P(\mathrm{M})) = \widehat{P(\mathrm{D})}$. Each feasible set of D is a basis of M which does not contain 0, and hence is a basis of $\mathrm{M} \setminus 0$, which proves that $\mathrm{env}(P(\mathrm{M} \setminus 0)) = \widehat{P(\mathrm{D})}$. □

Remark 6.9. Because the Weyl groups of type B and C root systems coincide, one may consider delta-matroids as type C Coxeter matroids, and consequently consider C_n-representability in terms of Lagrangian subspaces in a $2n$-dimensional space with a symplectic form. See [BGW98] or [BGW03, §3.4]. The proof of Proposition 6.8 shows that C_n-representable delta-matroids also have enveloping matroids.

We also show that delta-matroids arising from matroids admit enveloping matroids.

Proposition 6.10. Let M be a matroid on $[n]$. Then the delta-matroids $P(\mathrm{M})$ and $IP(\mathrm{M})$ have enveloping matroids.

Proof. For $P(\mathrm{M})$, we take the enveloping matroid $\mathrm{M} \oplus \overline{\mathrm{M}}^{\perp}$, where $\overline{\mathrm{M}}^{\perp}$ is the isomorphic image of $\mathrm{M}^{\perp}$ under $\overline{(\cdot)} : [n] \to [\bar{n}]$. Minkowski sums commute with linear projections, so

$$\begin{aligned}\mathrm{env}(P(\mathrm{M} \oplus \overline{\mathrm{M}}^{\perp})) &= \mathrm{env}(P(\mathrm{M}) + P(\overline{\mathrm{M}}^{\perp})) \\ &= P(\mathrm{M}) + (-P(\mathrm{M}^{\perp})) \\ &= P(\mathrm{M}) + (P(\mathrm{M}) - \mathbf{e}_{[n]}) = \widehat{P(\mathrm{M})}.\end{aligned}$$

For $IP(\mathrm{M})$ we take the free product $\mathrm{M} \mathbin{\square} \overline{\mathrm{M}}^{\perp}$ of [CS05], whose bases are the sets $S \cup \overline{T}$ of size $\mathrm{rank}\, \mathrm{M} + \mathrm{rank}\, \mathrm{M}^{\perp} = n$ with $S, T \subseteq [n]$ such that S is independent in M and T is spanning in $\mathrm{M}^{\perp}$. Write $SP(\mathrm{N})$ for the spanning set polytope of a matroid N, so $SP(\mathrm{N}^{\perp}) = -IP(\mathrm{N}) + \mathbf{e}_{[n]}$. We show that

$$P(\mathrm{M} \mathbin{\square} \overline{\mathrm{M}}^{\perp}) = (IP(\mathrm{M}) + SP(\overline{\mathrm{M}}^{\perp})) \cap H,$$

where H is the hyperplane $\{v \in \mathbb{R}^{2n} : \sum_{i\in[n,\bar{n}]} v_i = n\}$. For a polytope Q, any vertex of $Q \cap H$ is of the form $F \cap H$, where F is a vertex or edge of Q. The polytope $IP(\mathrm{M}) + SP(\overline{\mathrm{M}}^{\perp})$ is a lattice polytope whose edge directions all have the form $\mathbf{u}_i$ or $\mathbf{u}_i - \mathbf{u}_j$ for $i, j \in [n, \bar{n}]$ because each edge of a Minkowski sum is parallel to an edge of one of the two summands. As $\sum_{i\in[n,\bar{n}]} v_i$ takes values 0 or 1 on all of these direction vectors, if H intersects an edge of $IP(\mathrm{M}) + SP(\overline{\mathrm{M}}^{\perp})$ transversely, then the intersection is a lattice point. Therefore $(IP(\mathrm{M}) + SP(\overline{\mathrm{M}}^{\perp})) \cap H$ is a lattice polytope as well. By definition of the free product, $P(\mathrm{M} \,\Box\, \overline{\mathrm{M}}^{\perp})$ and this intersection have the same set of lattice points, so they are equal. Now as above

$$\begin{aligned} \mathrm{env}(P(\mathrm{M} \,\Box\, \overline{\mathrm{M}}^{\perp})) &\subseteq \mathrm{env}(IP(\mathrm{M}) + SP(\overline{\mathrm{M}}^{\perp})) \\ &= IP(\mathrm{M}) + (-SP(\mathrm{M}^{\perp})) \\ &= IP(\mathrm{M}) + (IP(\mathrm{M}) - \mathbf{e}_{[n]}) = \widehat{IP(\mathrm{M})}. \end{aligned}$$

The containment is an equality because every vertex of $\widehat{IP(\mathrm{M})}$ has the form $\mathbf{e}_S - \mathbf{e}_{\overline{E\setminus S}}$ for S an independent set of M, and this vertex has the preimage $(\mathbf{u}_S, \mathbf{u}_{E\setminus S})$ in $P(\mathrm{M} \,\Box\, \overline{\mathrm{M}}^{\perp})$. ☐

Example 6.11. In [Bou97, Section 4], Bouchet gives the example, which he attributes to Duchamp, of the delta-matroid D with the set of feasible sets

$$\begin{aligned} \mathcal{F} = \{&\{\bar{1},\bar{2},\bar{3},\bar{4}\},\{\bar{1},\bar{2},\bar{3},4\},\{\bar{1},2,3,\bar{4}\},\{1,\bar{2},3,\bar{4}\},\{1,2,\bar{3},\bar{4}\}, \\ &\{\bar{1},2,3,4\},\{1,\bar{2},3,4\},\{1,2,\bar{3},4\},\{1,2,3,4\}\}. \end{aligned}$$

This delta-matroid has the property that there is no matroid M on $[4,\bar{4}]$ whose set of admissible bases coincides with $\mathcal{F}$. In particular, D does not have an enveloping matroid.

7. Vector bundles and K-classes

We now define two types of equivariant vector bundles associated to realizations of delta-matroids, which we call *isotropic tautological bundles* and *enveloping tautological bundles* respectively. The isotropic tautological bundles are analogous to the bundles used in [BEST], and the enveloping tautological bundles are analogous to the bundles used in [EHL]. The construction of an isotropic tautological bundle depends on the choice of a B_n representation of a delta-matroid, and the construction of an enveloping tautological bundle depends on the choice of a realization of an enveloping matroid. The K-classes of the bundles will only depend on the delta-matroid, which leads to the construction of *isotropic tautological classes* and *enveloping tautological classes* for all delta-matroids, not necessarily with a B_n representation or a representable enveloping matroid.

In both cases, we will construct a T-equivariant map from X_{B_n} to a Grassmannian and define the bundles as pullbacks of certain universal bundles. Let us therefore prepare with a discussion of maps from X_{B_n} to Grassmannians. The discussion can be easily adapted to replace X_{B_n} with any smooth projective toric variety, but such generality won't be needed here.

7.1. **Maps into Grassmannians.** Let $L \subset \Bbbk^N$ be a linear space of dimension r, corresponding to a point $[L]$ of $Gr(r;N)$ and representing a matroid M of rank r on $[N]$. Let $\iota\colon T \to \mathbb{G}_{\mathrm{m}}^N$ be an inclusion of T into the torus acting on $Gr(r;N)$, and let $\iota^{\#}\colon \operatorname{Char}(\mathbb{G}_{\mathrm{m}}^N) \to \operatorname{Char}(T)$ be the pullback map on character lattices. Then $\iota^{\#}P(\mathrm{M})$ is a lattice polytope in $\operatorname{Char}(T)\otimes\mathbb{R}$. Suppose that Σ_{B_n} refines the normal fan of $\iota^{\#}(P(\mathrm{M}))$. For each $w \in \mathfrak{S}_n^B$ and any v in the interior of C_w, let B be any basis of M such that the corresponding vertex of $P(\mathrm{M})$ maps under $\iota^{\sharp}$ into $\operatorname{face}_v \iota^{\#}P(\mathrm{M})$.

Proposition 7.1. With the set-up as above, there is a unique T-equivariant morphism $\varphi_L\colon X_{B_n} \to Gr(r;N)$ such that the identity of $T \subset X_{B_n}$ is sent to $[L]$. The pullback $\varphi_L^*(\mathcal{S}_{\mathrm{univ}})$ of the tautological subbundle on $Gr(r;N)$ is a T-equivariant vector bundle on X_{B_n} such that for each $w \in \mathfrak{S}_n^B$, the localization value is

$$[\varphi_L^*(\mathcal{S}_{\mathrm{univ}})]_w = \sum_{i\in B} \iota^{\#}T_i.$$

Proof. The moment polytope (taken with respect to the Plücker embedding of the Grassmannian) of the $\mathbb{G}_{\mathrm{m}}^N$-orbit closure $\overline{\mathbb{G}_{\mathrm{m}}^N\cdot[L]} \subset Gr(r;N)$ is $P(\mathrm{M})$, so by Proposition 6.1(2), the moment polytope of the T-orbit closure $\overline{T\cdot[L]}$ is $\iota^{\#}P(\mathrm{M})$. Note that $\overline{T\cdot[L]}$ is a (possibly non-normal) toric variety whose embedded torus is $T/\operatorname{Stab}_T([L])$. The normalization of $\overline{T\cdot[L]}$ is a toric variety whose fan is the normal fan of $\iota^{\#}P(\mathrm{M})$ (considered in $\operatorname{Cochar}(T)\otimes\mathbb{R}$, possibly with lineality space), and whose lattice may be finer than the lattice in Σ_{B_n}. We therefore have a unique morphism $X_{B_n} \to \overline{T\cdot[L]} \hookrightarrow Gr(r;N)$ such that the identity of T is sent to $[L]$.

To compute the localization of $[\varphi_L^*(\mathcal{S}_{\mathrm{univ}})]$ to a fixed point of X_{B_n}corresponding to $w \in \mathfrak{S}_n^B$, we consider the image of this fixed point, $x_w \in Gr(r;N)$. Because pullbacks commute with pullbacks, it suffices to compute the pullback in T-equivariant K-theory of $[\mathcal{S}_{\mathrm{univ}}]$ to x_w. Note that x_w is a T-fixed point, which implies that $\overline{\mathbb{G}_{\mathrm{m}}^N\cdot x_w}$ is acted on trivially by T, so $K_T(\overline{\mathbb{G}_{\mathrm{m}}^N\cdot x_w}) = K(\overline{\mathbb{G}_{\mathrm{m}}^N\cdot x_w})\otimes \mathbb{Z}[T_1^{\pm1},\ldots,T_n^{\pm1}]$. Therefore the pullback in T-equivariant K-theory of $[\mathcal{S}_{\mathrm{univ}}]$ to any point of $\overline{\mathbb{G}_{\mathrm{m}}^N\cdot x_w}$ is the same element of $\mathbb{Z}[T_1^{\pm1},\ldots,T_n^{\pm1}]$. The $\mathbb{G}_{\mathrm{m}}^N$-fixed points of $\overline{\mathbb{G}_{\mathrm{m}}^N\cdot x_w}$ are exactly the vertices of $P(\mathrm{M})$ in the preimage of $\operatorname{face}_v \iota^{\#}P(\mathrm{M})$. The pullback in $\mathbb{G}_{\mathrm{m}}^N$-equivariant K-theory of $[\mathcal{S}_{\mathrm{univ}}]$ to a $\mathbb{G}_{\mathrm{m}}^N$-fixed point of $Gr(r;N)$ corresponding to $B\subset[N]$ is $\sum_{i\in B}T_i$. Applying $\iota^{\#}$ implies the result. □

For using Proposition 7.1, we set up some notation for a delta-matroid D and $w \in \mathfrak{S}_n^B$:

- Let $B_w(\mathrm{D})$ be the w**-minimal feasible set of** D, i.e., the feasible set corresponding to the vertex $\operatorname{face}_v P(\mathrm{D})$ of $P(\mathrm{D})$ on which any linear functional v in the interior of C_w achieves its minimum.
- Likewise, let $B_w^{\max}(\mathrm{D})$ be the w**-maximal** feasible set corresponding to the vertex of $P(\mathrm{D})$ on which any linear functional in the interior of C_w achieves its maximum.

Note that $\overline{B_w^{\max}(\mathrm{D})} = B_w(\mathrm{D}^{\perp})$. We omit (D) and simply write B_w if no confusion is expected.

7.2. **Construction of isotropic tautological bundles.** Let $\mathcal{O}_{OGr(n;2n+1)}^{\oplus 2n+1}$ be the rank $2n+1$ trivial bundle on $OGr(n;2n+1)$, which is equipped with the standard quadratic form, and which is a T-equivariant vector bundle with the action

$$(t_1,\ldots,t_n)\cdot(x_1,\ldots,x_n,x_{\overline{1}},\ldots,x_{\overline{n}},x_0) = (t_1x_1,\ldots,t_nx_n,t_1^{-1}x_{\overline{1}},\ldots,t_n^{-1}x_{\bar{n}},x_0). \tag{7.1}$$

Let $\mathcal{I}_{\text{univ}}$ be the universal isotropic subbundle of $\mathcal{O}^{\oplus 2n+1}_{OGr(n;2n+1)}$, whose fiber over a point of $OGr(n; 2n+1)$ corresponding to the maximal isotropic subspace $L \subset \Bbbk^{2n+1}$ is L. Under the inclusion $OGr(n; 2n+1) \subset Gr(n; 2n+1)$, the bundle $\mathcal{I}_{\text{univ}}$ is the T-equivariant subbundle of $\mathcal{O}^{\oplus 2n+1}_{OGr(n;2n+1)}$ obtained as the restriction of the universal subbundle on $Gr(n; 2n+1)$. Then the following proposition follows from Proposition 7.1 and the fact that $OGr(n; 2n+1)$ is a T-fixed subvariety of $Gr(n; 2n+1)$.

Proposition 7.2. For each B_n representation $L \subset \Bbbk^{2n+1}$ of a delta-matroid D, we have a T-equivariant map

$$X_{B_n} \to \overline{T \cdot [L]} \hookrightarrow OGr(n; 2n+1)$$

such that the identity of T is sent to $[L]$. For each $w \in \mathfrak{S}^B_n$, the pullback of $\mathcal{I}_{\text{univ}}$ localizes to $\sum_{i \in B_w} T_i$ at the T-fixed point of X_{B_n} corresponding to w.

Note our continued use of the convention that $T_{\bar{i}} = T_i^{-1}$ for $i \in [n]$.

Definition 7.3. Let L be a B_n representation of a delta-matroid D. Then the **isotropic tautological bundle** $\mathcal{I}_L$ on X_{B_n} is the pullback of $\mathcal{I}_{\text{univ}}$ under the map $X_{B_n} \to OGr(n; 2n+1)$ in Proposition 7.2.

Let $\mathcal{O}^{\oplus 2n+1}_{X_{B_n}}$ be the rank $2n+1$ trivial bundle with a T-equivariant structure given by the action of T on $\Bbbk^{2n+1}$ in (7.1). Note that $\mathcal{I}_L$ is the unique T-equivariant subbundle of $\mathcal{O}^{\oplus 2n+1}_{X_{B_n}}$ whose fiber at the identity of $T \subset X_{B_n}$ is the isotropic subspace L. In particular, its dual $\mathcal{I}_L^\vee$ is globally generated, and $\mathcal{I}_L$ is an anti-nef vector bundle. The equivariant K-class of $\mathcal{I}_L$ depends only on the delta-matroid D. Moreover, we show that this K-class is well-defined for any delta-matroid, not necessarily representable.

Proposition 7.4. For any delta-matroid D on $[n, \bar{n}]$, there is a class $[\mathcal{I}_{\mathrm{D}}] \in K_T(X_{B_n})$ defined by

$$[\mathcal{I}_{\mathrm{D}}]_w = \sum_{i \in B_w} T_i.$$

We define the **isotropic tautological class** $[\mathcal{I}_{\mathrm{D}}]$ of D by the above formula. Proposition 7.2 implies that $[\mathcal{I}_{\mathrm{D}}] = [\mathcal{I}_L]$ if L is a B_n representation of D.

Proof. We need to check that the above formula satisfies the compatibility condition in Theorem 3.1. Let $w \in \mathfrak{S}^B_n$, and set $w' = w\tau_{i,i+1}$. Then the cones corresponding to w and w' share a hyperplane whose normal vector is $\mathbf{e}_{w(i)} - \mathbf{e}_{w(i+1)}$. As the normal fan of $\widehat{P(\mathrm{D})}$ coarsens Σ_{B_n}, the w-minimal and w'-minimal vertices of $\widehat{P(\mathrm{D})}$ either coincide or differ by an edge parallel to $\mathbf{e}_{w(i)} - \mathbf{e}_{w(i+1)}$. This implies that $[\mathcal{I}_{\mathrm{D}}]_w - [\mathcal{I}_{\mathrm{D}}]_{w'} = \pm(T_{w(i)} - T_{w(i+1)})$ is divisible by $1 - T_{w(i)}T^{-1}_{w(i+1)}$.

Now set $w' = w\tau_n$. Then the cones corresponding to w and w' share a hyperplane whose normal vector is $\mathbf{e}_{w(n)}$. Again, that the normal fan of $\widehat{P(\mathrm{D})}$ coarsens Σ_{B_n} implies that $[\mathcal{I}_{\mathrm{D}}]_w - [\mathcal{I}_{\mathrm{D}}]_{w'} = \pm(1 - T_{w(n)})$ is divisible by $1 - T_{w(n)}$. □

Remark 7.5. We could also consider the quotient bundles $\mathcal{O}^{\oplus 2n+1}_{X_{B_n}}/\mathcal{I}_L$. However, one can verify that $[\mathcal{I}_{\mathrm{D}}] + [\mathcal{I}_{\mathrm{D}}]^\vee = [\mathcal{O}^{\oplus 2n}]$, and so $c([\mathcal{I}_{\mathrm{D}}]^\vee) = c(\mathcal{O}^{\oplus 2n+1}_{X_{B_n}}/\mathcal{I}_L)$. Therefore, studying the quotient bundle does not give any new elements of $A^\bullet(X_{B_n})$.

7.3. **Construction of enveloping tautological bundles.** From each realization $L \subset \Bbbk^{2n}$ of an enveloping matroid M of a delta-matroid D, we construct the enveloping tautological bundles $\mathcal{S}_L^E$ and $\mathcal{Q}_L^E$. As before, let $\pi_i \colon X_{B_n} \to \mathbb{P}^1$ denote the composition $X_{B_n} \to (\mathbb{P}^1)^n \to \mathbb{P}^1$, where the latter map is the projection onto the ith factor. Let us treat $\mathbb{P}^1$ as the toric variety of the fan in $\mathbb{R}$ consisting of the positive ray, negative ray, and the origin. $\mathbb{P}^1$ has two torus-fixed divisors ∞ and o that correspond respectively to the negative ray and the positive ray, and correspond respectively to the intervals $[0,1]$ and $[-1,0]$ under the standard correspondence between polytopes and base-point-free divisors on toric varieties [CLS11, Chapter 6]. Let $\mathcal{O}(1_\infty)$ and $\mathcal{O}(1_o)$ be the respective toric line bundles isomorphic to $\mathcal{O}_{\mathbb{P}^1}(1)$, and define

$$\mathcal{M} = \bigoplus_{i \in [n]} \pi_i^* \mathcal{O}(1_\infty) \oplus \pi_i^* \mathcal{O}(1_o).$$

We now show the existence of vector bundles $\mathcal{S}_L^E$ and $\mathcal{Q}_L^E$ on X_{B_n} that fit into a short exact sequence of T-equivariant vector bundles

$$0 \to \mathcal{S}_L^E \to \mathcal{M} \to \mathcal{Q}_L^E \to 0,$$

which is characterized by the property that the fiber over of the identity point of T is $0 \to L \to \Bbbk^{2n} \to \Bbbk^{2n}/L \to 0$. We prepare with a combinatorial lemma. Recall that $\square$ denotes the cube $[0,1]^n$, and the standard basis of $\mathbb{R}^{2n}$ is denoted $\mathbf{u}_1, \ldots, \mathbf{u}_n, \mathbf{u}_{\bar{1}}, \ldots, \mathbf{u}_{\bar{n}}$.

Lemma 7.6. Let M be an enveloping matroid of a delta-matroid D. Then

$$\operatorname{env}(IP(\mathrm{M})) = P(\mathrm{D}) + \square - \mathbf{e}_{[n]}.$$

Proof. First we note that $\operatorname{env}(IP(\mathrm{M}))$ is contained in $P(\mathrm{D}) + \square - \mathbf{e}_{[n]}$. Every vertex of $\operatorname{env}(IP(\mathrm{M}))$ can be written as $\frac{1}{2}\operatorname{env}(\mathbf{u}_B) + \frac{1}{2}\operatorname{env}(-\mathbf{u}_S)$ for some basis B of M and $S \subset B$. Then $\frac{1}{2}\operatorname{env}(\mathbf{u}_B) \in P(\mathrm{D}) - (\frac{1}{2}, \ldots, \frac{1}{2})$ and $\frac{1}{2}\operatorname{env}(-\mathbf{u}_S) \in \square - (\frac{1}{2}, \ldots, \frac{1}{2})$.

Now it suffices to show that every vertex of $P(\mathrm{D}) + \square - \mathbf{e}_{[n]}$ is contained in $\operatorname{env}(IP(\mathrm{M}))$. Let v be a vector in the interior of C_w. Then

$$\begin{aligned}
\operatorname{face}_v(P(\mathrm{D}) + \square - \mathbf{e}_{[n]}) &= \operatorname{face}_v(P(\mathrm{D}) - \tfrac{1}{2}\mathbf{e}_{[n]}) + \operatorname{face}_v(\square - \tfrac{1}{2}\mathbf{e}_{[n]}) \\
&= \tfrac{1}{2}\mathbf{e}_{B_w} + \tfrac{1}{2}\mathbf{e}_{w([\bar{n}])} \\
&= \tfrac{1}{2}\mathbf{e}_{B_w} - \tfrac{1}{2}\mathbf{e}_{w([n])}.
\end{aligned}$$

Because the normal fan of $P(\mathrm{D}) + \square - \mathbf{e}_{[n]}$ is a coarsening of Σ_{B_n}, every vertex is of the form $\frac{1}{2}\mathbf{e}_{B_w} - \frac{1}{2}\mathbf{e}_{w([n])}$ for some $w \in \mathfrak{S}_n^B$. We see that this is equal to $\operatorname{env}(\mathbf{u}_{B_w} - \mathbf{u}_{B_w \cap w([n])})$. Because $B_w \cap w([n]) \subset B_w$, this is contained in $IP(\mathrm{M})$. $\square$

We first construct the dual of the vector bundle $\mathcal{Q}_L^E$. Let $L^\perp$ be the dual space $(\Bbbk^{2n}/L)^\vee$, considered as a subspace of $\Bbbk^{2n}$ under the isomorphism $(\Bbbk^{2n})^\vee \simeq \Bbbk^{2n}$. It represents the dual matroid of the matroid represented by L. Let the torus T act on $\Bbbk^{4n} = \Bbbk^{2n} \times \Bbbk^{2n}$ by the usual action $(t_1x_1, \ldots, t_nx_n, t_1^{-1}x_{\bar{1}}, \ldots, t_n^{-1}x_{\bar{n}})$ on the first $\Bbbk^{2n}$ factor and trivially on the second $\Bbbk^{2n}$ factor. We let T act on $Gr(n; 4n)$ accordingly.

Proposition 7.7. For a representation L of an enveloping matroid M of a delta-matroid D, let $E_L \subset \Bbbk^{4n}$ be the image of $L^\perp$ under the diagonal embedding $\Bbbk^{2n} \hookrightarrow \Bbbk^{4n}$. Then there is a composition of T-equivariant maps

$$\varphi_L : X_{B_n} \to \overline{T \cdot [E_L]} \hookrightarrow Gr(n; 4n).$$

We define the **enveloping tautological quotient bundle** $\mathcal{Q}_L^E$ to be the *dual* of the pullback of the universal subbundle on $Gr(n; 4n)$ via the map φ_L.

Proof. Let $\widetilde{T}$ be the $2n$-dimensional torus $\mathbb{G}_m^{2n}$ with the action on $Gr(n; 4n)$ induced by

$$(t_1, \ldots, t_{2n}) \cdot (x_1, \ldots, x_{4n}) = (t_1 x_1, \ldots, t_{2n} x_{2n}, x_{2n+1}, \ldots, x_{4n}).$$

By [EHL, Proposition 3.16], the moment polytope of $\overline{\widetilde{T} \cdot [E_L]}$ is $IP(\mathrm{M}^\perp)$. By Proposition 6.1(2), the moment polytope of $\overline{T \cdot [E_L]}$ is $\operatorname{env}(IP(\mathrm{M}^\perp)) = P(\mathrm{D}^\perp) + \square - \mathbf{e}_{[n]}$. Note that the normal fan of $P(\mathrm{D}^\perp) + \square - \mathbf{e}_{[n]}$ coarsens Σ_{B_n}, so we conclude by Proposition 7.1. □

By construction, we have a surjection $\mathcal{O}_{X_{B_n}}^{\oplus 4n} \to \mathcal{Q}_L^E$. There is also a surjection $\mathcal{O}_{X_{B_n}}^{\oplus 4n} \to \mathcal{M}$, given by taking the direct sum over all $i = 1, \ldots, n$ of the surjections

$$\mathcal{O}_{X_{B_n}}^{\oplus 4} \simeq H^0(\mathbb{P}^1, \mathcal{O}(1_\infty) \oplus \mathcal{O}(1_o)) \otimes \mathcal{O}_{X_{B_n}} \to \pi_i^* \mathcal{O}(1_\infty) \oplus \pi_i^* \mathcal{O}(1_o),$$

whose kernel is $\pi_i^*(-1_\infty) \oplus \pi_i^*(-1_o)$.

Proposition 7.8. The composition

$$\bigoplus_{i \in [n]} \pi_i^*(-1_\infty) \oplus \pi_i^*(-1_o) \to \mathcal{O}_{X_{B_n}}^{\oplus 4n} \to \mathcal{Q}_L^E$$

is zero, so there is a map $\mathcal{M} \to \mathcal{Q}_L^E$.

We define the **enveloping subbundle** $\mathcal{S}_L^E$ to be the kernel of the map $\mathcal{M} \to \mathcal{Q}_L^E$.

Proof. It suffices to check this on the dense open torus $T \subset X_{B_n}$. By considering each factor of $T = \mathbb{G}_\mathrm{m}^n$ separately, the computation reduces to the case $n = 1$. Over a point $t \in \mathbb{G}_\mathrm{m}$, the fiber of $\pi_i^*(-1_\infty) \oplus \pi_i^*(-1_o) \subseteq \mathcal{O}_{\mathbb{P}^1}^{\oplus 4}$ is the subspace $\{(ta, t^{-1}b, a, b) \colon (a, b) \in k^2\} \subseteq k^4$. The form of E_L then implies the claim. □

We now compute the T-equivariant K-classes of $\mathcal{S}_L^E$ and $\mathcal{Q}_L^E$.

Proposition 7.9. The equivariant K-classes of $\mathcal{S}_L^E$ and $\mathcal{Q}_L^E$ are given by

$$[\mathcal{S}_L^E]_w = n - |\overline{B_w^{\max}} \cap w([n])| + \sum_{i \in w([n]), i \notin B_w^{\max}} T_i, \text{ and } [\mathcal{Q}_L^E]_w = |\overline{B_w^{\max}} \cap w([n])| + \sum_{i \in B_w^{\max} \cap w([n])} T_i.$$

Proof. Let v be a vector in the interior of C_w. We have noted that $\overline{B_w^{\max}}$ of D is equal to the w-minimal feasible set of $\mathrm{D}^\perp$. Then, as in the proof of Lemma 7.6, we have that

$$\operatorname{face}_v(P(\mathrm{D}^\perp) + \square - \mathbf{e}_{[n]}) = \tfrac{1}{2}\mathbf{e}_{B_w(\mathrm{D}^\perp)} - \tfrac{1}{2}\mathbf{e}_{w([n])} = \tfrac{1}{2}\mathbf{e}_{\overline{B_w^{\max}}} - \tfrac{1}{2}\mathbf{e}_{w([n])}.$$

In order to compute the localization of the pullback of $\mathcal{S}_{\mathrm{univ}}$, we find a preimage of $\operatorname{face}_v(P(\mathrm{D}^\perp) + \square - \mathbf{e}_{[n]})$ in the polytope of the matroid represented by E_L. A preimage in $IP(\mathrm{M}^\perp)$ of this vertex is $\mathbf{u}_{\overline{B_w^{\max}}} - \mathbf{u}_{\overline{B_w^{\max}} \cap w([n])}$. A preimage of this in the matroid polytope of the matroid represented by E_L

extends the independent set $\overline{B_w^{\max}} \setminus \overline{B_w^{\max}} \cap w([n])$ of $\mathrm{M}^\perp$ to a basis without adding any elements in $[2n]$. Proposition 7.1 then implies that the localization of the pullback of $\mathcal{S}_{\text{univ}}$ at the fixed point of X_{B_n} corresponding to w is

$$|\overline{B_w^{\max}} \cap w([n])| + \sum_{i \in \overline{B_w^{\max}} \setminus \overline{B_w^{\max}} \cap w([n])} T_i = |\overline{B_w^{\max}} \cap w([n])| + \sum_{i \in \overline{B_w^{\max} \cap w([n])}} T_i.$$

Because $\mathcal{Q}_L^E$ is the dual of the pullback of $\mathcal{S}_{\text{univ}}$, this gives the result for $\mathcal{Q}_L^E$. We note that $[\mathcal{M}]_w = n + \sum_{i \in w([n])} T_i$. As $[\mathcal{S}_L^E] = [\mathcal{M}] - [\mathcal{Q}_L^E]$, the result for $[\mathcal{S}_L^E]$ follows. □

In particular, the equivariant K-classes of $[\mathcal{S}_L^E]$ and $[\mathcal{Q}_L^E]$ depend only on the delta-matroid associated to L. For arbitrary delta-matroid D, the proof of Proposition 7.4 immediately adapts to show that we may define **enveloping tautological classes** $[\mathcal{S}_{\mathrm{D}}^E]$ and $[\mathcal{Q}_{\mathrm{D}}^E]$ in $K_T(X_{B_n})$ by the formulas in Proposition 7.9. Note that the enveloping tautological classes $[\mathcal{S}_{\mathrm{D}}^E]^\vee$ and $[\mathcal{Q}_{\mathrm{D}}^E]^\vee$ have "nice Chern roots" in the sense discussed above Proposition 3.8.

Remark 7.10. Arguing analogously to [BEST, Proposition 5.6], one can show that any fixed polynomial in the tautological classes of delta-matroids or their Chern classes is a valuative invariant of delta-matroids in the sense of [ESS21].

7.4. **Intersection computations.** We now compute several intersection numbers arising from the Chern and Segre classes of isotropic and enveloping tautological classes. We first do the computations with enveloping tautological classes, which are easier to work with because they are closely related to the exceptional isomorphisms ϕ^B and ζ^B introduced in Section 3. We then relate an intersection number of the Chern classes of the isotropic tautological classes to one involving enveloping tautological classes.

We begin by realizing both the interlace polynomial and the U-polynomial as intersection numbers of the enveloping tautological classes. Because the classes $[\mathcal{S}_{\mathrm{D}}^E]$ do not have any positivity properties, this does not give log-concavity properties for the interlace polynomial. But these results will form the basis for later intersection theory computations that prove Theorem B. In [EHL, Theorem 8.1], the analogous computation on X_{St_n} yields the rank-generating function of a matroid.

Theorem 7.11. We have that $\int_{X_{B_n}} c([\mathcal{S}_{\mathrm{D}}^E], u) \cdot c([\mathcal{Q}_{\mathrm{D}}^E], v) = v^n \operatorname{Int}_{\mathrm{D}}(u/v)$.

Proof. To compute $\int_{X_{B_n}} c([\mathcal{S}_{\mathrm{D}}^E], u) \cdot c([\mathcal{Q}_{\mathrm{D}}^E], v)$, we look at the degree n part of $c^T([\mathcal{S}_{\mathrm{D}}^E], u) \cdot c^T([\mathcal{Q}_{\mathrm{D}}^E], v)$. Let $S \in \mathsf{AdS}_n$, and consider the cone τ_S whose rays are $\{\mathbf{e}_i : i \in S\}$. Then τ_S is a maximal cone in the fan of $(\mathbb{P}^1)^n$. The linear function defined by $\mathbf{e}_S$ attains its maximum on a face F of $P(\mathrm{D})$, and every function in the interior of τ_S attains its maximum on a face of F because every cone of Σ_{B_n} which is contained in τ_S contains $\mathbf{e}_S$. Note any point x of F minimizes the distance to $\mathbf{e}_S$ from $P(\mathrm{D})$.

Note that $C_w \in \tau_S$ if and only if $S = w([n])$. For each $w \in \mathfrak{S}_n^B$ with $S = w([n])$, we have that

$$c^T([\mathcal{S}_{\mathrm{D}}^E])_w = \prod_{i \in S, i \notin B_w^{\max}} (1 + t_i), \quad \text{and } c^T([\mathcal{Q}_{\mathrm{D}}^E])_w = \prod_{i \in S \cap B_w^{\max}} (1 + t_i).$$

We see that the degree n part of $c^T([\mathcal{S}_{\mathrm{D}}^E], u)_w \cdot c^T([\mathcal{Q}_{\mathrm{D}}^E], v)_w$ is

$$(-1)^{|S \cap [\bar{n}]|} u^{d_{\mathrm{D}}(S)} v^{n - d_{\mathrm{D}}(S)} t_1 \cdots t_n.$$

Note that, for each $S \in \mathsf{AdS}_n$, the piecewise polynomial function that is $(-1)^{|S\cap[\bar{n}]|}t_1\cdots t_n$ on τ_S and vanishes otherwise is $c_n^T(\bigoplus_{i\in[n]}\pi_i^*\mathcal{O}(1))$, where we give $\mathcal{O}(1)$ on the ith copy of $\mathbb{P}^1$ the $\mathcal{O}(1_\infty)$ linearization if $i \in S$, and give it the $\mathcal{O}(1_o)$ linearization if $\bar{i} \in S$. Proposition 5.2 gives

$$\int_{X_{B_n}} c([\mathcal{S}_{\mathrm{D}}^E], u)\cdot c([\mathcal{Q}_{\mathrm{D}}^E], v) = \sum_{S\in\mathsf{AdS}_n} u^{d_{\mathrm{D}}(S)}v^{n-d_{\mathrm{D}}(S)}\int_{(\mathbb{P}^1)^n} c_n(\oplus\pi_i^*\mathcal{O}(1)) = v^n\,\mathrm{Int}_{\mathrm{D}}(u/v). \quad \square$$

We prepare to do more computations by studying how enveloping tautological classes restrict to smaller permutohedral varieties. The description of the fan of Σ_{B_n} implies that the closure of each coordinate $\mathbb{G}_{\mathrm{m}}^{n-1}\subset T$ in X_{B_n} can be identified with $X_{B_{n-1}}$. The inclusion is $\mathbb{G}_{\mathrm{m}}^{n-1}$-equivariant, so for each $i\in n$, we have a map $K_T(X_{B_n})\to K_{\mathbb{G}_{\mathrm{m}}^{n-1}}(X_{B_{n-1}})$ given by the composition of the forgetful map $K_T(X_{B_n})\to K_{\mathbb{G}_{\mathrm{m}}^{n-1}}(X_{B_n})$ and the restriction map. Recall that for a delta-matroid D and $I\subseteq[n]$, $\mathrm{D}(I)$ is the projection of D away from I.

Proposition 7.12. The images of $[\mathcal{S}_{\mathrm{D}}^E]$ and $[\mathcal{Q}_{\mathrm{D}}^E]$ under the map $K_T(X_{B_n})\to K_{\mathbb{G}_{\mathrm{m}}^{n-1}}(X_{B_{n-1}})$ are $1+[\mathcal{S}_{\mathrm{D}(i)}^E]$ and $1+[\mathcal{Q}_{\mathrm{D}(i)}^E]$ respectively.

Proof. Under the embedding $X_{B_{n-1}}\hookrightarrow X_{B_n}$, each $\mathbb{G}_{\mathrm{m}}^{n-1}$-fixed point of $X_{B_{n-1}}$ is the identity of the torus embedded into a T-fixed curve in X_{B_n} on which $\mathbb{G}_{\mathrm{m}}^{n-1}$ acts trivially. We may compute the $\mathbb{G}_{\mathrm{m}}^{n-1}$-equivariant localization at this fixed point by computing the T-equivariant localization at any T-fixed point of this curve, and then applying the forgetful map $K_T(\mathrm{pt})\to K_{\mathbb{G}_{\mathrm{m}}^{n-1}}(\mathrm{pt})$. Then the result follows from the definition of enveloping tautological classes. $\square$

Proposition 7.13. We have that

$$U_{\mathrm{D}}(u,v) = \int_{X_{B_n}} c(\boxplus\mathcal{O}(1),u)\cdot c([\mathcal{S}_{\mathrm{D}}^E],v)\cdot c([\mathcal{Q}_{\mathrm{D}}^E]).$$

Proof. The zero-locus of a general element of the complete linear system of $\pi_i^*\mathcal{O}(1)$ is $\overline{\{t\in T\colon t_i=\lambda\}}$ for some $\lambda\in\Bbbk^*$. As these divisor are all $\mathbb{G}_m$-translates of the closure of $\mathbb{G}_{\mathrm{m}}^{[n]\setminus i}$, the class $[X_{B_{n-1}}]\in A^1(X_{B_n})$ represents $c_1(\pi_i^*\mathcal{O}(1))$. Letting i vary, we see that $c(\boxplus\mathcal{O}(1))$ is the sum of the Chow classes of the closures of the coordinate subtori of T. The closure of each coordinate subtorus of T can be identified with a smaller X_{B_k}. By the projection formula and Proposition 7.12, we see that

$$\begin{aligned}\int_{X_{B_n}} c(\boxplus\mathcal{O}(1),u)\cdot c([\mathcal{S}_{\mathrm{D}}^E],v)\cdot c([\mathcal{Q}_{\mathrm{D}}^E],1) &= \sum_{I\subseteq[n]} u^{|I|}\int_{X_{B_{n-|I|}}} c([\mathcal{S}_{\mathrm{D}}^E],v)|_{X_{B_{n-|I|}}}\cdot c([\mathcal{Q}_{\mathrm{D}}^E],1)|_{X_{B_{n-|I|}}}\\ &= \sum_{I\subseteq[n]} u^{|I|}\int_{X_{B_{n-|I|}}} c([\mathcal{S}_{\mathrm{D}(I)}^E],v)\cdot c([\mathcal{Q}_{\mathrm{D}(I)}^E],1).\end{aligned}$$

The the result follows from Theorem 7.11 and Lemma 5.4. $\square$

Recall that γ is the first Chern class of the line bundle corresponding to the cross polytope $\Diamond$ and s denotes the Segre class. We now do the computation which underlies the proof of Theorem B(1.2).

Theorem 7.14. We have that

$$\int_{X_{B_n}} s([\mathcal{Q}_{\mathrm{D}}^E]^\vee,z)\cdot c([\mathcal{Q}_{\mathrm{D}}^E],w)\cdot\frac{1}{1-y\gamma}\cdot c(\boxplus\mathcal{O}(1),x) = (y+w)^n U_{\mathrm{D}}\left(\frac{2z+x}{y+w},\frac{y-z}{y+w}\right).$$

The key tools in the proof are the two exceptional isomorphisms and the Hirzebruch–Riemann–Roch-type formulas that they satisfy, which are a manifestation of Serre duality. This allows us to show the equality of certain intersection numbers, and leverage Theorem 7.11 to compute more intersection numbers.

Proof. We prove the theorem in three steps.
Step 1: we have that

$$\int_{X_{B_n}} s([\mathcal{Q}_{\mathrm{D}}^E]^\vee, z) \cdot c([\mathcal{Q}_{\mathrm{D}}^E]) = U_{\mathrm{D}}(2z, -z).$$

Because $[\mathcal{S}_{\mathrm{D}}^E] + [\mathcal{Q}_{\mathrm{D}}^E] = [\mathcal{M}] = [\boxplus\mathcal{O}(1)^{\oplus 2}]$, we have that $c([\mathcal{S}_{\mathrm{D}}^E], z) \cdot c([\mathcal{Q}_{\mathrm{D}}^E], z) = c(\boxplus\mathcal{O}(1)^{\oplus 2}, z) = c(\boxplus\mathcal{O}(1), 2z)$. as $s(\boxplus\mathcal{O}(1), u) = c(\boxplus\mathcal{O}(1)^\vee, u)$. So

$$s([\mathcal{Q}_{\mathrm{D}}^E]^\vee, z) = c([\mathcal{S}_{\mathrm{D}}^E], -z) \cdot c(\boxplus\mathcal{O}(1), 2z).$$

Then, using Proposition 7.12, we see that

$$\begin{aligned}\int_{X_{B_n}} s([\mathcal{Q}_{\mathrm{D}}^E]^\vee, z) \cdot c([\mathcal{Q}_{\mathrm{D}}^E], w) &= \int_{X_{B_n}} c([\mathcal{S}_{\mathrm{D}}^E], -z) \cdot c(\boxplus\mathcal{O}(1), 2z) \cdot c([\mathcal{Q}_{\mathrm{D}}^E], w) \\ &= \sum_{I \subseteq [n]} (2z)^{|I|} \int_{X_{B_{E\setminus I}}} c([\mathcal{S}_{\mathrm{D}(I)}^E], -z) \cdot c([\mathcal{Q}_{\mathrm{D}(I)}^E], w) \\ &= \sum_{I \subseteq [n]} (2z)^{|I|} w^{n-|I|} \operatorname{Int}_{\mathrm{D}(I)}(-z/w).\end{aligned}$$

Setting $w = 1$ and using Lemma 5.4 gives the result.
Step 2: we have that

$$\int_{X_{B_n}} s([\mathcal{Q}_{\mathrm{D}}^E]^\vee, z) \cdot c([\mathcal{Q}_{\mathrm{D}}^E], w) \cdot \frac{1}{1-\gamma} = (1+w)^n U_{\mathrm{D}}\left(\frac{2z}{1+w}, \frac{1-z}{1+w}\right).$$

Let $[\square]$ be the class of the line bundle corresponding to the cube $\square = [0,1]^n$. From Lemma 3.5 and [EHL, Corollary 6.5(1)], we have that both $\phi^B([\square]) = c(\boxplus\mathcal{O}(1))$ and $\zeta^B([\square]) = c(\boxplus\mathcal{O}(1))$. Applying Proposition 3.8, Proposition 3.7, and Theorem C, we get that

$$\begin{aligned}&\chi\left(\Big(\sum_{j\geq 0} \operatorname{Sym}^j [\mathcal{Q}_{\mathrm{D}}^E]^\vee z\Big)\Big(\sum_{i \geq 0} \wedge^i [\mathcal{Q}_{\mathrm{D}}^E]^\vee w\Big)[\square]\right) \\ &= \int_{X_{B_n}} \frac{1}{(1-z)^n} \cdot s\left([\mathcal{Q}_{\mathrm{D}}^E]^\vee, \frac{z}{z-1}\right) \cdot (w+1)^n \cdot c\left([\mathcal{Q}_{\mathrm{D}}^E]^\vee, \frac{w}{1+w}\right) \cdot \frac{1}{1-\gamma} \\ &= \int_{X_{B_n}} \frac{1}{(1-z)^n} s\left([\mathcal{Q}_{\mathrm{D}}^E], \frac{1}{1-z}\right) \cdot (w+1)^n \cdot c\left([\mathcal{Q}_{\mathrm{D}}^E], \frac{1}{1+w}\right) \cdot c(\boxplus\mathcal{O}(1), 2).\end{aligned}$$

Equating the two right-hand sides, canceling, and replacing w by $-\frac{w}{1+w}$ and z by $\frac{z}{z-1}$, we obtain

$$\int_{X_{B_n}} s([\mathcal{Q}_{\mathrm{D}}^E]^\vee, z) \cdot c([\mathcal{Q}_{\mathrm{D}}^E], w) \cdot \frac{1}{1-\gamma} = \int_{X_{B_n}} s([\mathcal{Q}_{\mathrm{D}}^E], 1-z) \cdot c([\mathcal{Q}_{\mathrm{D}}^E], w+1) \cdot c(\boxplus\mathcal{O}(1), 2).$$

Substituting in the result of Step 1 after homogenizing, we have that

$$\int_{X_{B_n}} s([\mathcal{Q}_{\mathrm{D}}^E]^\vee, z-1) \cdot c([\mathcal{Q}_{\mathrm{D}}^E], 1+w) = (1+w)^n U_{\mathrm{D}}\left(\frac{2(z-1)}{1+w}, \frac{1-z}{1+w}\right).$$

Therefore, using Lemma 5.4, we have that

$$\begin{aligned}\int_{X_{B_n}} s([\mathcal{Q}_{\mathrm{D}}^E]^\vee, z)\cdot c([\mathcal{Q}_{\mathrm{D}}^E], w)\cdot \frac{1}{1-\gamma} &= \sum_{I\subseteq[n]} 2^{|I|}(1+w)^{n-|I|}U_{\mathrm{D}(I)}\left(\frac{2(z-1)}{1+w}, \frac{1-z}{1+w}\right)\\ &= (1+w)^n\sum_{I\subseteq E}\left(\frac{2}{1+w}\right)^{|I|}U_{\mathrm{D}(I)}\left(\frac{2(z-1)}{1+w}, \frac{1-z}{1+w}\right)\\ &= (1+w)^n U_{\mathrm{D}}\left(\frac{2z}{1+w}, \frac{1-z}{1+w}\right)\end{aligned}$$

Step 3: we now prove the result. We compute:

$$\begin{aligned}&\int_{X_{B_n}} s([\mathcal{Q}_{\mathrm{D}}^E]^\vee, z)\cdot c([\mathcal{Q}_{\mathrm{D}}^E], w)\cdot \frac{1}{1-y\gamma}\cdot c(\boxplus\mathcal{O}(1), x)\\ &= \sum_{I\subseteq[n]} x^{|I|}\int_{X_{B_n}} s([\mathcal{Q}_{\mathrm{D}(I)}^E]^\vee, z)\cdot c([\mathcal{Q}_{\mathrm{D}(I)}^E], w)\cdot\frac{1}{1-y\gamma}\\ &= \sum_{I\subseteq[n]}(y+w)^{n-|I|}x^{|I|}U_{\mathrm{D}(I)}\left(\frac{2z}{y+w}, \frac{y-z}{y+w}\right)\\ &= (y+w)^n U_{\mathrm{D}}\left(\frac{2z+x}{y+w}, \frac{y-z}{y+w}\right).\end{aligned}$$

□

Theorem 7.15. Let D be a delta-matroid. We have that

$$\int_{X_{B_n}} c([\mathcal{I}_{\mathrm{D}}]^\vee, q)\cdot\frac{1}{1-y\gamma}\cdot c(\boxplus\mathcal{O}(1), x) = (y+q)^n U_{\mathrm{D}}\left(\frac{x}{y+q}, \frac{y-q}{y+q}\right).$$

We prove the above theorem by relating it to Theorem 7.14. We first recall the equivariant descriptions of $c^T([\mathcal{I}_{\mathrm{D}}]^\vee)$. Recall that if $i\in[n]$, then $t_{\bar{i}} := -t_i$. On a fixed point of X_{B_n} corresponding to $w\in\mathfrak{S}_n^B$, we have that

$$c^T([\mathcal{I}_{\mathrm{D}}]^\vee, q)_w = \prod_{i\in B_w}(1-t_i q) = \prod_{i\in\overline{B_w}}(1+t_i q).$$

Proof. We claim that

$$s([\mathcal{Q}_{\mathrm{D}^\perp}^E]^\vee, q)\cdot c([\mathcal{Q}_{\mathrm{D}^\perp}^E], q)\cdot c(\boxplus\mathcal{O}(1), -2q) = c([\mathcal{I}_{\mathrm{D}}]^\vee, q).$$

Note that $c(\boxplus\mathcal{O}(1), x)\cdot c(\boxplus\mathcal{O}(1), -2q) = c(\boxplus\mathcal{O}(1), x-2q)$. Then, assuming the claim, Theorem 7.14 implies the result as $U_{\mathrm{D}}(u,v) = U_{\mathrm{D}^\perp}(u,v)$. Observe that

$$c^T([\mathcal{Q}_{\mathrm{D}^\perp}^E], q)_w = \prod_{i\in\overline{B_w}\cap w([n])}(1+t_i q), \text{ and } s^T([\mathcal{Q}_{\mathrm{D}^\perp}^E]^\vee, q)_w = c^T([\mathcal{Q}_{\mathrm{D}^\perp}^E], -q)_w^{-1} = \prod_{i\in B_w\cap\overline{w([n])}}\frac{1}{1+t_i q}.$$

On $\mathbb{P}^1$, the piecewise polynomial function which is t on the cone $\{x<0\}$ and $-t$ on the cone $\{x>0\}$ is a linearization of $\mathcal{O}(-2)$. Therefore, with this linearization we have that

$$c^T(\boxplus\mathcal{O}(1), -2q)_w = \prod_{i\in\overline{w([n])}}(1+t_i q).$$

Then the claim follows from multiplying the above expressions together. □

8. LOG-CONCAVITY

In this section, we prove Theorem B. First we recall some definitions. Let $f \in \mathbb{R}[x_1, \dots, x_n]$ be a homogeneous polynomial of degree d. If $f = \sum a_{\mathbf{m}} x^{\mathbf{m}}$, then the **normalization** of f, denoted $N(f)$, is the polynomial $\sum a_{\mathbf{m}} \frac{x^{\mathbf{m}}}{\mathbf{m}!}$, where $\mathbf{m}! = m_1! \cdots m_n!$ if $\mathbf{m} = (m_1, \dots, m_n)$. We call f the **denormalization** of $N(f)$. We say that f is **strictly Lorentzian** if the coefficient of every monomial of degree d is positive, and every quadratic form obtained by taking $d-2$ partial derivatives of f is nondegenerate with signature $(+, -, \dots, -)$. We say that f is **Lorentzian** if it is a coefficientwise limit of strictly Lorentzian polynomials. It follows from [BH20, Example 2.26] and [BH20, Theorem 2.10] that a denormalized Lorentzian polynomial has a log-concave unbroken array of coefficients. Then we have the following corollary of Theorem B.

Corollary 8.1. Let D be a delta-matroid which has an enveloping matroid. Then the coefficients of $U_{\mathrm{D}}(u, 0)$, $U_{\mathrm{D}}(u, -1)$, $(y+1)^n U_{\mathrm{D}}(0, \frac{y-1}{y+1}) = (y+1)^n \operatorname{Int}_{\mathrm{D}}(\frac{y-1}{y+1})$, and $U_{\mathrm{D}}(2u, -u)$ each form a nonnegative log-concave sequence with no internal zeros, and in particular form a unimodal sequence.

Proof. To obtain the first three formulas, we set $y = q = 1/2$, then $y = 0, q = 1$, and $x = 0, q = 1$ respectively in (1.1). To obtain the last formula, set $x = y = 0, w = 1$ in (1.2). The result then follows from [BLP, Lemma 4.8]. □

Remark 8.2. For the adjacency delta-matroid $\mathrm{D}(G)$ of a graph G (Example 6.4), [ABS04] conjectured that the coefficients of $\operatorname{Int}_{\mathrm{D}(G)}(v-1)$ form a unimodal sequence, which was disproved by [DP10]. Both works conjectured that $\operatorname{Int}_{\mathrm{D}(G)}(v)$ has unimodal coefficients. We note that $\operatorname{Int}_{\mathrm{D}}(v)$ may not have unimodal coefficients even when D is an even delta-matroid with a D_n-representation, like $\mathrm{D}(G)$. See Example 8.3 below. We do not know of an example where $\operatorname{Int}_{P(\mathrm{M})}(v) = T_{\mathrm{M}}(v+1, v+1)$ is not log concave.

Example 8.3. Let $\mathrm{U}^{\circ}_{r,n}$ be the even delta-matroid on $[n, \overline{n}]$ whose feasible sets are

$$\{S \cup ([\overline{n}] \setminus \overline{S}) : S \subseteq [n] \text{ with } |S| \leq r \text{ and } |S| \equiv r \mod 2\}.$$

That is, the vertices of the polytope $P(\mathrm{U}^{\circ}_{r,n})$ are obtained from $IP(\mathrm{U}_{r,n})$ by taking only the vertices corresponding to subsets with parity equal to r. Then $\mathrm{U}^{\circ}_{r,n}$ has a D_n-representation by the row-span of the $n \times 2n$ matrix

$$\left[\begin{array}{cc|cc} I_r & A & B & 0 \\ 0 & 0 & -A^t & I_{n-r} \end{array}\right]$$

where I_k is the $k \times k$ identity matrix, A is a general $r \times (n-r)$ matrix, and B is a general $r \times r$ skew-symmetric matrix. In particular, $\mathrm{U}^{\circ}_{r,n}$ has an enveloping matroid. By the formula Proposition 5.2, we compute that the coefficients of $(1, v, v^2, v^3, \dots)$ in $\operatorname{Int}_{\mathrm{U}^{\circ}_{m-3,2m}}(v)$ are

$$\left(\sum_{\substack{0 \leq i \leq m-3 \\ i \equiv m-3 \mod 2}} \binom{2m}{i}, \sum_{\substack{0 \leq i \leq m-3 \\ i \not\equiv m-3 \mod 2}} \binom{2m}{i} + \binom{2m}{m-2}, \binom{2m}{m-1}, \binom{2m}{m}, \dots \right)$$

For large m, this sequence is not unimodal. For instance, at $m = 10$ the sequence reads

$$(94184, 169766, 167960, 184756, \dots).$$

In particular, the interlace polynomial of an even delta-matroids with a D_n-representation need not have unimodal or log-concave coefficients.

Remark 8.4. The nonnegativity of the coefficients of $U_{\mathrm{D}}(2u, -u)$, which is part of the content of Corollary 8.1, can be proven directly using the recursive definition of the U-polynomial. It would be interesting to give a combinatorial interpretation of the coefficient of $U_{\mathrm{D}}(2u, -u)$. We conjecture that the coefficients of $U_{\mathrm{D}}(2u, -u)$ form a *flawless* sequence, i.e., $a_i \leq a_{n-i}$ for $i \leq \lfloor n/2 \rfloor$.

8.1. **Motivation.** We exhibit the general strategy for constructing log-concave sequences from vector bundles, first used in [BEST, Section 8] and later placed into a general framework in [EHL]. We do this in the special case of showing that the coefficients of $(y+1)^n \operatorname{Int}_{\mathrm{D}}(\frac{y-1}{y+1})$ are log-concave when D has an enveloping matroid.

Setting $x = 0$ and $q = 1$ in Theorem 7.15, we have the equality

$$\int_{X_{B_n}} c([\mathcal{I}_D]^\vee) \cdot \frac{1}{1-y\gamma} = (y+1)^n \operatorname{Int}_{\mathrm{D}}\left(\frac{y-1}{y+1}\right).$$

Suppose first we are in the special case that D has a B_n representation $L \subset \Bbbk^{2n+1}$. The first step will be rewriting this intersection to involve Segre classes rather than Chern classes. As $\mathcal{I}_L$ is a subbundle of $\mathcal{O}_{X_{B_n}}^{2n+1}$, by dualizing we obtain a short exact sequence

$$0 \to \mathcal{K}_L \to \mathcal{O}_{X_{B_n}}^{2n+1} \to \mathcal{I}_L^\vee \to 0$$

for some vector bundle $\mathcal{K}_L$. Then $c(\mathcal{I}_L^\vee) = s(\mathcal{K}_L)$, and so

$$\int_{X_{B_n}} c(\mathcal{I}_L^\vee)\,\frac{1}{1-\gamma y} = \int_{X_{B_n}} s(\mathcal{K}_L)\,\frac{1}{1-\gamma y} = \sum_k y^k \int_{X_{B_n}} s_{n-k}(\mathcal{K}_L)\gamma^k = \sum_k y^k \int_{\mathbb{P}(\mathcal{K}_L)} \delta^{2n-k}\gamma^k$$

where δ is the first Chern class of $\mathcal{O}(1)$ on $\mathbb{P}(\mathcal{K}_L)$. The Khovanskii-Teissier inequality implies the coefficient sequence is log-concave. To establish this log-concavity beyond the case that D is B_n representable, we note that we may rewrite the last equation as

$$\sum_{k=0}^{n} y^k \int_{\mathbb{P}(\mathcal{K}_L)} \delta^{2n-k}\gamma^k = \sum_{k=0}^{n} y^k \int_{X_{B_n}\times\mathbb{P}^{2n}} [\mathbb{P}(\mathcal{K}_L)]\delta^{2n-k}\gamma^k,$$

where $[\mathbb{P}(\mathcal{K}_L)] = \sum_{i=0}^n c_{n-i}(\mathcal{I}_L^\vee)\delta^i \in A^\bullet(X_{B_n} \times \mathbb{P}^{2n}) = A^\bullet(X_{B_n})[\delta]/(\delta^{2n+1})$ is the fundamental class of $\mathbb{P}(\mathcal{K}_L) \subset X_{B_n} \times \mathbb{P}^{2n}$. The formula for this class makes sense for any delta-matroid, and one can formally define $[\mathbb{P}(\mathcal{K}_{\mathrm{D}})] = \sum_{i=0}^n c_{n-i}([\mathcal{I}_{\mathrm{D}}^\vee])\delta^i \in A^\bullet(X_{B_n} \times \mathbb{P}^{2n})$. By Theorem 7.15, $\int_{X\times\mathbb{P}^{2n}} [\mathbb{P}(\mathcal{K}_{\mathrm{D}})]\delta^{2n-k}\gamma^k$ still computes the coefficients of $(y+1)^n \operatorname{Int}_{\mathrm{D}}(\frac{y-1}{y+1})$.

In order to deduce log-concavity, we need to know that the Chow class $[\mathbb{P}(\mathcal{K}_{\mathrm{D}})]$ has Hodge-theoretic properties resembling those of an irreducible subvariety. The framework of [EHL, Section 8.3] constructs classes which are associated to any *matroid* which have good Hodge-theoretic properties.[3] When D has an enveloping matroid M, we can use this to deduce that $[\mathbb{P}(\mathcal{K}_{\mathrm{D}})]$ has good Hodge-theoretic properties.

[3]For technical reasons we actually work with classes in $A^\bullet(X_{B_n} \times \mathbb{P}^{2n-1})$ instead of $A^\bullet(X_{B_n} \times \mathbb{P}^{2n})$ which more naturally extend to all rank n matroids, but the underlying idea is the same.

8.2. **Proof of log-concavity.** Before proving Theorem B, we prove a log-concavity statement for an arbitrary matroid of rank n on $[n, \bar{n}]$ (Theorem 8.8) by using the framework in [EHL, Section 8.3], which is based on [BEST, Section 9]. Afterwards, we relate this log-concavity statement to Theorem B. Using Proposition 7.1, we construct two types of vector bundles on X_{B_n} that are associated to a realization of a matroid of rank n on $[n, \bar{n}]$. First we give a definition (cf. Definition 2.3).

Definition 8.5. Let A be an abelian group. A function

$$\varphi\colon \{\text{matroids of rank } r \text{ on } [n]\} \to A$$

is **valuative** if it factors through the map $\mathrm{M} \mapsto \mathbf{1}(P_{\mathrm{M}})$. That is, for any matroids $\mathrm{M}_1, \dots, \mathrm{M}_k$ and integers $a_1, \dots, a_k$ such that $\sum a_i \mathbf{1}(P(\mathrm{M}_i)) = 0$, we have that $\sum a_i \varphi(\mathrm{M}_i) = 0$.

Let T act on $\Bbbk^{4n}$ by $(t_1 x_1, t_2 x_2, \dots, t_n x_n, t_1^{-1} x_{n+1}, \dots, t_n^{-1} x_{2n}, x_{2n+1}, \dots, x_{4n})$. Let $L \subset \Bbbk^{2n}$ be a linear space of dimension n. Let E_L be the image of $L^{\perp}$ under the diagonal embedding of $\Bbbk^{2n}$ into $\Bbbk^{4n}$ and consider the point $[E_L] \in Gr(n; 4n)$. The fan of the normalization of $\overline{T \cdot [E_L]}$ is the normal fan of $\mathrm{env}(IP(\mathrm{M}))$. Every edge of $\mathrm{env}(IP(\mathrm{M}))$ is parallel to $\mathbf{e}_i$ or $\mathbf{e}_i \pm \mathbf{e}_j$, so Σ_{B_n} is a coarsening of the normal fan of $\mathrm{env}(IP(\mathrm{M}))$. Therefore there is a toric morphism $X_{\Sigma_{B_n}} \to Gr(n; 4n)$. Set $\mathcal{S}_{\mathrm{univ}}$ and $\mathcal{Q}_{\mathrm{univ}}$ to be the universal subbundle and quotient bundle respectively on $Gr(n; 4n)$. Let $\tilde{\mathcal{K}}_L^E$ and $\tilde{\mathcal{Q}}_L^E$ be the duals of the pullbacks of $\mathcal{S}_{\mathrm{univ}}$ and $\mathcal{Q}_{\mathrm{univ}}$ respectively.

Lemma 8.6. For each $w \in \mathfrak{S}_n^B$, let I_w be any independent set of $\mathrm{M}^{\perp}$ such that any functional in the interior of C_w achieves its minimum on the corresponding vertex of $\mathrm{env}(IP(\mathrm{M}^{\perp}))$. Then

$$[\tilde{\mathcal{Q}}_L^E]_w = n - |I_w \cap w([n])| + \sum_{i \in I_w \cap w([n])} T_i, \text{ and } [\tilde{\mathcal{K}}_L^E]_w = n + |I_w \cap w([n])| + \sum_{i \notin w([n]) \cap I_w} T_i.$$

Note that the classes $[\tilde{\mathcal{Q}}_L^E]$ and $[\tilde{\mathcal{K}}_L^E]$ only depend on the matroid M that L represents. For any matroid M of rank n on $[n, \bar{n}]$, we define classes $[\tilde{\mathcal{Q}}_{\mathrm{M}}^E]$ and $[\tilde{\mathcal{K}}_{\mathrm{M}}^E]$ in $K_T(X_{B_n})$; the proof of Proposition 7.4 adapts to show that these are indeed well-defined. The proof of [BEST, Proposition 5.6] shows that any function that maps a matroid M of rank n on $[n, \bar{n}]$ to a fixed polynomial expression in the Chern classes of $[\tilde{\mathcal{Q}}_{\mathrm{M}}^E]$ and $[\tilde{\mathcal{K}}_{\mathrm{M}}^E]$ is a valuative invariant of matroids of rank n on $[n, \bar{n}]$.

We now construct analogues of isotropic tautological bundles. Consider a matroid M of rank n on $[n, \bar{n}]$ represented by $L \subset \Bbbk^{2n}$. Then L determines a geometric point of $Gr(n; 2n)$. We have a T-action on $Gr(n; 2n)$ given by

$$(t_1, \dots, t_n) \cdot (x_1, \dots, x_n, x_{\bar{1}}, \dots, x_{\bar{n}}) = (t_1 x_1, \dots, t_n x_n, t_1^{-1} x_{\bar{1}}, \dots, t_n^{-1} x_{\bar{n}}).$$

The fan of the normalization of $\overline{T \cdot [L]}$ is the toric variety with normal fan $\mathrm{env}(P(\mathrm{M}))$, which is a coarsening of Σ_{B_n}. This determines a morphism $X_{B_n} \to Gr(n; 2n)$; define $\tilde{\mathcal{K}}_L$ to be dual of the pullback of the universal quotient bundle $\mathcal{Q}_{\mathrm{univ}}$ under this map. Proposition 7.1 implies the following lemma.

Lemma 8.7. For $w \in \mathfrak{S}_n^B$, let B_w be a basis corresponding to any vertex in the preimage of the vertex of $\mathrm{env}(P(\mathrm{M}))$ that any functional in the interior of C_w achieves its minimum on. Then

$$[\tilde{\mathcal{K}}_L]_w = \sum_{i \in B_w} T_i.$$

Note that the above description of the equivariant K-class depends only on the matroid M. Define $[\tilde{\mathcal{K}}_{\mathrm{M}}] \in K_T(X_{B_n})$ by the above formula for any M; the proof of Proposition 7.4 adapts to show that these are indeed well-defined. The proof of [BEST, Proposition 5.6] shows that any function that maps a matroid M of rank n on $[n, \bar{n}]$ to a fixed polynomial expression in the Chern classes of $[\tilde{\mathcal{K}}_{\mathrm{M}}]$ is a valuative invariant of matroids of rank n on $[n, \bar{n}]$. Then the following result follows from the framework [EHL, Section 8.3], which establishes log-concavity properties for classes constructed in this way associated to loop-free and coloop-free matroids M.

Theorem 8.8. Let M be a loop-free and coloop-free matroid of rank n on $[n, \bar{n}]$. Then the polynomials

$$\int_{X_{B_n}} s([\tilde{\mathcal{Q}}_{\mathrm{M}}^E]^\vee, z) \cdot s([\tilde{\mathcal{K}}_{\mathrm{M}}^E], w) \cdot \frac{1}{1 - y\gamma} \cdot c(\boxplus \mathcal{O}(1), x) \text{ and } \int_{X_{B_n}} s([\tilde{\mathcal{K}}_{\mathrm{M}}], q) \cdot \frac{1}{1 - y\gamma} \cdot c(\boxplus \mathcal{O}(1), x)$$

are denormalized Lorentzian.

Proof of Theorem B. We first do (1.2). Consider the case when D is loop-free and coloop-free. By Lemma 6.7, the enveloping matroid M of D is loop-free and coloop-free. Then $[\tilde{\mathcal{Q}}_{\mathrm{M}}^E] = [\mathcal{Q}_{\mathrm{D}}^E]$, so $s([\tilde{\mathcal{Q}}_{\mathrm{M}}^E]^\vee, z) = s([\mathcal{Q}_{\mathrm{D}}^E]^\vee, z)$. Also, $s([\tilde{\mathcal{K}}_{\mathrm{M}}^E], w) = c([\tilde{\mathcal{Q}}_{\mathrm{M}}^E], w) = c([\mathcal{Q}_{\mathrm{D}}^E], w)$. We see that

$$\begin{aligned}
&\int_{X_{B_n}} s([\tilde{\mathcal{Q}}_{\mathrm{M}}^E]^\vee, z) \cdot s([\tilde{\mathcal{K}}_{\mathrm{M}}^E], w) \cdot \frac{1}{1 - y\gamma} \cdot c(\boxplus \mathcal{O}(1), x) \\
&= \int_{X_{B_n}} s([\mathcal{Q}_{\mathrm{D}}^E]^\vee, z) \cdot c([\mathcal{Q}_{\mathrm{D}}^E], w) \cdot \frac{1}{1 - y\gamma} \cdot c(\boxplus \mathcal{O}(1), x) = (y + w)^n U_{\mathrm{D}}\left(\frac{2z + x}{y + w}, \frac{y - z}{y + w}\right)
\end{aligned}$$

by Theorem 7.14. So when D is loop-free and coloop-free, Theorem 8.8 gives that the above polynomial is denormalized Lorentzian. In general, we can write $\mathrm{D} = \mathrm{D}' \times P(U_{0,k}) \times P(U_{\ell,\ell})$ for some k and ℓ, where D' is loop-free and coloop-free. Using the behavior of the U-polynomial for delta-matroids with loops, we have that

$$\begin{aligned}
&(y + w)^n U_{\mathrm{D}}\left(\frac{2z + x}{y + w}, \frac{y - z}{y + w}\right) = \\
&\left((y + w)^{n-k-\ell} U_{\mathrm{D}'}\left(\frac{2z + x}{y + w}, \frac{y - z}{y + w}\right)\right) \cdot \\
&\left((y + w)^k U_{P(U_{0,k})}\left(\frac{2z + x}{y + w}, \frac{y - z}{y + w}\right)\right) \cdot \left((y + w)^\ell U_{P(U_{\ell,\ell})}\left(\frac{2z + x}{y + w}, \frac{y - z}{y + w}\right)\right) \\
&= \left((y + w)^{n-k-\ell} U_{\mathrm{D}'}\left(\frac{2z + x}{y + w}, \frac{y - z}{y + w}\right)\right) \cdot (z + 3y + w)^{k+\ell}
\end{aligned}$$

As product of denormalized Lorentzian polynomials are denormalized Lorentzian [BH20, Corollary 3.8], we see that (1.2) is denormalized Lorentzian for all delta-matroids D that has an enveloping matroid.

The proof that (1.1) is denormalized Lorentzian is identical: one shows that, when M is an enveloping matroid of a loop-free and coloop-free delta-matroid D,

$$\begin{aligned}
\int_{X_{B_n}} s([\tilde{\mathcal{K}}_{\mathrm{M}}], q) \cdot \frac{1}{1 - y\gamma} \cdot c(\boxplus \mathcal{O}(1), x) &= \int_{X_{B_n}} c([\mathcal{I}_{\mathrm{D}}]^\vee, q) \cdot \frac{1}{1 - y\gamma} \cdot c(\boxplus \mathcal{O}(1), x) \\
&= (y + q)^n U_{\mathrm{D}}\left(\frac{x}{y + q}, \frac{y - q}{y + q}\right)
\end{aligned}$$

by Theorem 7.15. One then deduces the general case using the behavior of the U-polynomial under products. □

REFERENCES

[AA] Marcelo Aguiar and Federico Ardila. Hopf monoids and generalized permutahedra. *Mem. Amer. Math. Soc. (to appear)*. 1

[ABD10] Federico Ardila, Carolina Benedetti, and Jeffrey Doker. Matroid polytopes and their volumes. *Discrete Comput. Geom.*, 43(4):841–854, 2010. 3

[ABS04] Richard Arratia, Béla Bollobás, and Gregory B. Sorkin. The interlace polynomial of a graph. *J. Combin. Theory Ser. B*, 92(2):199–233, 2004. 4, 32, 43

[ACEP20] Federico Ardila, Federico Castillo, Christopher Eur, and Alexander Postnikov. Coxeter submodular functions and deformations of Coxeter permutahedra. *Adv. Math.*, 365:107039, 2020. 2, 3, 8

[ADH] Federico Ardila, Graham Denham, and June Huh. Lagrangian geometry of matroids. *Jour. Amer. Math. Soc. (to appear)*. 4

[AHK18] Karim Adiprasito, June Huh, and Eric Katz. Hodge theory for combinatorial geometries. *Ann. of Math. (2)*, 188(2):381–452, 2018. 2

[AvdH04] Martin Aigner and Hein van der Holst. Interlace polynomials. *Linear Algebra Appl.*, 377:11–30, 2004. 32

[Bas21] Jose Bastidas. The polytope algebra of generalized permutahedra. *Algebr. Comb.*, 4(5):909–946, 2021. 3

[BBGS00] Richard F. Booth, Alexandre V. Borovik, Israel Gelfand, and David A. Stone. Lagrangian matroids and cohomology. *Ann. Comb.*, 4(2):171–182, 2000. 32

[BEST] Andrew Berget, Christopher Eur, Hunter Spink, and Dennis Tseng. Tautological classes of matroids. arXiv:2103.08021. 2, 6, 14, 19, 21, 34, 39, 44, 45, 46

[BGW98] Alexandre V. Borovik, Israel Gelfand, and Neil White. Symplectic matroids. *J. Algebraic Combin.*, 8(3):235–252, 1998. 33

[BGW03] Alexandre V. Borovik, Israel Gelfand, and Neil White. *Coxeter matroids*, volume 216 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 2003. 2, 4, 9, 11, 31, 32, 33

[BH14] Robert Brijder and Hendrik Jan Hoogeboom. Interlace polynomials for multimatroids and delta-matroids. *European J. Combin.*, 40:142–167, 2014. 4, 28

[BH20] Petter Brändén and June Huh. Lorentzian polynomials. *Ann. of Math. (2)*, 192(3):821–891, 2020. 5, 43, 46

[BHM$^+$] Tom Braden, June Huh, Jacob Matherne, Nicholas Proudfoot, and Botong Wang. Singular Hodge theory for combinatorial geometries. arXiv:2010.06088. 20

[BHM$^+$22] Tom Braden, June Huh, Jacob P. Matherne, Nicholas Proudfoot, and Botong Wang. A semi-small decomposition of the Chow ring of a matroid. *Adv. Math.*, 409:Paper No. 108646, 2022. 20, 23

[Bjö84] Anders Björner. Some combinatorial and algebraic properties of Coxeter complexes and Tits buildings. *Adv. Math.*, 52(3):173–212, 1984. 25

[BLP] Petter Brändén, Jonathan Leake, and Igor Pak. Lower bounds for contingency tables via Lorentzian polynomials. *Isr. Jour. Math. (to appear)*. 5, 43

[Bou87] André Bouchet. Greedy algorithm and symmetric matroids. *Math. Programming*, 38(2):147–159, 1987. 4

[Bou97] André Bouchet. Multimatroids. I. Coverings by independent sets. *SIAM J. Discrete Math.*, 10(4):626–646, 1997. 27, 34

[Bre94] Francesco Brenti. q-Eulerian polynomials arising from Coxeter groups. *European J. Combin.*, 15(5):417–441, 1994. 25

[BST] Andrew Berget, Hunter Spink, and Dennis Tseng. Log-concavity of matroid h-vectors and mixed Eulerian numbers. arXiv:2005.01937. 2

[CLS11] David A. Cox, John B. Little, and Henry K. Schenck. *Toric varieties*, volume 124 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2011. 5, 6, 8, 21, 30, 37

[CMNR19a] Carolyn Chun, Iain Moffatt, Steven D. Noble, and Ralf Rueckriemen. Matroids, delta-matroids and embedded graphs. *J. Combin. Theory Ser. A*, 167:7–59, 2019. 4, 32

[CMNR19b] Carolyn Chun, Iain Moffatt, Steven D. Noble, and Ralf Rueckriemen. On the interplay between embedded graphs and delta-matroids. *Proc. Lond. Math. Soc. (3)*, 118(3):675–700, 2019. 4, 32

[CS05] Henry Crapo and William Schmitt. The free product of matroids. *European J. Combin.*, 26(7):1060–1065, 2005. 33

[DL94] Igor Dolgachev and Valery Lunts. A character formula for the representation of a Weyl group in the cohomology of the associated toric variety. *J. Algebra*, 168(3):741–772, 1994. 2

[DP10] Lars Eirik Danielsen and Matthew G. Parker. Interlace polynomials: enumeration, unimodality and connections to codes. *Discrete Appl. Math.*, 158(6):636–648, 2010. 43

[Duc92] Alain Duchamp. Delta matroids whose fundamental graphs are bipartite. *Linear Algebra Appl.*, 160:99–112, 1992. 32

[EG98] Dan Edidin and William Graham. Equivariant intersection theory. *Invent. Math.*, 131(3):595–634, 1998. 16

[EHL] Christopher Eur, June Huh, and Matt Larson. Stellahedral geometry of matroids. arXiv:2207.10605. 2, 6, 8, 16, 19, 20, 24, 34, 38, 39, 41, 44, 45, 46

[ESS21] Christopher Eur, Mario Sanchez, and Mariel Supina. The universal valuation of Coxeter matroids. *Bull. Lond. Math. Soc.*, 53(3):798–819, 2021. 14, 21, 39

[Ful93] William Fulton. *Introduction to toric varieties*, volume 131 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1993. The William H. Roever Lectures in Geometry. 5, 25, 26, 30

[GS87] Israel Gelfand and Vera Serganova. Combinatorial geometries and the strata of a torus on homogeneous compact manifolds. *Uspekhi Mat. Nauk*, 42(2(254)):107–134, 287, 1987. 31

[Ham17] Simon Hampe. The intersection ring of matroids. *J. Combin. Theory Ser. B*, 122:578–614, 2017. 2, 14, 25

[HK12] June Huh and Eric Katz. Log-concavity of characteristic polynomials and the Bergman fan of matroids. *Math. Ann.*, 354(3):1103–1116, 2012. 2

[Kly95] Alexander Klyachko. Toric varieties and flag spaces. volume 208, pages 139–162. 1995. 2

[LdMRS20] Lucía López de Medrano, Felipe Rincón, and Kristin Shaw. Chern-Schwartz-MacPherson cycles of matroids. *Proc. Lond. Math. Soc. (3)*, 120(1):1–27, 2020. 2

[McM89] Peter McMullen. The polytope algebra. *Adv. Math.*, 78(1):76–130, 1989. 8

[McM09] Peter McMullen. Valuations on lattice polytopes. *Adv. Math.*, 220(1):303–323, 2009. 8

[Mor93] Robert Morelli. The K-theory of a toric variety. *Adv. Math.*, 100(2):154–182, 1993. 8

[Mor17] Ada Morse. The interlace polynomial. In *Graph polynomials*, Discrete Math. Appl. (Boca Raton), pages 1–23. CRC Press, Boca Raton, FL, 2017. 4

[Pay06] Sam Payne. Equivariant Chow cohomology of toric varieties. *Math. Res. Lett.*, 13(1):29–41, 2006. 16

[Pos09] Alexander Postnikov. Permutohedra, associahedra, and beyond. *Int. Math. Res. Not. IMRN*, (6):1026–1106, 2009. 1, 2, 3

[PPR] Arnau Padrol, Vincent Pilaud, and Julian Ritter. Shard polytopes. *Int. Math. Res. Not. IMRN (to appear)*. 3

[Pro90] C. Procesi. The toric variety associated to Weyl chambers. In *Mots*, Lang. Raison. Calc., pages 153–161. Hermès, Paris, 1990. 2

[Sch03] Alexander Schrijver. *Combinatorial optimization. Polyhedra and efficiency. Vol. B*, volume 24 of *Algorithms and Combinatorics*. Springer-Verlag, Berlin, 2003. Matroids, trees, stable sets, Chapters 39–69. 10

[Sot03] Frank Sottile. Toric ideals, real toric varieties, and the moment map. In *Topics in algebraic geometry and geometric modeling*, volume 334 of *Contemp. Math.*, pages 225–240. Amer. Math. Soc., Providence, RI, 2003. 30

[Ste94] John R. Stembridge. Some permutation representations of Weyl groups associated with the cohomology of toric varieties. *Adv. Math.*, 106(2):244–301, 1994. 2

[TW15] E. Tsukerman and L. Williams. Bruhat interval polytopes. *Adv. Math.*, 285:766–810, 2015. 31

[VV03] Gabriele Vezzosi and Angelo Vistoli. Higher algebraic K-theory for actions of diagonalizable groups. *Invent. Math.*, 153(1):1–44, 2003. 16

HARVARD UNIVERSITY. CAMBRIDGE, MA. USA.
Email address: ceur@math.harvard.edu

School of Mathematical Sciences, Queen Mary University of London, London E1 4NS, UK
Email address: a.fink@qmul.ac.uk

Stanford University. Stanford, CA. USA.
Email address: mwlarson@stanford.edu

Stanford University. Stanford, CA. USA.
Email address: hspink@stanford.edu