The Tate Thesis
aka Automorphic Forms for GL₁

The plan:
1) Define the Schwartz space of functions to integrate
2) Figure out the local characters of \( \mathbb{Q}_v \)
3) Lift classical Dirichlet characters to characters of \( \mathbb{Q}_v \setminus \mathbb{A} \)
4) Define a global zeta function and prove a global functional equation
5) Factorize the global zeta function as a product of local zeta functions
6) Prove local functional equations
7) See what this all means for classical L-functions of Dirichlet characters
$$\mathbb{A} = \mathbb{A}_\mathbb{Q} = \prod_{v \leq \infty} \mathbb{Q}_v \quad (\mathbb{Q}_\infty = \mathbb{R})$$

Our goal: Fourier analysis on $\mathbb{A}$.

We fix an additive character of $\mathbb{A}$:

$$e_v: \mathbb{Q}_v \to \mathbb{C}^\times, \quad e_v(x) = \begin{cases} e^{-2\pi i v x} & \text{if } v < \infty \\ e^{2\pi i v x} & \text{if } v = \infty \end{cases}$$

$$e: \mathbb{A} \to \mathbb{C}^\times, \quad e(x) = \prod_v e_v(x_v)$$

Note: $e$ is trivial on $\mathbb{Q}$.

These induce isomorphisms (where $\hat{G} = \text{Hom}(G, \mathbb{C}^\times)$)

$$\mathbb{Q}_v \sim \hat{\mathbb{Q}}_v, \quad a \mapsto (x \mapsto e_v(ax))$$

$$\mathbb{A} \sim \hat{\mathbb{A}}, \quad a \mapsto (x \mapsto e(ax))$$

The latter induces $\mathbb{Q} \sim \hat{\mathbb{A}} / \mathbb{Q}$.

We equip $\mathbb{R}, \mathbb{Q}_p$, and $\mathbb{A}$ with the additive Haar measures $dx_\infty, dx_p, dx = \prod_v dx_v$.

(as usual) ($\mathbb{Z}_p$ has measure 1)

We equip $\mathbb{R}_x, \mathbb{Q}_p^x, \mathbb{A}^x$ with the multiplicative Haar measures $d^x x_\infty = \frac{dx_\infty}{1 x_\infty 1_\infty}, \quad d^x x_p = \frac{1}{1 - p^{-1}} \frac{dx_p}{1 x_p 1_p}$

($\mathbb{Z}_p^x$ has measure 1)

$$d^x x = \prod_v d^x_{x_v}$$
We will define the Fourier transform for functions in the Schwartz space.

For IR, the Schwartz space $S(\mathbb{R})$ is:
\{smooth functions $\Phi_\infty: \mathbb{R} \to \mathbb{C}$ s.t. $\sup_{x \in \mathbb{R}} |x^a \frac{d^b}{dx^b} \Phi(x)| < \infty \ \forall a, b \geq 0\}.

For $\mathbb{Q}_p$, the Schwartz space $S(\mathbb{Q}_p)$ is:
\{locally constant functions $\Phi_p: \mathbb{Q}_p \to \mathbb{C}$ with compact support\}

For $\mathbb{A}$, the Schwartz space $S(\mathbb{A})$ is:
\{finite linear combinations of functions of the form\}
\{$\Phi(x) = \prod_v \Phi_v(x_v)$ with $\Phi_v \in S(\mathbb{Q}_v)$, $\Phi_p = 1_{\mathbb{Z}_p}$ a.e.\}

Def: For $\Phi \in S(\mathbb{A})$, the Fourier transform of $\Phi$ is
\[\hat{\Phi}(x) = \int_{\mathbb{A}} \Phi(y) e(-xy) \, dy.\]

(and similarly for $\mathbb{Q}_v$)

Note:
1) For $v=\infty$, this is the usual Fourier transform
2) $\hat{1}_p = 1_p$
3) If $\Phi = \prod_v \Phi_v$, then $\hat{\Phi} = \prod \hat{\Phi}_v$.

Note:
1) For $v=\infty$, this is the usual Fourier transform
2) $\hat{1}_p = 1_p$
3) If $\Phi = \prod_v \Phi_v$, then $\hat{\Phi} = \prod \hat{\Phi}_v$. 
**Thm (Fourier inversion):** For any $\Phi \in S(\mathbb{A})$, $\hat{\Phi} \in S(\mathbb{A})$, and $\hat{\Phi}(x) = \overline{\Phi}(-x)$.

**Thm (Poisson summation):** For any $\Phi \in S(\mathbb{A})$,

$$
\sum_{\alpha \in \mathbb{A}} \Phi(\alpha x) = \frac{1}{|A|} \sum_{\alpha \in \mathbb{A}} \hat{\Phi}(\frac{\alpha}{x}) \quad \forall x \in \mathbb{A}^*.
$$

**Proof:**

The Fourier transform of $x \mapsto \Phi(\alpha x)$ is $x \mapsto \frac{1}{|A|} \hat{\Phi}(\frac{\alpha}{x})$, so assume $x = 1$.

$F(x) := \sum_{\alpha \in \mathbb{A}} \Phi(\alpha x)$ is a function on $\mathbb{A}/\mathbb{Q}$, so it has a Fourier expansion $F(x) = \sum_{\beta \in \mathbb{Q}} c_\beta e(\beta x)$.

We compute $c_\beta = \int_{\mathbb{A}/\mathbb{Q}} F(x)e(-\beta x) \, dx$

$$
= \int_{\mathbb{A}/\mathbb{Q}} \sum_{\alpha \in \mathbb{A}} \Phi(\alpha x) e(-\beta x) \, dx \\
= \int_{\mathbb{A}} \Phi(x)e(-\beta x) \, dx \\
= \hat{\Phi}(\beta),
$$

so $F(x) = \sum_{\beta \in \mathbb{Q}} \hat{\Phi}(\beta)e(\beta x)$. Set $x = 0$. \qed
Def: A Hecke character is a continuous unitary character $\omega: \mathbb{A}^* \to \mathbb{C}^*$ which is trivial on $\mathbb{Q}^*$.

A Hecke character $\omega$ factorizes as $\prod_v \omega_v$ with $\omega_v$ a continuous unitary character on $\mathbb{Q}_v^*$, so we need to:

(a) Understand continuous unitary characters on $\mathbb{Q}_v^*$
(b) Understand when they patch together to be trivial on $\mathbb{Q}^*$

For $\mathbb{R}^*$, continuous unitary characters are $1 \times 1_{\infty}$ and $\text{sgn}(x) 1 \times 1_{\infty}$ for $x \in \mathbb{R}$.

For $\mathbb{Q}_p^*$, we say $\omega_p$ is unramified if $\omega_p|_{\mathbb{Z}_p^*}$ is trivial.

If $\omega_p$ is unramified, it is determined by $\omega_p(p)$.

If $\omega_p$ is ramified, it is trivial on $1 + p^k \mathbb{Z}_p$ for some $k$ (the least such $p^k$ is the conductor), and then

$$\omega_p(p^k(j + p^k \mathbb{Z}_p)) = \omega_p(p)^k x(j)$$

for some Dirichlet character $x$ on $(\mathbb{Z}/p^k \mathbb{Z})^*$. 
Let $\chi$ be a primitive Dirichlet character on $(\mathbb{Z}/N\mathbb{Z})^\times$.

We will lift $\chi$ to a Hecke character $\tilde{\chi} = \prod \tilde{\chi}_v$ on $\mathbb{A}^\times$.

If $M \mid N$, write $\chi^{(m)}$ for the following Dirichlet character on $(\mathbb{Z}/M\mathbb{Z})^\times$:

$$
(\mathbb{Z}/M\mathbb{Z})^\times \to (\mathbb{Z}/M\mathbb{Z})^\times \times (\mathbb{Z}/N\mathbb{Z})^\times \xrightarrow{\chi} \mathbb{C}^\times
$$

$\tilde{\chi}_\infty$ is the character $\tilde{\chi}_\infty(\chi) = \chi(\text{sgn}(\chi(\infty)))$ on $\mathbb{R}^\times$.

If $p \nmid N$, $\tilde{\chi}_p$ is the unramified character on $\mathbb{Q}_p^\times$ with $\tilde{\chi}_p(p) = x(p)$.

If $p^k \mid N$ (with $k \geq 1$), $\tilde{\chi}_p$ is the ramified character of conductor $p^k$ on $\mathbb{Q}_p^\times$ with

$$
\tilde{\chi}_p(p) = \chi^{(N/p^k)}(p),
$$

$$
\tilde{\chi}_p(u) = \chi^{(p^k)}(u \mod 1 + p^k \mathbb{Z}_p)^{-1} \quad \text{if } u \in \mathbb{Z}_p^\times.
$$

Claim: $\tilde{\chi} = \prod \tilde{\chi}_v$ is a Hecke character (i.e. trivial on $\mathbb{Q}^\times$)

Lma: Every Hecke character is uniquely of the form $\tilde{\chi}(x) 1^{\infty}\epsilon$ for $\chi$ a primitive Dirichlet character and $\epsilon \in \mathbb{R}$.

Proof: Consider the isomorphism

$$
\mathbb{A}^\times \sim \mathbb{Q}^\times \times \mathbb{Z}^\times \times \mathbb{R}_+^\times
$$

$$
\times 1 \mapsto \left( \frac{\chi_\infty}{1^{\infty}}, \frac{1^{\infty}}{\chi_\infty}, \chi_{\text{finite}}, 1^{\infty} \right)
$$
Given a Hecke character $\omega$ and $\Phi \in \mathcal{S}(\mathbb{A})$, we define

$$S(s, \omega, \Phi) = \int_{\mathbb{A}} \Phi(x) \omega(x) 1 \cdot 1^s d^\times x.$$ 

Since $S(s, \omega 1 \cdot 1^s, \Phi) = S(s + i \epsilon, \omega, \Phi)$, we may assume $\omega$ is a lift of a Dirichlet character, so we write $\chi$ instead of $\omega$.

**Thm (Global functional equation):** $S(s, \chi, \Phi)$ has a meromorphic continuation to all of $\mathbb{C}$, entire unless $\chi = 1$, in which case the poles are at $s = 0, 1$. Moreover,

$$S(s, \chi, \Phi) = S(1 - s, \overline{\chi}, \overline{\Phi}).$$

**Proof:** Write $S(s, \chi, \Phi) = \int_{\mathbb{A}} \Phi(x) \chi(x) 1 \cdot 1^s d^\times x + \int_{\mathbb{A}} \Phi(x) \chi(x) 1 \cdot 1^s d^\times x$.

The local terms $S_1(s, \chi, \Phi)$ and $S_0(s, \chi, \Phi)$ are defined as follows:

- $S_1(s, \chi, \Phi)$ is entire, but $S_0$ only converges for $\Re s > 0$.
- \( S_0(s, \chi, \Phi) = \int_{\mathbb{A}^* / \mathbb{Q}^*} \left( \sum_{\alpha \in \mathbb{Q}^*} \Phi(\alpha x) \right) \chi(x) 1 \cdot 1^s d^\times x \)

By leveraging the Fourier transform, we can simplify the expression as follows:

$$S_0(s, \chi, \Phi) = \int_{\mathbb{A}^* / \mathbb{Q}^*} \left( \sum_{\alpha \in \mathbb{Q}^*} \Phi(\alpha x) \right) \chi(x) 1 \cdot 1^s d^\times x - \Phi(0) \int_{\mathbb{A}^* / \mathbb{Q}^*} \chi(x) 1 \cdot 1^s d^\times x$$

Further manipulation leads to:

$$S_0(s, \chi, \Phi) = \int_{\mathbb{A}^* / \mathbb{Q}^*} \left( \sum_{\alpha \in \mathbb{Q}^*} \widehat{\Phi} \left( \frac{\alpha}{x} \right) \right) \chi(x) 1 \cdot 1^{s-1} d^\times x + \frac{\Phi(0)}{s}$$
\[
\sum_{\alpha \in \mathcal{Q}}^{\mathfrak{a}} \left( \sum_{\alpha \in \mathcal{Q}}^{\star} \frac{\hat{\Phi}(\alpha x)}{1+\alpha} \right) \chi(x) \right) \left( \prod_{\alpha \in \mathcal{Q}} \frac{\hat{\Phi}(\alpha x)}{\alpha} \right) \chi(x) = \sum_{\alpha \in \mathcal{Q}}^{\mathfrak{a}} \left( \frac{\hat{\Phi}(0)}{s-1} - \frac{\Phi(0)}{s} \right)
\]

Altogether,

\[
S(s, \chi, \Phi) = S_1(s, \chi, \Phi) + S_1(1-s, \chi, \hat{\Phi}) + 1_{\alpha=1} \left( \frac{\hat{\Phi}(0)}{s-1} - \frac{\Phi(0)}{s} \right)
\]

This is meromorphic and invariant under

\[
s \to 1-s, \quad \chi \to \chi, \quad \Phi \to \hat{\Phi}.
\]

What does this mean for \( L \)-functions?

Let us choose a specific \( \Phi = \prod \Phi_v \) and see.

We have

\[
S(s, \chi, \Phi) = \prod_v S_v(s, \chi_v, \Phi_v)
\]

where

\[
S_v(s, \chi_v, \Phi_v) = \int_{\mathfrak{a}^* \Phi_v(x_v)} \chi_v(x_v) d^* x_v.
\]

Therefore,

\[
\prod_v S_v(s, \chi_v, \Phi_v) = \prod_v S_v(1-s, \chi_v, \hat{\Phi}_v).
\]

We will choose \( \Phi_v \) depending on \( \chi_v \).
For example, if $v=p<\infty$ and $x_p$ is unramified, choose $\hat{\Phi}_p = 1_{\mathbb{Z}_p}$.

We have $S_p(s, x_p, 1_{\mathbb{Z}_p}) = \int_{\mathbb{Q}_p^*} 1_{\mathbb{Z}_p}(x) x_p(x) \, d^* x_p$

$$= \int_{\mathbb{Z}_p^*} x_p(x) \, d^* x_p$$

$$= \sum_{k=0}^{\infty} \int_{\mathbb{Z}_p^*} x_p(x) \, d^* x_p$$

$$= \sum_{k=0}^{\infty} x_p(x)^k p^{-ks} \int_{\mathbb{Z}_p^*} d^* x_p$$

$$= \sum_{k=0}^{\infty} x_p(x)^k p^{-ks}$$

$$= \frac{1}{1-x_p(x)p^{-s}}$$

Since $\hat{\mathbb{Z}_p} = 1_{\mathbb{Z}_p}$, $S_p(1-s, x_p, \hat{\mathbb{Z}_p}) = \frac{1}{1-x_p(p) p^{-(1-s)}}$.

In general, we have the following:

<table>
<thead>
<tr>
<th>$v$</th>
<th>$X_v$</th>
<th>$\Phi_v(x)$</th>
<th>$\hat{\Phi}_v$</th>
<th>$S_v(s, x_v, \hat{\Phi}_v)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\infty$</td>
<td>$1$</td>
<td>$e^{-\pi x^2}$</td>
<td>$\Phi_\infty$</td>
<td>$\pi^{-s/2} \Gamma(s/2)$</td>
</tr>
<tr>
<td>$\infty$</td>
<td>$\text{sgn}(x)$</td>
<td>$xe^{-\pi x^2}$</td>
<td>$-i \Phi_\infty$</td>
<td>$\pi^{-s/2} \Gamma(s+1/2)$</td>
</tr>
<tr>
<td>$p&lt;\infty$</td>
<td>unramified</td>
<td>$1_{\mathbb{Z}_p}(x)$</td>
<td>$\hat{\Phi}_p$</td>
<td>$\frac{1}{1-x_p(p)p^{-s}}$</td>
</tr>
<tr>
<td>$p&lt;\infty$</td>
<td>ramified with conductor $p^k$</td>
<td>$e_p(x) \mathbb{1}_{p^{-k}\mathbb{Z}_p}(x)$</td>
<td>$p^k \mathbb{1}_{1+p^k\mathbb{Z}_p}$</td>
<td>$\frac{p^k(s-1)x_p(p)^k}{1-p^{-s}} \sum_{x \in (\mathbb{Z}/p^k \mathbb{Z})^*} e^{2\pi x_i p^{-s}}$</td>
</tr>
</tbody>
</table>

* In the last row, $S_p(1-s, x_p, \hat{\Phi}_p) = \frac{1}{1-p^{-s}}$. 
Putting this all together, we have shown that if $\chi$ is a primitive Dirichlet character modulo $N$, then

$$
\Lambda(s, \chi) = \frac{\tau(\chi)}{i^\varepsilon \sqrt{N}} \Lambda(1-s, \overline{\chi})
$$

where

$$
\Lambda(s, \chi) = \left( \frac{\pi}{N} \right)^{-s+\varepsilon} \prod \left( \frac{s+\varepsilon}{2} \right) \Lambda(s, \chi)
$$

$$
\Lambda(s, \chi) = \prod_{\rho \in \text{P} \times \text{N}} \frac{1}{1-\chi(\rho)}
$$

$$
\tau(\chi) = \sum_{j \in (\mathbb{Z}/N\mathbb{Z})^*} \chi(j) e^{2\pi i j/N}
$$

$$
\varepsilon = \begin{cases} 
0 & \text{if } \chi(-1) = 1 \\
1 & \text{if } \chi(-1) = -1
\end{cases}
$$

What if we had chosen $\Phi$ differently?

It turns out it doesn't matter!
Thm (Local functional equation): There is a meromorphic function \( \gamma_v(s, x_v) \) independent of \( \Phi_v \), s.t.
\[
S_v(s, x_v, \Phi_v) = \gamma_v(s, x_v) S_v(1-s, x_v, \widehat{\Phi}_v)
\]
for all \( \Phi_v \in S(Q_v) \).

Proof: We must show that \( S_v(s, x_v, \Phi_v) S_v(1-s, x_v, \widehat{\Phi}_v) \) is symmetric in \( \Phi_v \) and \( \Psi_v \).
\[
S_v(s, x_v, \Phi_v) S_v(1-s, x_v, \widehat{\Phi}_v)
= \int_{Q_v^x} \int_{Q_v^x} \Phi_v(x) x_v(x) 1 \times 1^{1-s} \hat{\Phi}_v(y) \mathcal{A} \gamma_v(y) 1 y_v^1 d^x y d^x
\]
\[
= \int_{Q_v^x} \int_{Q_v^x} \Phi_v(x) \hat{\Psi}_v(y) x_v(xy^{-1}) 1 \times 1^{1-s} 1 y_v^1 d^x y d^x
\]
\[
= \int_{Q_v^x} \int_{Q_v^x} \Phi_v(x) \hat{\Psi}_v(u) x_v(u^{-1}) 1 \times 1^{1-s} d^x u d^x
\]
\[
= \int_{Q_v^x} \int_{Q_v^x} \int_{Q_v^x} \Phi_v(x) \Psi_v(z) e_v(-uxz) x_v(u^{-1}) 1 \times 1^{1-s} d^x z d^x u d^x
\]
\[
= \int_{Q_v} \int_{Q_v} \int_{Q_v} \Phi_v(x) \Psi_v(z) e_v(-uxz) x_v(u^{-1}) 1^{1-s} d^x z d^x u d^x
\]
\[
\frac{1}{p-1} \text{ if } v=p<\infty
\]
\[
1 \text{ if } v=\infty
\]
Since $\gamma_v(s, x_v)$ is independent of $\Phi_v$, we can compute it explicitly using our choice of $\Phi_v$ from earlier.

If $v=\infty$ and $x_\infty(x) = 1$,
$$
\gamma_\infty(s, x_\infty) = \frac{\pi^{-s/2} \Gamma(s/2)}{\pi^{-1/2} \Gamma(1/2)}
$$

If $v=\infty$ and $x_\infty(x) = \text{sgn}(x)$,
$$
\gamma_\infty(s, x_\infty) = i \frac{\pi^{-s+1/2} \Gamma(s+1/2)}{\pi^{s} \Gamma(1/2)}
$$

If $v=p<\infty$ and $x_p$ is unramified,
$$
\gamma_p(s, x_p) = \frac{1 - x_p(p)p^{-1-s}}{1 - x_p(p)p^{-s}}
$$

If $v=p<\infty$ and $x_p$ is ramified with conductor $p^k$,
$$
\gamma_p(s, x_p) = p^{k(s-1)} x_p(p)^k \sum_{j \in (\mathbb{Z}/p^k \mathbb{Z})^*} x_p(j) e^{2\pi i \frac{u}{p^k}}
$$

In particular, $\gamma_v(s, x_v)$ is always meromorphic. \qed
The plan going forward:

1) Define the Schwartz space of functions to integrate

2) Figure out the local characters of $\mathbb{Q}_v^\times$ representations of $GL_2(\mathbb{Q}_v)$

3) Lift classical Dirichlet characters to characters of $\mathbb{Q}_v^\times \backslash \mathbb{A}$ modular/Maass forms automorphic forms on $GL_2(\mathbb{A}) \backslash GL_2(\mathbb{A})$

4) Define a global zeta function and prove a global functional equation

5) Factorize the global zeta function as a product of local zeta functions (via the tensor product theorem)

6) Prove local functional equations

7) See what this all means for classical $L$-functions of Dirichlet characters modular/Maass forms
The Local Functional Equation for $GL_2$

1) Review theory of admissible representations of $GL_2(\mathbb{Q}_p)$
   - Contragradient representations ($\pi \cong \omega \otimes \pi$)
   - Whittaker/Kirillov models (existence & uniqueness)
   - Jacquet modules

2) Review classification of irreducible admissible representations of $GL_2(\mathbb{Q}_p)$
   - Explicit Jacquet modules
   - Explicit Kirillov models

3) Define the local zeta function, and see what these zeta functions look like

4) Check that the spherical representations give the familiar $L$-factor

5) Prove the local functional equation

6) Explain the analogous results for $GL_2(\mathbb{R})$
A representation \((\pi, V)\) is **admissible** if:

(a) The stabilizer of each \(v \in V\) is open

(b) The space \(V^K\) of vectors stabilized by \(K\) is finite-dimensional for each open subgroup \(K \leq \text{GL}_2(\mathbb{Q}_p)\)

Note: the compact open subgroups \(\{g \in \text{GL}_2(\mathbb{Z}_p) | g \equiv 1 \text{ (mod } p^k)\}\) are a basis of opens at the identity.

**Lemma (Schur):** If \((\pi, V)\) is an irreducible admissible representation of \(\text{GL}_2(\mathbb{Q}_p)\), then \(\text{Hom}_{\text{GL}_2(\mathbb{Q}_p)}(V, V) \cong \mathbb{C}\).

**Proof:** By (a), \(V^K \neq 0\) for some open subgroup \(K\).

By (b), \(V^K\) is finite-dimensional.

If \(L \in \text{Hom}_{\text{GL}_2(\mathbb{Q}_p)}(V, V)\), \(L|_{V^K}\) has an eigenvalue \(\lambda\).

Since \(\ker(L-\lambda)\) is nonzero, it must be all of \(V\). \(\square\)

**Corollary:** If \((\pi, V)\) is an irr. ad. rep.,

the center \(\mathbb{Q}_p^*\) of \(\text{GL}_2(\mathbb{Q}_p)\) acts via some character \(\omega: \mathbb{Q}_p^* \to \mathbb{C}^*\), i.e. \(\pi((t \epsilon)) v = \omega(t) v\).
Lemma: The finite-dimensional, irr. ad. reps. of $GL_2(\mathbb{Q}_p)$ are all of the form $\chi \cdot \det$ for $\chi$ a character of $\mathbb{Q}_p$.

We abbreviate $\chi \cdot \det$ by writing $\chi$.

Given a representation $\Pi$ and character $\chi$, we can form the representation $\Pi \otimes \chi$, given by
\[(\Pi \otimes \chi)(g)v = \chi(\det g) \Pi(g)v.\]

Def: Let $(\pi, V)$ be an admissible rep. of $GL_2(\mathbb{Q}_p)$. A linear functional $\hat{V}: V \to \mathbb{C}$ is \underline{smooth} if $\hat{V} \circ \pi(g) = \hat{V}$ for $g$ in a neighborhood of the identity.

The \underline{contragradient} rep. of $(\pi, V)$ is $(\hat{\pi}, \hat{V})$ where $\hat{V}$ is the space of smooth linear functionals and $\hat{\pi}(g) \hat{V} = \hat{V} \circ \pi(g')$.

Properties:
1) If $(\pi, V)$ is admissible, then so is $(\hat{\pi}, \hat{V})$.
2) If $(\pi, V)$ is irreducible, then so is $(\hat{\pi}, \hat{V})$.
3) $\hat{\pi} \cong \pi$
Theorem: If \( \Pi \) is an irreducible ad. rep. of \( \text{GL}_2(\mathbb{Q}_p) \) with central character \( \omega : \mathbb{Q}_p^\times \to \mathbb{C}^\times \), then \( \hat{\Pi} \cong \omega^{-1} \otimes \Pi \).

Example: For a one-dimensional rep. \( \chi \), the central character is \( \chi^2 \) and \( \hat{\chi} \cong \chi^{-1} \cong (\chi^2)^{-1} \otimes \chi \).

Example: If \( \Pi \) has central character \( \omega \), then \( \omega^{-1} \otimes \Pi \) has central character \( \omega^{-1} \), so \( \hat{\Pi} \cong \omega \otimes \hat{\Pi} \cong \omega \otimes \omega^{-1} \otimes \Pi \cong \Pi \).

Proof (for \( \text{GL}_2(\mathbb{F}_p) \)): We must show \( \omega^{-1} \otimes \Pi \) and \( \hat{\Pi} \) have the same character. If \( \Pi \) has character \( \chi \), then \( \hat{\Pi} \) has character \( g \mapsto \chi(g^{-1}) \).

For \( g = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \text{GL}_2(\mathbb{F}_p) \),
\[
g^{-1} = \frac{1}{\det g} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{\det g} \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} g^T \begin{pmatrix} -1 & -1 \end{pmatrix}.
\]

Every matrix is conjugate to its transpose, so \( \chi(g^{-1}) = \omega(\det g)^{-1} \chi(g) \). \( \square \)
Let \((\pi, V)\) be an admissible representation of \(\text{GL}_2(\mathbb{Q}_p)\)

A Whittaker model for \((\pi, V)\) is a realization of \((\pi, V)\) as a space of functions \(W: \text{GL}_2(\mathbb{Q}_p) \to \mathbb{C}\) s.t.

\[
W((x^i, y)g) = e_p(x)W(g) \quad \forall g \in \text{GL}_2(\mathbb{Q}_p), x \in \mathbb{Q}_p,
\]

with \(\text{GL}_2(\mathbb{Q}_p)\) acting via right multiplication.

**Example:** Consider a one-dimensional representation \(X\).

A Whittaker model for \(X\) would require a (nonzero) function \(W: \text{GL}_2(\mathbb{Q}_p) \to \mathbb{C}\) s.t.

\[
W((x^i, y)g) = e_p(x)W(g) \text{ and } W(g) = x(\det g)W(1).
\]

No such function exists, so \(X\) has no Whittaker model.

**Theorem:** If \((\pi, V)\) is an infinite-dimensional irr. ad. rep, then it has a unique Whittaker model.

**Proof (Sketch for \(\text{GL}_2(\mathbb{F}_p)\)):**

Let \(N = \{(x^i, y) | x \in \mathbb{F}_p\}\).

Consider the character \(\Psi: N \to \mathbb{C}^\times, (x^i, y) \mapsto e_p(x)\).

A Whittaker model is the same as a map \(\pi \to \text{Ind}_{N}^{\text{GL}_2(\mathbb{F}_p)} \Psi\).

For uniqueness, we must show \(\text{End}_{\text{GL}_2(\mathbb{F}_p)}(\text{Ind}_{N}^{\text{GL}_2(\mathbb{F}_p)} \Psi)\) is commutative.

Existence follows by e.g. counting dimensions.
If \((\pi, V)\) is an infinite-dimensional irr. ad. rep. we write \(W(\pi)\) for its Whittacker model. The Kirillov model is the space
\[K(\pi) = \{ \phi : \mathbb{Q}_p^x \to \mathbb{C} \mid \phi(x) = W((x, 1)) \text{ for some } W \in W(\pi) \}\]
Note: \(W((x, 1))\) is never identically 0, so \(K(\pi) \cong W(\pi) \cong (\pi, V)\).

What is the action of \(GL_2(\mathbb{Q}_p)\)?
\[
\begin{align*}
(\pi(\alpha, 1) \phi)(x) &= (\pi(\alpha, 1) W)((x, 1)) = W((\alpha x, 1)) = \phi(\alpha x) \\
(\pi(1, b) \phi)(x) &= (\pi(1, b) W)((x, 1)) = W((1, b^x)(x, 1)) = e_p(b x) \phi(x) \\
(\pi(\omega, \epsilon) \phi)(x) &= \omega(\epsilon) \phi(x)
\end{align*}
\]
\[\{(\alpha, 1), (1, b), (\omega, \epsilon), (-1, 1)\}^2\] generate \(GL_2(\mathbb{Q}_p)\).
Only the action of \((-1, 1)\) is mysterious.
What is $K(\pi)$ as a space of functions on $\mathbb{Q}_p^\times$?

Take $\phi \in K(\pi)$.

Since $\pi$ is admissible and $(\pi(\varphi) \phi)(x) = \phi(\alpha x)$, $\phi$ is locally constant.

Since $\pi$ is admissible and $(\pi(\varphi) \phi)(x) = e_p(b x) \phi(x)$, $\phi$ has bounded support.

Therefore, $K(\pi) \subseteq C_c^\infty(\mathbb{Q}_p^\times)^*$, the space of locally constant functions with bounded support.

Also, $K(\pi) \supseteq C_c^\infty(\mathbb{Q}_p^\times)^*$, the space of locally constant functions with compact support, since $(\ast \ast) \supseteq C_c^\infty(\mathbb{Q}_p^\times)^*$ is irreducible.

In fact, $C_c^\infty(\mathbb{Q}_p^\times) = \{ \pi(\psi) \phi - \phi \mid \phi \in K(\pi), x \in \mathbb{Q}_p \}$.

In other words, we have an exact sequence

$$0 \rightarrow C_c^\infty(\mathbb{Q}_p^\times) \rightarrow K(\pi) \rightarrow J(K(\pi)) \rightarrow 0$$

where $J(V) = \bigvee_{\psi(\mathbf{v})} \psi(\mathbf{v}) \bigvee_{\psi(\mathbf{v})} \psi(\mathbf{v}) \bigvee_{\psi(\mathbf{v})} \psi(\mathbf{v})$, $x \in \mathbb{Q}_p$ is the Jacquet module of $V$.

Thus, to know the Jacquet module is to know the Kirillov model.

*The Schwartz space $S(\mathbb{Q}_p)$ is $C_c^\infty(\mathbb{Q}_p)$, inside which $C_c^\infty(\mathbb{Q}_p^\times)$ has codimension 1.
For characters $\chi_1$ and $\chi_2$ of $Q_p^x$, we define the representation $\mathcal{B}(\chi_1, \chi_2)$ to be the space of all $f \in C^\infty(GL_2(Q_p))$ s.t.

$$f((x_1, y_1)g) = \left|\frac{x_1}{x_2}\right|^\frac{1}{2} \chi_1(x_1) \chi_2(x_2) f(g),$$

with $GL_2(Q_p)$ acting via right translation.

If $\chi_1 \chi_2^{-1} \neq 1$, $\mathcal{B}(\chi_1, \chi_2)$ is irreducible. We denote its isomorphism class by $\Pi(\chi_1, \chi_2)$.

If $\chi_1 \chi_2^{-1} = 1$, $\mathcal{B}(\chi_1, \chi_2)$ has an irreducible invariant subspace $\sigma(\chi_1, \chi_2)$ of codimension 1.

If $\chi_1 \chi_2^{-1} = 1^{-1}$, $\mathcal{B}(\chi_1, \chi_2)$ has a one-dimensional invariant subspace irreducible quotient $\sigma(\chi_1, \chi_2)$.

**Theorem:** Let $(\pi, V)$ be an irr. ad. rep. of $GL_2(Q_p)$.

$$\dim J(V) = \begin{cases} 2 & \text{if } (\pi, V) \cong \Pi(\chi_1, \chi_2) \\ 1 & \text{if } (\pi, V) \cong \sigma(\chi_1, \chi_2) \text{ (or 1-dimensional)} \\ 0 & \text{otherwise} \end{cases}$$

$\Pi(\chi_1, \chi_2)$ are the principal series representations.

$\sigma(\chi_1, \chi_2)$ are the special representations.

The representations $(\pi, V)$ with $J(V) = 0$ are supercuspidal.
Let \((\pi, V)\) be an infinite-dimensional irr. ad. rep.

For \(\phi_1, \phi_2 \in C_0^\infty(Q_p^x)\), we write \(\phi_1(t) \sim \phi_2(t)\) if \(\phi_1(t) = \phi_2(t)\) for \(|t|\) sufficiently small.

If \((\pi, V) \cong \Pi(x_1, x_2)\), then

\[
\mathcal{K}(\pi) = \left\{ \phi \in C_0^\infty(Q_p^x) \mid \phi(t) \sim C_1 |t|^{\frac{n}{2}} x_1(t) + C_2 |t|^{\frac{n}{2}} x_2(t) \right\}
\]

if \(x_1 \neq x_2\)

\[
\mathcal{K}(\pi) = \left\{ \phi \in C_0^\infty(Q_p^x) \mid \phi(t) \sim (C_1 + C_2 v(t)) |t|^{\frac{n}{2}} x_1(t) \right\}
\]

if \(x_1 = x_2\)

If \((\pi, V) \cong \sigma(x_1, x_2)\) with \(x_1 x_2^{-1} = 1\), then

\[
\mathcal{K}(\pi) = \left\{ \phi \in C_0^\infty(Q_p^x) \mid \phi(t) \sim C |t|^{\frac{n}{2}} x_1(t) \right\}
\]

If \((\pi, V)\) is supercuspidal, then \(\mathcal{K}(\pi) = C_0^\infty(Q_p^x)\).
Let \((\pi, V)\) be an infinite-dimensional irr. ad. rep.

For \(\phi \in \mathcal{H}(\pi)\), we define the local zeta function

\[
\zeta(s, \phi) = \int_{\mathbb{Q}_p^*} \phi(x) |x|^{s-\frac{1}{2}} d^x \chi.
\]

Note: \(\mathcal{H}(\mathbb{F}_q \otimes \pi) = \mathcal{H}(\pi)\), so \(\int_{\mathbb{Q}_p^*} \phi(x) \zeta(x) |x|^{s-\frac{1}{2}} d^x \chi\) is a zeta function for \(\mathcal{H}(\mathbb{F}_q \otimes \pi)\).

Gelbart defines \(\zeta(g, \xi, \phi, s) = \int_{\mathbb{Q}_p^*} (g \phi)(x) \zeta(x) |x|^{s-\frac{1}{2}} d^x \chi\).

In our notation, this is just \(\zeta(s, \xi g \phi)\).

What can \(\zeta(s, \phi)\) look like?

\[
\zeta(s, 1_p \mathbb{Q}_p) = \int_{\mathbb{Q}_p^*} |x|^{s-\frac{1}{2}} d^x \chi = p^{-\frac{s-1}{2}}.
\]

\[
\zeta(s, 1_\mathbb{Q}_p^* \chi \mathbb{Q}_p) = \int_{\mathbb{Q}_p^*} \chi(x) |x|^{s} d^x \chi = \sum_{k=0}^{\infty} \int_{\mathbb{Q}_p^*} \chi(x) |x|^{s} d^x \chi
\]

\[
= \sum_{k=0}^{\infty} \chi(p)^k p^{-ks} \int_{\mathbb{Q}_p^*} \chi(x) d^x \chi
\]

\[
= \begin{cases} 
\frac{1}{1-\chi(p)p^{-s}} & \text{if } \chi \text{ is unramified} \\
0 & \text{if } \chi \text{ is ramified.}
\end{cases}
\]
Hence, \( \{ S(s, \phi) \mid \phi \in \mathcal{H}(\pi) \} = L(s, \pi) \mathbb{C}[p^s, p^{-s}] \) where

\[
L(s, \pi) = \begin{cases} 
\frac{1}{(1-\alpha_1 p^{-s})(1-\alpha_2 p^{-s})} & \text{if } (\pi, \nu) \cong \pi(x_1, x_2) \text{ with } x_1 x_2 \neq 1 \\
\frac{1}{1-\alpha_i p^{-s}} & \text{if } (\pi, \nu) \cong \sigma(x_1, x_2) \text{ with } x_1 x_2 = 1 \\
1 & \text{if } (\pi, \nu) \text{ is supercuspidal}
\end{cases}
\]

where \( \alpha_i = \begin{cases} x_i(p) & \text{if } x_i \text{ is unramified} \\
0 & \text{if } x_i \text{ is ramified.}
\end{cases} \)

In particular, \( S(s, \phi) \) always has a meromorphic continuation to all of \( \mathbb{C} \), with no more poles than \( L(s, \pi) \).

Example: Recall that a representation is called spherical (or unramified) if it has a \( \text{GL}_2(\mathbb{Z}_p) \)-fixed vector.

If \( (\pi, \nu) \) is spherical, then it is of the form \( \pi(x_1, x_2) \) with \( x_1, x_2 \) unramified, and the space of \( \text{GL}_2(\mathbb{Z}_p) \)-fixed vectors is one-dimensional.

Let \( \phi^0 \) be the \( \text{GL}_2(\mathbb{Z}_p) \)-fixed vector in \( \mathcal{H}(\pi) \) with \( \phi^0(1) = 1 \). It turns out that (with \( \alpha_i = x_i(p) \))

\[
\phi^0(p^k) = \begin{cases} 
p^{-k/2} \frac{\alpha_i^{k+1} - \alpha_j^{k+1}}{\alpha_i - \alpha_j} & \text{if } k \geq 0 \\
0 & \text{if } k < 0,
\end{cases}
\]

so \( S(s, \phi^0) = \sum_{k=0}^{\infty} p^{-k/2} \frac{\alpha_i^{k+1} - \alpha_j^{k+1}}{\alpha_i - \alpha_j} = \frac{1}{(1-\alpha_1 p^{-s})(1-\alpha_2 p^{-s})} = L(s, \pi) \).
**Theorem (local functional equation):**

Let \((\pi, V)\) be an irreducible admissible representation of \(\text{GL}_2(\mathbb{Q}_p)\) with central character \(\omega\).

For some meromorphic function \(\zeta(s, \pi)\) independent of \(\Phi \in \mathcal{H}(\pi)\),

\[
\zeta(s, \Phi) = \zeta(s, \pi) \zeta(1-s, \omega^{-1} \pi(-, \Phi))
\]

is an element of \(\mathcal{H}(\omega^{-1} \otimes \pi) = \mathcal{H}(\tilde{\pi})\).

**Proof:** For fixed \(s\), consider the linear functionals

\[
\Lambda_1(\Phi) = \zeta(s, \Phi) \quad \text{and} \quad \Lambda_2(\Phi) = \zeta(1-s, \omega^{-1} \pi(-, \Phi))
\]

We will show that \(\Lambda_1\) and \(\Lambda_2\) are (usually) proportional.

For all \(a \in \mathbb{Q}_p^\times\) and \(\Phi \in \mathcal{H}(\pi)\),

\[
\Lambda_2(\pi(a, \cdot) \Phi) = \int_{\mathbb{Q}_p^\times} \left[ \pi((-1, \cdot)(a, \cdot)) \Phi \right](x) \omega^{-1}(x) d_{\mathbb{Q}_p^\times}^x
\]

\[
= \int_{\mathbb{Q}_p^\times} \left[ \pi((-1, \cdot)(a, \cdot)) \Phi \right](x) \omega^{-1}(x) d_{\mathbb{Q}_p^\times}^x
\]

\[
= \omega(a) \int_{\mathbb{Q}_p^\times} \left[ \pi((-1, \cdot)) \Phi \right](a^{-1} x) \omega^{-1}(x) d_{\mathbb{Q}_p^\times}^x
\]

\[
= \omega(a) \int_{\mathbb{Q}_p^\times} \left[ \pi((-1, \cdot)) \Phi \right](a^{-1} x) \omega^{-1}(a^{-1} y) a^{-s + i \frac{1}{2}} d_{\mathbb{Q}_p^\times}^x
\]

\[
= \omega(a) \int_{\mathbb{Q}_p^\times} \left[ \pi((-1, \cdot)) \Phi \right](y) \omega^{-1}(y) y^{-s + i \frac{1}{2}} d_{\mathbb{Q}_p^\times}^y
\]

\[
= \omega(a) \Lambda_2(\Phi).
\]
Restrict \( \Lambda_1 \) and \( \Lambda_2 \) to \( C^\infty_c(Q^*_p) = \ker(H(K(\pi)) \to J(K(\pi))) \).

We will show that for \( \phi \in C^\infty_c(Q^*_p) \),

\[
\Lambda_2(\phi) = \Lambda_2(1_{Z_p^*}) \Lambda_1(\phi).
\]

Given \( \phi \in C^\infty_c(Q^*_p) \), choose \( k > 0 \) s.t. \( \phi(a \pi) = \phi(x) \) if \( a \equiv 1 \pmod{p^k} \)

\[
\phi = \sum_{j \in (Z/p^kZ)} \sum_{m \in Z} \Phi(p^m J) 1_{p^m J(1+p^kZ_p)}
\]

\[
= \sum_{j \in (Z/p^kZ)} \sum_{m \in Z} \Phi(p^m J) \Pi\left( (p^m J)^{-1} \right) 1_{1+p^kZ_p},
\]

so \( \Lambda_2(\phi) = \sum_{j \in (Z/p^kZ)} \sum_{m \in Z} \Phi(p^m J) p^{m(-s+1)} \Lambda_2(1_{1+p^kZ_p}) \Lambda_1(\phi) \Lambda_2(1_{1+p^kZ_p}).
\]

Letting \( \phi = 1_{Z_p^*} \), we find \( \Lambda_2(1_{Z_p^*}) = \Lambda_2(1_{1+p^kZ_p}), \) giving

Let \( \Lambda = \Lambda_2 - \Lambda_2(1_{Z_p^*}) \Lambda_1 \), which induces a map on \( J(K(\pi)) \).

In fact, \( \Lambda \) spans a \((*,*)\)-invariant space where \((\ell^*_1, \ell^*_2)\) acts as \( \omega(\ell^*_2)^{-1} |\ell^*_1|^{s-\frac{1}{2}}. \) Since \( \dim J(K(\pi)) \leq 2 \),

\( \Lambda = 0 \) for all but two values of \( s \pmod{\frac{2\pi \ell^*_2}{\log \ell^*_p}}. \)
In other words, for $s$ outside a discrete set,\[ \exists \gamma(s, \pi) \text{ s.t. } \Lambda_1(\phi) = \gamma(s, \pi) \Lambda_2(\phi). \]

Since zeta functions are meromorphic, so is $\gamma(s, \pi)$. \[\square\]

For principal series,\[ \gamma(s, \pi\pi(x, x)) = \gamma(s, x) \gamma(s, x_2) \quad (x, x_2^{-1} \neq 1, 1^{+}) \]

For special representations,\[ \gamma(s, \sigma(x, x)) = \gamma(s, x) \gamma(s, x_2) \quad (x, x_2^{-1} = 1, 1^{+}) \]

For supercuspidal representations, $\gamma(s, \pi) \in \mathbb{C}[p^3, p^3]$ is nonvanishing, so $\gamma(s, \pi) = Ap^{-ms}$ for some $A \in \mathbb{C}, m \in \mathbb{Z}$.

**Proposition:** Suppose $\pi_1$ and $\pi_2$ have the same central character. If $\gamma(s, s \otimes \pi_1) = \gamma(s, s \otimes \pi_2)$ for all characters $s$ of $\mathbb{Q}_p$, then $\pi_1 \cong \pi_2$. 
Proof:
We must show $\pi_i(-1') \Phi = \pi_2(-1') \Phi \forall \Phi \in \Gamma(\pi_1) \cap \Gamma(\pi_2)$.
Replacing $\Phi$ with $(\alpha \Phi)$, it is enough to show
$$(\pi_i(-1') \Phi)(1) = (\pi_2(-1') \Phi)(1).$$

Let $\phi_i = \pi_i(-1') \Phi$.

Given a character $\xi$ of $\mathbb{Q}_p^\times$ and $n \in \mathbb{Z}$, let
$$F_\xi(n) = \int_{|x|=p^{-n}} (\phi_1(x) - \phi_2(x)) \xi(x) dx.$$ 

By Fourier inversion on $\mathbb{Z}_p^\times$,
$$\phi_1(1) - \phi_2(1) = \sum_{\xi \in \hat{\mathbb{Z}_p^\times}} F_\xi(0),$$
so it suffices to show $F_\xi(0) = 0 \forall \xi$.

Since $\phi_i \in C_0(\mathbb{Q}_p^\times)$, $F_\xi(n) = 0 \forall n << 0$.

By hypothesis, $s(s, \xi \phi_1) = s(s, \xi \phi_2)$, so
$$\sum_{n \in \mathbb{Z}} F_\xi(n) \rho^{-ns} = s(s, \xi \phi_1) - s(s, \xi \phi_2) = 0$$
for $\text{Re} s >> 0$.

Hence, $F_\xi(n) = 0 \forall n \in \mathbb{Z}$, and in particular $F_\xi(0) = 0$. 
What about for $\text{GL}_2(\mathbb{R})$?

Let $(\pi, V)$ be an irreducible admissible $(\text{GL}_2(\mathbb{R}), O(2))$-module.

A Whittaker model for $(\pi, V)$ is a realization of $(\pi, V)$ as a space of smooth functions $W: \text{GL}_2(\mathbb{R}) \to \mathbb{C}$ s.t.

(i) $W((\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}) g) = e^{\infty(x)} W(g)$ $\forall g \in \text{GL}_2(\mathbb{R}), x \in \mathbb{R},$

(ii) $W(g) = O(y^N)$ for some $N$, where

$$g = (\begin{smallmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{smallmatrix})(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})(\begin{smallmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{smallmatrix}),$$

with $\text{GL}_2(\mathbb{R})$ acting via right multiplication.

If $\zeta$ and $\Delta$ act as $\mu$ and $\lambda$, and $W$ is in the weight $k$ space where $(\cos \theta \sin \theta)$ acts as $e^{ik\theta}$, then (for $\det g > 1$)

$$W(g) = \zeta^\mu e^{\infty(x)} e^{ik\theta} w(y)$$

where

$$w'' + (-4\pi^2 + \frac{2nk}{y} + \frac{\lambda}{y^2})w = 0.$$
$(\mathfrak{gl}_2(\mathbb{R}), O(2))$-modules are determined by:

- The $SO(2)$-weights which appear
- The eigenvalue $\mu$ of $\mathbb{Z}$
- The eigenvalue $\lambda$ of $\Delta$
- The action of $(-1,1)$

This proves uniqueness in the following:

**Theorem**: If $(\pi, V)$ is infinite-dimensional, then it has a unique Whittaker model $\mathcal{W}(\pi)$.

The **Kirillov model** is the space

$$\mathcal{K}(\pi) = \{ \phi : \mathbb{Q}_p^* \to \mathbb{C} \mid \phi(x) = W((x,1)) \text{ for some } W \in \mathcal{W}(\pi) \}$$

Note that we still have

$$\mathcal{K}(\mathfrak{sl}_2 \otimes \pi) = \mathfrak{sl} \mathcal{K}(\pi) \quad \text{and} \quad \hat{\pi} \cong \omega^{-1} \otimes \pi.$$
Theorem (local functional equation):
Let $(\pi, V)$ be an infinite-dimensional irreducible admissible $\mathfrak{gl}_2(\mathbb{R}), O(2))$-module with central character $\omega$.
For some meromorphic function $\gamma(s, \pi)$ independent of $\phi \in \mathcal{H}(\pi)^+$
$$\gamma(s, \pi) = \gamma(s, \pi) \gamma(1-s, \omega^{-1} \pi(-1, 1) \phi).$$

For principal series,
$$\gamma(s, \pi(X_1, X_2)) = \gamma(s, X_1) \gamma(s, X_2) \quad (x_1 x_2^{-1} \neq \text{sgn}(x) x^4)$$

For discrete series,
$$\gamma(s, \sigma(X_1, X_2)) = \gamma(s, X_1) \gamma(s, X_2) \quad (x_1 x_2^{-1} = \text{sgn}(x) x^4)$$

Proposition: Suppose $\pi_1$ and $\pi_2$ have the same central character.
If $\gamma(s, \xi \otimes \pi_1) = \gamma(s, \xi \otimes \pi_2)$ for all characters $\xi$ of $\mathbb{R}^+$, then $\pi_1 \cong \pi_2$. 
The Global Functional Equation for GL₂

1) Review the local theory from last time

2) Define global Whittaker models, and deduce uniqueness

3) Recall Whittaker models of automorphic representations

4) Deduce (strong) multiplicity one

5) Prove the global functional equation (and converse theorem)

6) See what this all means for modular/Maas forms
Last time:

Let \( v \leq \infty \) be a place of \( \mathbb{Q} \).

Let \((\pi, V)\) be an infinite-dimensional irreducible admissible representation of \( GL_a (\mathbb{Q}_p) \) if \( v = p < \infty \), \((gl_a (\mathbb{R}), O(a))\)-module if \( v = \infty \).

**Thm 1:** \( \pi \cong \omega \otimes \hat{\pi} \) (where \( \omega \) is the central character)

**Thm 2:** \((\pi, V)\) has a unique Whittaker model, i.e. a realization of \((\pi, V)\) as a space \( W(\pi) \) of smooth functions \( W: GL_a (\mathbb{Q}_v) \rightarrow \mathbb{C} \) s.t.
\[
W((1, x)g) = e_v(x) W(g)
\]
and \( W((y, 1)) = O(y^N) \) for some \( N \) if \( v = \infty \).

The Kirillov model is \( K(\pi) = \{ W_{(x, 1)} | W \in W(\pi) \} \).

Note: \( C_0^\infty (\mathbb{Q}_p^*) \subseteq K(\pi) \) if \( v = p < \infty \).
For $\Phi \in K(\pi)$, we define $\zeta_v(s, \Phi) = \int_{Q_v^*} \Phi(x)|x|^{s-1/2}dx$.

If $v = p < \infty$, $x_1$ and $x_2$ are unramified, and $\Phi^0$ is the $GL_d(\mathbb{Z}_p)$-fixed vector in $\Pi_p(x_1, x_2)$, then

$$\zeta_p(s, \Phi^0) = \zeta_p(s, \Pi_p(x_1, x_2)) = \frac{1}{(1-x_1(p))(1-x_2(p))}$$

**Thm 3:** For some meromorphic $\gamma_v(s, \pi)$,

$$\zeta_v(s, \Phi) = \gamma_v(s, \pi)\zeta(1-s, \omega^{-1}\Pi(-, \cdot)\Phi) \quad \forall \Phi \in K(\pi).$$

Specifically,

$$\gamma_v(s, \Pi_v(x_1, x_2)) = \gamma_v(s, x_1)\gamma_v(s, x_2)$$

$$\gamma_v(s, \sigma_v(x_1, x_2)) = \gamma_v(s, x_1)\gamma_v(s, x_2)$$

**Thm 4:** If $\Pi_1$ and $\Pi_2$ have the same central character and $\gamma_v(s, \xi \otimes \Pi_1) = \gamma_v(s, \xi \otimes \Pi_2)$ for all characters $\xi$ of $\mathbb{Q}_v^*$, then $\Pi_1 \cong \Pi_2$. 
Let \((\pi, V)\) be an irreducible admissible 
\((\mathfrak{gl}_2(\mathbb{R}), O(2)) \times \text{GL}_2(\mathbb{A}_f)\)-module

By the tensor product theorem, \(\pi \cong \hat{\otimes} \pi_v\) with each \(\pi_v\) irreducible admissible, and \(\pi_v\) spherical for almost all \(v\).

**Thm 1:** Let \(\hat{\pi} = \otimes \hat{\pi}_v\).

If \(\pi\) has central character \(\omega\). Then, \(\hat{\pi} \cong \omega^{-1} \otimes \pi\)

If \(\pi \subset \mathcal{A}_0(\text{GL}_2(\mathbb{Q}) \backslash \text{GL}_2(\mathbb{A}), \omega)\),
then \(\hat{\pi} \subset \mathcal{A}_0(\text{GL}_2(\mathbb{Q}) \backslash \text{GL}_2(\mathbb{A}), \omega^{-1})\).

A **Whittaker model** of \(\pi\) is a realization of \(\pi\) as a space of smooth functions \(W: \text{GL}_2(\mathbb{A}) \rightarrow \mathbb{C}\) s.t.

\[
W((i, i^*)g) = e(x) W(g)
\]

\[
W((x, i)) = O(1|x|^N)\] for some \(N\).

This is the same as a Whittaker model \(W(\pi_v)\) for each \(\pi_v\), and taking \(\text{Span}\{\pi_v W_v(g_v) | W_v \in \mathcal{A}_0(\pi_v), W_v \text{ spherical a.e.} v\}\).

**Thm 2:** \(\pi\) has a Whittaker model i f f each \(\pi_v\) does, (i.e. each \(\pi_v\) is \(\infty\)-dimensional), in which case it is unique.
Let \( \phi \in A_c(GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}^f), \omega) \) be an automorphic form. For fixed \( g \in GL_2(\mathbb{A}^f) \), \( \phi((^{1\,i}_{\ast\,\ast})g) \) is a function on \( \mathbb{A}/\mathbb{Q} \), so it has a Fourier expansion

\[
\phi((^{1\,i}_{\ast\,\ast})g) = \sum_{\alpha \in \mathbb{Q}^*} \hat{\phi}_\alpha(g) e(\alpha x) = \sum_{\alpha \in \mathbb{Q}^*} \hat{\phi}_\alpha(g) e(\alpha x)
\]

where \( \hat{\phi}_\alpha(g) = \int_{\mathbb{A}/\mathbb{Q}} \phi((^{1\,i}_{\ast\,\ast})g) e(-\alpha x) \, dx \).

Note that

\[
\hat{\phi}_\alpha(g) = \int_{\mathbb{A}/\mathbb{Q}} \phi((^{\alpha\,i}_{1\,1})g) e(-\alpha x) \, dx
\]

\[
= \int_{\mathbb{A}/\mathbb{Q}} \phi((^{1\,i}_{\ast\,\ast})(^{\alpha\,1}_{1\,i})g) e(-\alpha x) \, dx
\]

\[
= \int_{\mathbb{A}/\mathbb{Q}} \phi((^{1\,i}_{\ast\,\ast})(^{\alpha\,1}_{1\,i})g) e(-x) \, dx
\]

\[
= \hat{\phi}_1((^{\alpha\,1}_{1\,i})g)
\]

Hence, \( \phi(g) = \sum_{\alpha \in \mathbb{Q}^*} \hat{\phi}_1((^{\alpha\,1}_{1\,i})g) \).

Note: \( \hat{\phi}_1((^{1\,i}_{\ast\,\ast})g) = e(\alpha) \hat{\phi}_1(g) \) and \( \hat{\phi}_1((^{\alpha\,1}_{1\,i})) = O(1_{\mathbb{A}^f}) \), so \( \hat{\phi}_1 \) is a Whittaker function!
In summary: If \((\pi, V)\) is a cuspidal automorphic representation, then \(\{\hat{\phi}, \phi \in V\}\) is a ready-made Whittaker model.

We can recover \(\phi\) from \(\hat{\phi}\), via \(\phi(g) = \sum_{\alpha \in \mathbb{Q}^*} \hat{\phi}_i (\alpha \cdot i) g\).

**Thm (Multiplicity one):** If \((\Pi^1, V^1)\) and \((\Pi^2, V^2)\) are cuspidal automorphic representations and \(\Pi^1 \cong \Pi^2\), then \(V^1 = V^2\) (inside \(\mathbb{A}_0 (GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}))\)).

**Thm (Strong multiplicity one):** If \((\Pi^1, V^1)\) and \((\Pi^2, V^2)\) are cuspidal automorphic representations and \(\Pi^1_v \cong \Pi^2_v\), except for finitely many finite \(v\), then \(V^1 = V^2\).

**Proof:** It suffices to show \(V^1 \cap V^2 \neq \emptyset\).

Define \(W^1 = \bigcap_v \Pi W^1_v\) and \(W^2 = \bigcap_v \Pi W^2_v\) so that

(i) \(W^i_v \in \mathcal{H}(\Pi^i)\),

(ii) \(W^1_v = W^2_v\) if \(\Pi^1_v \cong \Pi^2_v\)

(iii) \(W^i_v ((x, 1)) = 1_{\mathbb{Z}_p^*} (x)\) if \(\Pi^1_v \not\cong \Pi^2_v\) and

(iv) \(W^i_v\) is spherical for almost all \(v\).
Let $\phi^i(g) = \sum_{a \in \mathfrak{a}} W^i(a, g) \in V^i$. Then, $\phi^1(g) = \phi^3(g)$ for some open compact $K_0 \subset GL_2(\mathbb{A}_f)$. For $g \in GL_2(\mathbb{Q})(\mathbb{A}_f) \backslash GL_2(\mathbb{R})$, $K_0 = GL_2(\mathbb{A})$.

**Thm (Very strong multiplicity one):** If $(\Pi^1, V^1)$ and $(\Pi^3, V^3)$ are automorphic cuspidal representations and $\Pi^1_v \cong \Pi^3_v$ for all but finitely many $v$, then $V^1 = V^3$.

For example, define $\alpha_1(p), \alpha_2(p) \in \mathbb{C}$ for $p \notin S$ ($S$ finite). We can ask, “Is there a cuspidal automorphic representation $\Pi$ such that for $p \notin S$, $\Pi_p$ is unramified with parameters $\alpha_1(p), \alpha_2(p)$?”

Automorphic rep. of $GL_2$ $\xrightarrow{\text{operation}}$ Automorphic rep. of $GL_n$

$\alpha_1(p), \alpha_2(p)$ for $p \notin S$ $\xrightarrow{\text{some operation}}$ $\beta_1(p), \ldots, \beta_n(p)$ for $p \notin S$
Let \((\pi, V)\) be an irreducible admissible unitarizable \((\mathfrak{g}\mathfrak{l}_2(\mathbb{R}), \mathbb{O}(2)) \times \mathbb{G}L_2(\mathbb{A}_f)\)-module with a Whittaker model.

Given \(W \in \mathcal{W}(\pi)\), let \(S(s, W) = \int_{\mathbb{A}_f^*} W((* \; 1)) \, l(x) \, x^{s - \frac{1}{2}} \, dx\).

If \(W = T \pi W_v\), then \(S(s, W) = \prod_v S_v(s, W_v)\) \(\mid (*)_{\lambda}\)

\(S_v(s, W_v)\) converges for \(\text{Res} > \frac{1}{2}\), and \(S_v(s, W_v) = L(s, \pi_v)\) a.e. \(v\), so the product converges for \(\text{Res} > \frac{3}{2}\).

Suppose \((\pi, V)\) is automorphic, so \(W = \hat{\phi}\), for some automorphic form \(\phi\).

Then, \(S(s, W) = \int_{\mathbb{A}_f^* / \mathbb{Q}_f^*} \sum_{\alpha \in \mathbb{Q}_f^*} W((\alpha \; 1)(* \; 1)) \, l(x) \, x^{s - \frac{1}{2}} \, dx\)

\(= \int_{\mathbb{A}_f^* / \mathbb{Q}_f^*} \phi((* \; 1)) \, l(x) \, x^{s - \frac{1}{2}} \, dx\)

\(= \int_{\mathbb{A}_f^* / \mathbb{Q}_f^*} \phi((-1 \; \prime)(* \; 1)) \, l(x) \, x^{s - \frac{1}{2}} \, dx\)

\(= \int_{\mathbb{A}_f^* / \mathbb{Q}_f^*} \phi((x \; 1)(-1 \; \prime)) \, l(x) \, x^{s - \frac{1}{2}} \, dx\)

\(= \int_{\mathbb{A}_f^* / \mathbb{Q}_f^*} \phi((x^{-1} \; 1)(-1 \; \prime)) \, \omega(x) \, l(x) \, x^{s - \frac{1}{2}} \, dx\)

\(\xrightarrow{x \rightarrow x^{-1}}\)

\(= \int_{\mathbb{A}_f^* / \mathbb{Q}_f^*} \phi((\prime \; x)(-1 \; \prime)) \, l(x) \, x^{s - \frac{1}{2}} \, dx\)

\(= S(1 - s, \omega^{-1}\pi(-1 \; \prime)W)\).
Altogether, we have shown
\[
\text{global F.E. } S(s, W) = S(1-s, \omega^{-1} \pi(\cdot, \cdot)^W)
\]
\[
\text{local F.E. } \prod_v S_v(s, W_v) = \prod_v \gamma_v(s, \pi_v) S_v(1-s, \omega^{-1} \pi_v(\cdot, \cdot)^W_v)
\]
Thus, \(\prod_v \gamma_v(s, \pi_v) = 1\).

More palatably, if \(\pi_v\) is unramified and \(W_v\) is spherical for \(v \in S\), and if \(L_S(s, \pi) = \prod_{v \in S} L(s, \pi_v)\), then
\[
L_S(1-s, \hat{\pi}) = \prod_{v \in S} \gamma_v(s, \pi_v) L_S(s, \pi)
\]
We also have a converse theorem.

**Thm:** \(\pi\) is automorphic iff for every Hecke character \(\xi\),

1. \(L_S(s, \xi \otimes \pi)\) extends to an entire function,
   bounded in vertical strips, and
   \[
   (\cdot, \cdot)\xi(\alpha) = \sum_{\gamma \in \Gamma \setminus \Gamma_0^\infty} \gamma \xi^{-1}(\gamma \alpha)\xi(\gamma)\]
   is an automorphic form.
2. Automorphy with respect to \((\alpha, i), (1, i)\) and \((\tau, \epsilon)\) are clear.
   It suffices to show \((-1, 1)\phi = \phi\). This is the same as
   Thm 4 from last time \(\gamma_v(s, \xi \otimes \pi_v)\) determines \(\pi_v\).

\[\square\]
Let $\mathcal{C}^\infty(\text{PGL}_2(\mathbb{R})^+, k)$ consist of those $F \in \mathcal{C}^\infty(\text{GL}_2(\mathbb{R})^+)$ such that $F(ugk_0) = e^{ik\theta}F(g)$ for $u \in \mathbb{R}^\times$, $g \in \text{GL}_2(\mathbb{R})^+$, $k_0 = (\cos \theta, \sin \theta) \in \text{SO}(2)$.

Let $\mathcal{H} = \{x+iy \mid y > 0 \}$. $\text{GL}_2(\mathbb{R})^+ \ltimes \mathcal{H}$ via $(a \, b) z = \frac{az + b}{cz + d}$.

The stabilizer of $i \in \mathcal{H}$ is $\mathbb{R}^\times \text{SO}(2)$, so $\text{PGL}_2(\mathbb{R})^+/\text{SO}(2) \sim \mathcal{H}$, and we can interpret $\mathcal{C}^\infty(\text{PGL}_2(\mathbb{R})^+, k)$ as functions on $\mathcal{H}$:

\[
\mathcal{C}^\infty(\text{PGL}_2(\mathbb{R})^+, k) \leftrightarrow \mathcal{C}^\infty(\mathcal{H})
\]

\[
F \quad \mapsto \quad f(x+iy) = F((y^{1/2}, xy^{-1/2}))
\]

\[
F(g) = (\frac{c+id}{ic+di})^{-k} f(gi) \leftrightarrow f
\]

$\mathcal{C}^\infty(\text{PGL}_2(\mathbb{R})^+, k)$ has a right $\text{GL}_2(\mathbb{R})^+$-action via left multiplication. On $\mathcal{C}^\infty(\mathcal{H})$, this corresponds to

\[
(f \mid_{k} g)(z) = (\frac{cz+d}{icz+di})^{-k} f(gz).
\]

The condition $LF = 0$ corresponds to $y^{-k/2}f$ being holomorphic.

On holomorphic functions, the action looks like

\[
(f \mid_{k} g)(z) = (\text{det}g)^{k/2} (cz+d)^{-k} f(gz).
\]
Suppose $F$ is a Maass form of weight $k$, eigenvalue $\lambda$, and Nebentypus $\chi$ for $\Gamma_0(N)$, so
\[
\Delta F = \lambda F \quad \text{and} \\
\gamma \lambda \chi = \chi(d) F \quad \text{for} \quad \gamma = (a \ b \ c \ d) \in \Gamma_0(N).
\]
Then, $F(g) = (\gamma \lambda \chi(g))$ satisfies
\[
\Delta F = \lambda F \quad \text{and} \\
F(\gamma g k_\theta) = e^{ik\theta} \chi(\gamma) F(g) \quad \text{for} \quad \gamma \in \Gamma_0(N), \quad \theta \in \mathbb{R}, \quad k_\theta \in SO(2).
\]
We define $\tilde{f} \in \mathcal{A}_0(\text{GL}_2(\mathbb{Q}) \backslash \text{GL}_2(\mathbb{A}), \tilde{\chi})$ by
\[
\tilde{f}(\gamma g_\infty k_\theta) = (\gamma \lambda \chi(g_\infty)) \chi(d \pmod{N})^{-1}
\]
for $\gamma \in \text{GL}_2(\mathbb{Q})$, $g_\infty \in \text{GL}_2(\mathbb{R})^+$, $k_\theta = (a \ b) \in K_0(N) = \{g \in \text{GL}_2(\mathbb{Z}) \mid g \equiv (\ast \ast) \pmod{N}\}$.
Suppose \( T_p \mathcal{F} = \alpha_p \mathcal{F} \ \forall \ p \nmid N \), and \((\mathbb{T}, \mathcal{V}) = \bigotimes \mathbb{T}_v \) is an automorphic representation with \( \mathcal{F} \mid \mathcal{V} \neq 0 \).

If \( p \nmid N \), \( \mathbb{T}_p \) is unramified, with parameters \( \alpha, \alpha_2 \) the roots of \( X^2 - \rho^{-n} \alpha_p + x(p) \).

By strong multiplicity one, there is only one choice of \( \mathcal{V} \), so \( \mathcal{F} \in \mathcal{V} \!\!\!\!\!\!\!\!

If \( S = \{ p \mid p \nmid N \} \), then

\[ L_S(s, \mathbb{T}) = \prod_{p \mid N} \frac{1}{(1 - \alpha_1 p^{-s})(1 - \alpha_2 p^{-s})} = \prod_{p \mid N} \frac{1}{1 - \alpha p^{-s} - x(p) p^{-2s}} \]

has an analytic continuation and functional equation.
Example: Let $f \in S_k(\text{SL}_2(\mathbb{Z}))$ be a Hecke eigenform.

$\tilde{f}$ is in the automorphic representation $\Pi = \bigotimes \Pi_v$

where $\Pi_\infty = \sigma(sgn^k 1.1_{\frac{k-1}{2}}, 1.1_{\frac{-k}{2}}) = \mathfrak{D}_0(k)$

$\Pi_p = \Pi(x_1, x_2)$ with $x_i$ unramified

and if $\alpha_i = x_i(p)$, then $\alpha_1, \alpha_2$ are the roots of $X^2 - p^{\frac{k}{2}} a_p + 1$.

For $S = \{\infty\}$, $L_S(\Pi) = \prod_{p < \infty} \frac{1}{1 - a_p p^{-s + k/2} + p^{-2s}} = L(s + \frac{k-1}{2}, f)$

and $\gamma_\infty(s, \Pi_\infty) = i^k (2\pi)^{1-2s} \frac{\Gamma(s + \frac{k-1}{2})}{\Gamma(1 - (s + \frac{k-1}{2}))}$

so

$\Lambda(s, f) = i^k \Lambda(k - s, f)$

where $\Lambda(s, f) = (2\pi)^{-s} \Gamma(s) L(s, f)$,

as expected.