The Tate Thesis

aka Automorphic Forms for GL_1

The plan

1) Define the Schwartz space of functions to integrate
2) Figure out the local characters of $\mathbb{Q}_v^\times$
3) Lift classical Dirichlet characters to characters of $\mathbb{Q}^\times \backslash \mathbb{A}^\times$
4) Define a global zeta function and prove a global functional equation
5) Factorize the global zeta function as a product of local zeta functions
6) Prove local functional equations
7) See what this all means for classical L-functions of Dirichlet characters

$\mathbb{A} = \mathbb{A}_{\mathbb{Q}} = \prod_{v \leq \infty} \mathbb{Q}_v \quad (\mathbb{Q}_\infty = \mathbb{R})$

Our goal: Fourier analysis on $\mathbb{A}$

We fix an additive character of $\mathbb{A}$.

$$e_v : \mathbb{Q}_v \to \mathbb{C}^\times, \quad e_v(x) = \begin{cases} e^{-2\pi i \{x\}} & \text{if } v < \infty \\ e^{2\pi i x} & \text{if } v = \infty \end{cases}$$

$$e : \mathbb{A} \to \mathbb{C}^\times, \quad e(x) = \prod_v e_v(x_v)$$

Note e is trivial on $\mathbb{Q}$

These induce isomorphisms (where $\hat{G} = \mathrm{Hom}(G, \mathbb{C}^\times)$)

$$\mathbb{Q}_v \xrightarrow{\sim} \hat{\mathbb{Q}}_v, \quad a \mapsto (x \mapsto e_v(ax))$$

$$\mathbb{A} \xrightarrow{\sim} \hat{\mathbb{A}}, \quad a \mapsto (x \mapsto e(ax))$$

The latter induces $\mathbb{Q} \xrightarrow{\sim} \widehat{\mathbb{A}/\mathbb{Q}}$.

We equip $\mathbb{R}$, $\mathbb{Q}_p$, and $\mathbb{A}$ with the additive Haar measures $\underset{\text{(as usual)}}{dx_\infty}$, $\underset{(\mathbb{Z}_p \text{ has measure } 1)}{dx_p}$, $dx = \prod_v dx_v$

We equip $\mathbb{R}^\times$, $\mathbb{Q}_p^\times$, $\mathbb{A}^\times$ with the multiplicative Haar measures $d^\times x_\infty = \frac{dx_\infty}{|x_\infty|_\infty}$, $\underset{(\mathbb{Z}_p^\times \text{ has measure } 1)}{d^\times x_p} = \frac{1}{1-p^{-1}} \frac{dx_p}{|x_p|_p}$,

$$d^\times x = \prod_v d^\times x_v$$

We will define the Fourier transform for functions in the <u>Schwartz space</u>

For $\mathbb{R}$, the Schwartz space $S(\mathbb{R})$ is

$$\left\{\text{smooth functions } \Phi_\infty \ \mathbb{R}\to\mathbb{C} \text{ s.t. } \sup_{x\in\mathbb{R}} \left|x^a \frac{d^b}{dx^b} f(x)\right| < \infty \ \forall a,b \geq 0\right\}$$

For $\mathbb{Q}_p$, the Schwartz space $S(\mathbb{Q}_p)$ is

$$\left\{\text{locally constant functions } \Phi_p \ \mathbb{Q}_p\to\mathbb{C} \text{ with compact support}\right\}$$

For $\mathbb{A}$, the Schwartz space $S(\mathbb{A})$ is

$$\left\{\begin{array}{l}\text{finite linear combinations of functions of the form} \\ \Phi(x)=\prod_v \Phi_v(x_v) \text{ with } \Phi_v \in S(\mathbb{Q}_v),\ \Phi_p = \mathbb{1}_{\mathbb{Z}_p} \text{ a.e.}\end{array}\right\}$$

<u>Def</u>: For $\Phi \in S(\mathbb{A})$, the <u>Fourier transform</u> of Φ is

$$\hat{\Phi}(x) = \int_{\mathbb{A}} \Phi(y)\, e(-xy)\, dy$$

(and similarly for $\mathbb{Q}_v$)

Note.

1) For $v=\infty$, this is the usual Fourier trasform
2) $\hat{\mathbb{1}}_p = \mathbb{1}_p$
3) If $\Phi = \prod_v \Phi_v$, then $\hat{\Phi} = \prod_v \hat{\Phi}_v$

Thm (Fourier inversion) For any $\Phi \in S(\mathbb{A})$, $\hat{\Phi} \in S(\mathbb{A})$, and $\hat{\hat{\Phi}}(x) = \Phi(-x)$

Thm (Poisson summation) For any $\Phi \in S(\mathbb{A})$,

$$\sum_{\alpha \in \mathbb{Q}} \Phi(\alpha t) = \frac{1}{|t|} \sum_{\alpha \in \mathbb{Q}} \hat{\Phi}\left(\frac{\alpha}{t}\right) \quad \forall t \in \mathbb{A}^\times$$

Proof

The Fourier transform of $x \mapsto \Phi(tx)$ is $x \mapsto \frac{1}{|t|}\hat{\Phi}\left(\frac{x}{t}\right)$, so assume $t = 1$

$F(x) = \sum_{\alpha \in \mathbb{Q}} \Phi(\alpha + x)$ is a function on $\mathbb{A}/\mathbb{Q}$, so it has a Fourier expansion $F(x) = \sum_{\beta \in \mathbb{Q}} c_\beta e(\beta x)$

We compute
$$\begin{aligned} c_\beta &= \int_{\mathbb{A}/\mathbb{Q}} F(x) e(-\beta x)\, dx \\ &= \int_{\mathbb{A}/\mathbb{Q}} \sum_{\alpha \in \mathbb{Q}} \Phi(x+\alpha)\, e(-\beta x)\, dx \\ &= \int_{\mathbb{A}} \Phi(x) e(-\beta x)\, dx \\ &= \hat{\Phi}(\beta), \end{aligned}$$

so $F(x) = \sum_{\beta \in \mathbb{Q}} \hat{\Phi}(\beta) e(\beta x)$ Set $x = 0$ $\square$

<u>Def</u> A <u>Hecke character</u> is a continuous unitary character $\omega\ \mathbb{A}^\times \to \mathbb{C}^\times$ which is trivial on $\mathbb{Q}^\times$

A Hecke character ω factorizes as $\prod_v \omega_v$ with ω_v a continuous unitary character on $\mathbb{Q}_v^\times$, so we need to

(a) Understand continuous unitary characters on $\mathbb{Q}_v^\times$

(b) Understand when they patch together to be trivial on $\mathbb{Q}^\times$

For $\mathbb{R}^\times$, continuous unitary characters are

$|x|_\infty^{it}$ and $\operatorname{sgn}(x)\,|x|_\infty^{it}$ for $t \in \mathbb{R}$

For $\mathbb{Q}_p^\times$, we say ω_p is <u>unramified</u> if $\omega_p|_{\mathbb{Z}_p^\times}$ is trivial

If ω_p is unramified, it is determined by $\omega_p(p)$

If ω_p is ramified, it is trivial on $1+p^k\mathbb{Z}_p$ for some k (the least such p^k is the conductor), and then

$$\omega_p(p^m(j+p^k\mathbb{Z}_p)) = \omega_p(p)^m\,\chi(j)$$

for some Dirichlet character χ on $(\mathbb{Z}/p^k\mathbb{Z})^\times$

Let χ be a primitive Dirichlet character on $(\mathbb{Z}/N\mathbb{Z})^\times$

We will lift χ to a Hecke character $\tilde{\chi} = \prod_v \tilde{\chi}_v$ on $\mathbb{A}^\times$

If $M \| N$, write $\chi^{(M)}$ for the following Dirichlet character on $(\mathbb{Z}/M\mathbb{Z})^\times$

$$(\mathbb{Z}/M\mathbb{Z})^\times \hookrightarrow (\mathbb{Z}/M\mathbb{Z})^\times \times (\mathbb{Z}/\tfrac{N}{M}\mathbb{Z})^\times = (\mathbb{Z}/N\mathbb{Z})^\times \xrightarrow{\chi} \mathbb{C}^\times$$

$\tilde{\chi}_\infty$ is the character $\tilde{\chi}_\infty(x_\infty) = \chi(\operatorname{sgn}(x_\infty))$ on $\mathbb{R}^\times$

If $p \nmid N$, $\tilde{\chi}_p$ is the unramified character on $\mathbb{Q}_p^\times$ with $\tilde{\chi}_p(p) = \chi(p)$

If $p^k \| N$ (with $k \geq 1$), $\tilde{\chi}_p$ is the ramified character of conductor p^k on $\mathbb{Q}_p^\times$ with

$$\tilde{\chi}_p(p) = \chi^{(N/p^k)}(p),$$
$$\tilde{\chi}_p(u) = \chi^{(p^k)}(u \mod 1+p^k\mathbb{Z}_p)^{-1} \quad \text{if } u \in \mathbb{Z}_p^\times$$

<u>Claim</u> $\tilde{\chi} = \prod_v \tilde{\chi}_v$ is a Hecke character (ie trivial on $\mathbb{Q}^\times$)

<u>Lma</u> Every Hecke character is uniquely of the form $\tilde{\chi}(x)|x|^{it}$ for χ a primitive Dirichlet character and $t \in \mathbb{R}$

<u>Proof</u> Consider the isomorphism

$$\mathbb{A}^\times \xrightarrow{\sim} \mathbb{Q}^\times \times \hat{\mathbb{Z}}^\times \times \mathbb{R}_+^\times$$
$$x \longmapsto \left(\frac{x_\infty}{|x|}, \frac{|x|}{x_\infty} x_{\text{finite}}, |x|\right)$$

$\square$

Given a Hecke character ω and $\Phi \in S(\mathbb{A})$, we define

$$\zeta(s,\omega,\Phi) = \int_{\mathbb{A}^\times} \Phi(x)\,\omega(x)\,|x|^s\,d^\times x$$

Since $\zeta(s,\omega|\,|^{it},\Phi) = \zeta(s+it,\omega,\Phi)$, we may assume ω is a lift of a Dirichlet character, so we write χ instead of ω

<u>Thm (Global functional equation)</u> $\zeta(s,\chi,\Phi)$ has a meromorphic continuation to all of $\mathbb{C}$, entire unless $\chi=1$, in which case the poles are at $s=0,1$ Moreover,

$$\zeta(s,\chi,\Phi) = \zeta(1-s,\bar{\chi},\hat{\Phi})$$

<u>Proof</u> Write

$$\zeta(s,\chi,\Phi) = \underbrace{\int_{\substack{\mathbb{A}^\times \\ |x|>1}} \Phi(x)\,\chi(x)\,|x|^s\,d^\times x}_{\zeta_1(s,\chi,\Phi)} + \underbrace{\int_{\substack{\mathbb{A}^\times \\ |x|<1}} \Phi(x)\,\chi(x)\,|x|^s\,d^\times x}_{\zeta_0(s,\chi,\Phi)}$$

ζ_1 is entire, but ζ_0 only converges for $\operatorname{Re} s \gg 0$

$$\zeta_0(s,\chi,\Phi) = \int_{\substack{\mathbb{A}^\times/\mathbb{Q}^\times \\ |x|<1}} \Big(\sum_{\alpha\in\mathbb{Q}^\times} \Phi(\alpha x)\Big)\,\chi(x)\,|x|^s\,d^\times x$$

$$= \int_{\substack{\mathbb{A}^\times/\mathbb{Q}^\times \\ |x|<1}} \Big(\sum_{\alpha\in\mathbb{Q}} \Phi(\alpha x)\Big)\,\chi(x)\,|x|^s\,d^\times x - \Phi(0)\int_{\substack{\mathbb{A}^\times/\mathbb{Q}^\times \\ |x|<1}} \chi(x)\,|x|^s\,d^\times x$$

$$= \int_{\substack{\mathbb{A}^\times/\mathbb{Q}^\times \\ |x|<1}} \Big(\sum_{\alpha\in\mathbb{Q}} \hat{\Phi}\big(\tfrac{\alpha}{x}\big)\Big)\,\chi(x)\,|x|^{s-1}\,d^\times x - \mathbb{1}_{\chi=1}\,\frac{\Phi(0)}{s}$$

$$= \int_{\substack{\mathbb{A}^\times/\mathbb{Q}^\times \\ |x|<1}} \Big(\sum_{\alpha\in\mathbb{Q}^\times} \hat{\Phi}\big(\tfrac{\alpha}{x}\big)\Big)\,\chi(x)\,|x|^{s-1}\,d^\times x + \mathbb{1}_{\chi=1}\Big(\frac{\hat{\Phi}(0)}{s-1} - \frac{\Phi(0)}{s}\Big)$$

$$= \int_{\substack{\mathbb{A}^\times/\mathbb{Q}^\times \\ |x|>1}} \Big(\sum_{\alpha \in \mathbb{Q}^\times} \hat{\Phi}(\alpha x) \Big) \overline{\chi}(x) |x|^{1-s} d^\times x + \mathbb{1}_{\chi=1} \left(\frac{\hat{\Phi}(0)}{s-1} - \frac{\Phi(0)}{s} \right)$$

$$= \zeta_1(1-s, \overline{\chi}, \hat{\Phi}) + \mathbb{1}_{\chi=1} \left(\frac{\hat{\Phi}(0)}{s-1} - \frac{\Phi(0)}{s} \right)$$

Altogether,

$$\zeta(s, \chi, \Phi) = \zeta_1(s, \chi, \Phi) + \zeta_1(1-s, \overline{\chi}, \hat{\Phi}) + \mathbb{1}_{\chi=1} \left(\frac{\hat{\Phi}(0)}{s-1} - \frac{\Phi(0)}{s} \right)$$

This is meromorphic and invariant under

$$s \to 1-s, \quad \chi \to \overline{\chi}, \quad \Phi \to \hat{\Phi} \qquad \square$$

What does this mean for L-functions?

Let us choose a specific $\Phi = \prod_v \Phi_v$ and see

We have $\zeta(s, \chi, \Phi) = \prod_v \zeta_v(s, \chi_v, \Phi_v)$

where $\zeta_v(s, \chi_v, \Phi_v) = \int_{\mathbb{Q}_v^\times} \Phi_v(x_v) \chi_v(x_v) |x_v|_v^s d^\times x_v$

Therefore,

$$\boxed{\prod_v \zeta_v(s, \chi_v, \Phi_v) = \prod_v \zeta_v(1-s, \overline{\chi}_v, \hat{\Phi}_v)}$$

We will choose Φ_v depending on χ_v

For example, if $v=p<\infty$ and χ_p is unramified, choose $\Phi_p = \mathbb{1}_{\mathbb{Z}_p}$

We have
$$\begin{aligned}
\zeta_p(s,\chi_p,\mathbb{1}_{\mathbb{Z}_p}) &= \int_{\mathbb{Q}_p^\times} \mathbb{1}_{\mathbb{Z}_p}(x)\chi_p(x)|x_p|_p^s \, d^\times x_p \\
&= \int_{\mathbb{Z}_p-\{0\}} \chi_p(x)|x_p|_p^s \, d^\times x_p \\
&= \sum_{k=0}^{\infty} \int_{p^k\mathbb{Z}_p^\times} \chi_p(x)|x_p|_p^s \, d^\times x_p \\
&= \sum_{k=0}^{\infty} \chi_p(p)^k p^{-ks} \int_{p^k\mathbb{Z}_p^\times} d^\times x_p \\
&= \sum_{k=0}^{\infty} \chi_p(p)^k p^{-ks} \\
&= \frac{1}{1-\chi_p(p)p^{-s}}
\end{aligned}$$

Since $\hat{\mathbb{1}}_{\mathbb{Z}_p} = \mathbb{1}_{\mathbb{Z}_p}$, $\zeta_p(1-s,\overline{\chi}_p,\hat{\mathbb{1}}_{\mathbb{Z}_p}) = \dfrac{1}{1-\overline{\chi}_p(p)\,p^{-(1-s)}}$

In general, we have the following

v	χ_v	$\Phi_v(x)$	$\hat{\Phi}_v$	$\zeta_v(s,\chi_v,\Phi_v)$
∞	1	$e^{-\pi x^2}$	Φ_∞	$\pi^{-s/2}\Gamma(s/2)$
∞	$\mathrm{sgn}(x)$	$x e^{-\pi x^2}$	$-i\Phi_\infty$	$\pi^{-\frac{s+1}{2}}\Gamma\left(\frac{s+1}{2}\right)$
$p<\infty$	unramified	$\mathbb{1}_{\mathbb{Z}_p}(x)$	Φ_p	$\dfrac{1}{1-\chi_p(p)p^{-s}}$
$p<\infty$	ramified with conductor p^k	$e_p(x)\mathbb{1}_{p^{-k}\mathbb{Z}_p}(x)$	$p^k\mathbb{1}_{1+p^k\mathbb{Z}_p}$	$\dfrac{p^{k(s-1)}\chi_p(p)^{-k}}{1-p^{-1}}\sum_{u\in(\mathbb{Z}/p^k\mathbb{Z})^\times}\chi_p(u)e^{-2\pi i\frac{u}{p^k}}$

$*$ In the last row, $\zeta_p(1-s,\overline{\chi}_p,\hat{\Phi}_p) = \dfrac{1}{1-p^{-1}}$

Putting this all together, we have shown that if χ is a primitive Dirichlet character modulo N, then

$$\Lambda(s,\chi) = \frac{\tau(\chi)}{i^{\varepsilon}\sqrt{N}}\Lambda(1-s,\overline{\chi})$$

where $\Lambda(s,\chi) = \left(\frac{\pi}{N}\right)^{-\frac{s+\varepsilon}{2}}\Gamma\left(\frac{s+\varepsilon}{2}\right)L(s,\chi)$

$$L(s,\chi) = \prod_{p\nmid N}\frac{1}{1-\chi(p)}$$

$$\tau(\chi) = \sum_{j\in(\mathbb{Z}/N\mathbb{Z})^{\times}}\chi(j)e^{2\pi i\frac{j}{N}}$$

$$\varepsilon = \begin{cases} 0 & \text{if } \chi(-1)=1 \\ 1 & \text{if } \chi(-1)=-1 \end{cases}$$

What if we had chosen Φ differently?

It turns out it doesn't matter!

<u>Thm (Local functional equation)</u> There is a meromorphic function $\gamma_v(s,\chi_v)$ independent of Φ_v s.t.

$$\zeta_v(s,\chi_v,\Phi_v)=\gamma_v(s,\chi_v)\,\zeta_v(1-s,\overline{\chi}_v,\hat{\Phi}_v)$$

for all $\Phi_v\in S(\mathbb{Q}_v)$

<u>Proof</u> We must show that $\zeta_v(s,\chi_v,\Phi_v)\zeta_v(1-s,\overline{\chi}_v,\hat{\Psi}_v)$ is symmetric in Φ_v and Ψ_v

$$\zeta_v(s,\chi_v,\Phi_v)\zeta_v(1-s,\overline{\chi}_v,\hat{\Psi}_v)$$

$$=\int_{\mathbb{Q}_v^\times}\int_{\mathbb{Q}_v^\times}\Phi_v(x)\chi_v(x)|x|_v^s\,\hat{\Psi}_v(y)\overline{\chi}_v(y)|y|_v^{1-s}\,d^\times y\,d^\times x$$

$$=\int_{\mathbb{Q}_v^\times}\int_{\mathbb{Q}_v^\times}\Phi_v(x)\hat{\Psi}_v(y)\chi_v(xy^{-1})|x|_v^s|y|_v^{1-s}\,d^\times y\,d^\times x$$

$$=\int_{\mathbb{Q}_v^\times}\int_{\mathbb{Q}_v^\times}\Phi_v(x)\hat{\Psi}_v(ux)\chi_v(u^{-1})|x|_v|u|_v^{1-s}\,d^\times u\,d^\times x \qquad (u=\tfrac{y}{x})$$

$$=\int_{\mathbb{Q}_v^\times}\int_{\mathbb{Q}_v^\times}\int_{\mathbb{Q}_v}\Phi_v(x)\Psi_v(z)e_v(-uxz)\chi_v(u^{-1})|x|_v|u|_v^{1-s}\,dz\,d^\times u\,d^\times x$$

$$=\int_{\mathbb{Q}_v}\int_{\mathbb{Q}_v^\times}\int_{\mathbb{Q}_v}\Phi_v(x)\Psi_v(z)e_v(-uxz)\chi_v(u^{-1})|u|_v^{1-s}\,dz\,d^\times u\,dx$$

$$\begin{cases}\frac{p}{p-1} & \text{if } v=p<\infty\\ 1 & \text{if } v=\infty\end{cases}$$

Since $\gamma_v(s,\chi_v)$ is independent of Φ_v, we can compute it explicitly using our choice of Φ_v from earlier

If $v=\infty$ and $\chi_\infty(x)=1$,

$$\gamma_\infty(s,\chi_\infty)=\frac{\pi^{-s/2}\Gamma(s/2)}{\pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)}$$

If $v=\infty$ and $\chi_\infty(x)=\operatorname{sgn}(x)$,

$$\gamma_\infty(s,\chi_\infty)=i\frac{\pi^{-\frac{s+1}{2}}\Gamma\left(\frac{s+1}{2}\right)}{\pi^{-\frac{(1-s)+1}{2}}\Gamma\left(\frac{(1-s)+1}{2}\right)}$$

If $v=p<\infty$ and χ_p is unramified,

$$\gamma_p(s,\chi_p)=\frac{1-\overline{\chi_p(p)}\,p^{-(1-s)}}{1-\chi_p(p)\,p^{-s}}$$

If $v=p<\infty$ and χ_p is ramified with conductor p^k,

$$\gamma_p(s,\chi_p)=p^{k(s-1)}\chi_p(p)^{-k}\sum_{j\in(\mathbb{Z}/p^k\mathbb{Z})^\times}\chi_p(j)\,e^{-2\pi i\frac{j}{p^k}}$$

In particular, $\gamma_v(s,\chi_v)$ is always meromorphic $\square$

The plan going forward

1) Define the ~~Schwartz space~~ Kirillov model of functions to integrate

2) Figure out the local ~~characters of $\mathbb{Q}_v^\times$~~ representations of $GL_2(\mathbb{Q}_v)$

3) Lift classical ~~Dirichlet characters~~ modular/Maass forms to ~~characters of $\mathbb{Q}^\times \backslash \mathbb{A}^\times$~~ automorphic forms on $GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A})$

4) Define a global zeta function and prove a global functional equation

5) Factorize the global zeta function as a product of local zeta functions (via the tensor product theorem)

6) Prove local functional equations

7) See what this all means for classical L-functions of ~~Dirichlet characters~~ modular/Maass forms

The Local Functional Equation for GL_2

1) Review theory of admissible representations of $GL_2(\mathbb{Q}_p)$
   - Contragradient representations ($\hat{\pi} \cong \omega^{-1} \otimes \pi$)
   - Whittacker/Kirillov models (existence & uniqueness)
   - Jacquet modules
2) Review classification of irreducible admissible representations of $GL_2(\mathbb{Q}_p)$
   - Explicit Jacquet modules
   - Explicit Kirillov models
3) Define the local zeta function, and see what these zeta functions look like
4) Check that the spherical representations give the familiar L-factor
5) Prove the local functional equation
6) Explain the analogous results for $GL_2(\mathbb{R})$

A representation (π, V) is <u>admissible</u> if

(a) The stabilizer of each $v \in V$ is open

(b) The space V^K of vectors stabilized by K is finite-dimensional for each open subgroup $K \subseteq GL_2(\mathbb{Q}_p)$

Note the compact open subgroups $\{g \in GL_2(\mathbb{Z}_p) \mid g \equiv 1 \pmod{p^k}\}$ are a basis of opens at the identity

<u>Lemma (Schur)</u> If (π, V) is an irreducible admissible representation of $GL_2(\mathbb{Q}_p)$, then $\mathrm{Hom}_{GL_2(\mathbb{Q}_p)}(V, V) \cong \mathbb{C}$

<u>Proof</u> By (a), $V^K \neq 0$ for some open subgroup K

By (b), V^K is finite-dimensional

If $L \in \mathrm{Hom}_{GL_2(\mathbb{Q}_p)}(V, V)$, $L|_{V^K}$ has an eigenvalue λ

Since $\ker(L - \lambda)$ is nonzero, it must be all of V $\square$

<u>Corollary</u> If (π, V) is an irr ad rep, the center $\mathbb{Q}_p^*$ of $GL_2(\mathbb{Q}_p)$ acts via some character $\omega\ \mathbb{Q}_p^\times \to \mathbb{C}^\times$, i.e. $\pi\left(\begin{pmatrix} t & \\ & t \end{pmatrix}\right) v = \omega(t) v$

<u>Lemma</u> The finite-dimensional irr ad reps of $GL_2(\mathbb{Q}_p)$ are all of the form $\chi \circ \det$ for χ a character of $\mathbb{Q}_p^\times$

We abbreviate $\chi \circ \det$ by writing χ

Given a representation π and character χ, we can form the representation $\pi \otimes \chi$, given by

$$(\pi \otimes \chi)(g)v = \chi(\det g)\,\pi(g)v$$

<u>Def</u> Let (π, V) be an admissible rep of $GL_2(\mathbb{Q}_p)$

A linear functional $\hat{v}\ V \to \mathbb{C}$ is <u>smooth</u> if $\hat{v} \circ \pi(g) = \hat{v}$ for g in a neighborhood of the identity

The <u>contragradient</u> rep of (π, V) is $(\hat{\pi}, \hat{V})$ where $\hat{V}$ is the space of smooth linear functionals and

$$\hat{\pi}(g)\hat{v} = \hat{v} \circ \pi(g^{-1})$$

Properties

1) If (π, V) is admissible, then so is $(\hat{\pi}, \hat{V})$
2) If (π, V) is irreducible, then so is $(\hat{\pi}, \hat{V})$
3) $\hat{\hat{\pi}} \cong \pi$

<u>Theorem</u> If π is an irr ad rep of $GL_2(\mathbb{Q}_p)$ with central character ω $\mathbb{Q}_p^\times \to \mathbb{C}^\times$, then $\hat{\pi} \cong \omega^{-1} \otimes \pi$

<u>Example</u> For a one-dimensional rep χ, the central character is χ^2, and $\hat{\chi} \cong \chi^{-1} \cong (\chi^2)^{-1} \otimes \chi$

<u>Example</u> If π has central character ω, then $\omega^{-1} \otimes \pi$ has central character ω^{-1}, so $\hat{\hat{\pi}} \cong \omega \otimes \hat{\pi} \cong \omega \otimes \omega^{-1} \otimes \pi \cong \pi$

<u>Proof (for $GL_2(\mathbb{F}_p)$)</u>

We must show $\omega^{-1} \otimes \pi$ and $\hat{\pi}$ have the same character

If π has character χ, then $\hat{\pi}$ has character $g \mapsto \chi(g^{-1})$

For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{F}_q)$,

$$g^{-1} = \frac{1}{\det g}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{\det g}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} g^T \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$$

Every matrix is conjugate to its transpose, so

$$\chi(g^{-1}) = \omega(\det g)^{-1}\chi(g)$$

$\square$

Let (π, V) be an admissible representation of $GL_2(\mathbb{Q}_p)$

A <u>Whittacker model</u> for (π, V) is a realization of (π, V) as a space of functions $W\ GL_2(\mathbb{Q}_p) \to \mathbb{C}$ st

$$W\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) = e_p(x) W(g) \quad \forall g \in GL_2(\mathbb{Q}_p),\ x \in \mathbb{Q}_p,$$

with $GL_2(\mathbb{Q}_p)$ acting via right multiplication

<u>Example</u> Consider a one-dimensional representation χ

A Whittacker model for $\mathbb{C}$ would require a (nonzero) function $W\ GL_2(\mathbb{Q}_p) \to \mathbb{C}$ st

$$W\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) = e_p(x) W(g) \text{ and}$$

$$W(g) = \chi(\det g) W(1)$$

No such function exists, so χ has no Whittacker model

<u>Theorem</u> If (π, V) is an infinite-dimensional irr ad rep, then it has a unique Whittacker model

<u>Proof (Sketch for $GL_2(\mathbb{F}_p)$)</u>

Let $N = \left\{ \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \middle| x \in \mathbb{F}_p \right\}$

Consider the character $\psi\ N \to \mathbb{C}^\times$, $\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \mapsto e_p(x)$

A Whittacker model is the same as a map $\pi \to \mathrm{Ind}_N^{GL_2(\mathbb{F}_q)} \psi$

For uniqueness, we must show $\mathrm{End}_{GL_2(\mathbb{F}_q)}\left(\mathrm{Ind}_N^{GL_2(\mathbb{F}_q)} \psi\right)$ is commutative

Existence follows by eg counting dimensions $\square$

If (π, V) is an infinite-dimensional irr ad rep we write $\mathcal{W}(\pi)$ for its Whittacker model

The <u>Kirillov model</u> is the space

$$\mathcal{K}(\pi) = \left\{ \phi \ \mathbb{Q}_p^\times \to \mathbb{C} \;\middle|\; \phi(x) = W\left(\begin{pmatrix} x & \\ & 1 \end{pmatrix}\right) \text{ for some } W \in \mathcal{W}(\pi) \right\}$$

Note $W\left(\begin{pmatrix} x & \\ & 1 \end{pmatrix}\right)$ is never identically 0, so $\mathcal{K}(\pi) \cong \mathcal{W}(\pi) \cong (\pi, V)$

What is the action of $GL_2(\mathbb{Q}_p)$?

$$\left(\pi\begin{pmatrix} a & \\ & 1 \end{pmatrix}\phi\right)(x) = \left(\pi\begin{pmatrix} a & \\ & 1 \end{pmatrix}W\right)\left(\begin{pmatrix} x & \\ & 1 \end{pmatrix}\right) = W\left(\begin{pmatrix} ax & \\ & 1 \end{pmatrix}\right) = \phi(ax)$$

$$\left(\pi\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}\phi\right)(x) = \left(\pi\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}W\right)\left(\begin{pmatrix} x & \\ & 1 \end{pmatrix}\right) = W\left(\begin{pmatrix} 1 & bx \\ & 1 \end{pmatrix}\begin{pmatrix} x & \\ & 1 \end{pmatrix}\right) = e_p(bx)\phi(x)$$

$$\left(\pi\begin{pmatrix} t & \\ & t \end{pmatrix}\phi\right)(x) = \omega(t)\,\phi(x)$$

$\left\{ \begin{pmatrix} a & \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}, \begin{pmatrix} t & \\ & t \end{pmatrix}, \begin{pmatrix} -1 & \\ & 1 \end{pmatrix} \right\}$ generate $GL_2(\mathbb{Q}_p)$

Only the action of $\begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$ is mysterious

What is $K(\pi)$ as a space of functions on $\mathbb{Q}_p^\times$?

Take $\phi \in K(\pi)$

Since π is admissible and $(\pi\begin{pmatrix} a & \\ & 1 \end{pmatrix}\phi)(x) = \phi(ax)$,

ϕ is locally constant

Since π is admissible and $(\pi\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}\phi)(x) = e_p(bx)\,\phi(x)$,

ϕ has bounded support

Therefore, $K(\pi) \subseteq C_0^\infty(\mathbb{Q}_p^\times)$*, the space of locally constant functions with bounded support

Also, $K(\pi) \supseteq C_c^\infty(\mathbb{Q}_p^\times)$*, the space of locally constant functions with compact support, since $\begin{pmatrix} * & * \\ & 1 \end{pmatrix} \circlearrowleft C_c^\infty(\mathbb{Q}_p^\times)$ is irreducible

In fact, $C_c^\infty(\mathbb{Q}_p^\times) = \{\pi\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\phi - \phi \mid \phi \in K(\pi), x \in \mathbb{Q}_p\}$

In other words, we have an exact sequence

$$0 \to C_c^\infty(\mathbb{Q}_p^\times) \to K(\pi) \to J(K(\pi)) \to 0$$

where $J(V) = V / \langle \pi\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}v - v \mid v \in V, x \in \mathbb{Q}_p \rangle$

is the Jacquet module of V

Thus, to know the Jacquet module is to know the Kirillov model!

*The Schwartz space $S(\mathbb{Q}_p)$ is $C_c^\infty(\mathbb{Q}_p)$, inside which $C_c^\infty(\mathbb{Q}_p^\times)$ has codimension 1

For characters χ_1 and χ_2 of $\mathbb{Q}_p^\times$, we define the representation $\mathcal{B}(\chi_1,\chi_2)$ to be the space of all $f \in C^\infty(GL_2(\mathbb{Q}_p))$ s.t.

$$f\left(\begin{pmatrix} x_1 & y \\ & x_2 \end{pmatrix} g\right) = \left|\frac{x_1}{x_2}\right|^{1/2} \chi_1(x_1)\chi_2(x_2) f(g),$$

with $GL_2(\mathbb{Q}_p)$ acting via right translation

If $\chi_1\chi_2^{-1} \neq |\,|^{\pm 1}$, $\mathcal{B}(\chi_1,\chi_2)$ is irreducible. We denote its isomorphism class by $\pi(\chi_1,\chi_2)$

If $\chi_1\chi_2^{-1} = |\,|$, $\mathcal{B}(\chi_1,\chi_2)$ has an irreducible invariant subspace $\sigma(\chi_1,\chi_2)$ of codimension 1

If $\chi_1\chi_2^{-1} = |\,|^{-1}$, $\mathcal{B}(\chi_1,\chi_2)$ has a one-dimensional invariant subspace irreducible quotient $\sigma(\chi_1,\chi_2)$

<u>Theorem</u>: Let (π, V) be an irr ad rep of $GL_2(\mathbb{Q}_p)$

$$\dim J(V) = \begin{cases} 2 & \text{if } (\pi,V) \cong \pi(\chi_1,\chi_2) \\ 1 & \text{if } (\pi,V) \cong \sigma(\chi_1,\chi_2) \text{ (or 1-dimensional)} \\ 0 & \text{otherwise} \end{cases}$$

$\pi(\chi_1,\chi_2)$ are the <u>principal series</u> representations

$\sigma(\chi_1,\chi_2)$ are the <u>special</u> representations

The representations (π, V) with $J(V) = 0$ are <u>supercuspidal</u>

Let (π, V) be an infinite-dimensional irr ad rep

For $\phi_1, \phi_2 \in C_0^\infty(\mathbb{Q}_p^\times)$, we write $\phi_1(t) \approx \phi_2(t)$ if $\phi_1(t) = \phi_2(t)$ for $|t|$ sufficiently small

If $(\pi, V) \cong \pi(\chi_1, \chi_2)$, then

$$K(\pi) = \{\phi \in C_0^\infty(\mathbb{Q}_p^\times) \mid \phi(t) \approx c_1 |t|^{1/2} \chi_1(t) + c_2 |t|^{1/2} \chi_2(t)\} \text{ if } \chi_1 \neq \chi_2$$

$$K(\pi) = \{\phi \in C_0^\infty(\mathbb{Q}_p^\times) \mid \phi(t) \approx (c_1 + c_2 v(t)) |t|^{1/2} \chi_1(t)\} \text{ if } \chi_1 = \chi_2$$

the valuation $\mathbb{Q}_p^\times \to \mathbb{Z}$

If $(\pi, V) \cong \sigma(\chi_1, \chi_2)$ with $\chi_1 \chi_2^{-1} = |\ |$, then

$$K(\pi) = \{\phi \in C_0^\infty(\mathbb{Q}_p^\times) \mid \phi(t) \approx c |t|^{1/2} \chi_1(t)\}$$

If (π, V) is supercuspidal, then $K(\pi) = C_c^\infty(\mathbb{Q}_p^\times)$

Let (π, V) be an infinite-dimensional irr ad rep

For $\phi \in \mathcal{K}(\pi)$, we define the local zeta function

$$\zeta(s,\phi) = \int_{\mathbb{Q}_p^\times} \phi(x)|x|^{s-\frac{1}{2}} d^\times x$$

Note $\mathcal{K}(\xi \otimes \pi) = \xi \mathcal{K}(\pi)$, so $\int_{\mathbb{Q}_p^\times} \phi(x)\xi(x)|x|^{s-\frac{1}{2}} d^\times x$ is a zeta function for $\mathcal{K}(\xi \otimes \pi)$

Gelbart defines $\zeta(g, \xi, \phi, s) = \int_{\mathbb{Q}_p^\times} (g\phi)(x)\xi(x)|x|^{s-\frac{1}{2}} d^\times x$

In our notation, this is just $\zeta(s, \xi g \phi)$

What can $\zeta(s,\phi)$ look like?

$$\zeta(s, \mathbb{1}_{p^k \mathbb{Z}_p^\times}) = \int_{p^k \mathbb{Z}_p^\times} |x|^{s-\frac{1}{2}} d^\times x = p^{-k(s-\frac{1}{2})}$$

$$\begin{aligned}
\zeta(s, |t|^{1/2}\chi \mathbb{1}_{\mathbb{Z}_p}) &= \int_{\mathbb{Z}_p} \chi(x)|x|^s d^\times x \\
&= \sum_{k=0}^{\infty} \int_{p^k \mathbb{Z}_p^\times} \chi(x)|x|^s d^\times x \\
&= \sum_{k=0}^{\infty} \chi(p)^k p^{-ks} \int_{\mathbb{Z}_p^\times} \chi(x) d^\times x \\
&= \begin{cases} \frac{1}{1-\chi(p)p^{-s}} & \text{if } \chi \text{ is unramified} \\ 0 & \text{if } \chi \text{ is ramified} \end{cases}
\end{aligned}$$

Hence, $\{\mathcal{Z}(s,\phi) \mid \phi \in \mathcal{K}(\pi)\} = L(s,\pi)\mathbb{C}[p^{-s}, p^{s}]$ where

$$L(s,\pi) = \begin{cases} \frac{1}{(1-\alpha_1 p^{-s})(1-\alpha_2 p^{-s})} & \text{if } (\pi,V) \cong \pi(\chi_1,\chi_2) \text{ with } \chi_1\chi_2^{-1} \neq |\,|^{\pm 1} \\ \frac{1}{1-\alpha_1 p^{-s}} & \text{if } (\pi,V) \cong \sigma(\chi_1,\chi_2) \text{ with } \chi_1\chi_2^{-1} = |\,| \\ 1 & \text{if } (\pi,V) \text{ is supercuspidal} \end{cases}$$

where $\alpha_i = \begin{cases} \chi_i(p) & \text{if } \chi_i \text{ is unramified} \\ 0 & \text{if } \chi_i \text{ is ramified} \end{cases}$

In particular, $\mathcal{Z}(s,\phi)$ always has a meromorphic continuation to all of $\mathbb{C}$, with no more poles than $L(s,\pi)$

<u>Example</u> Recall that a representation is called <u>spherical</u> (or unramified) if it has a $GL_2(\mathbb{Z}_p)$-fixed vector

If (π, V) is spherical, then it is of the form $\pi(\chi_1,\chi_2)$ with χ_1,χ_2 unramified, and the space of $GL_2(\mathbb{Z}_p)$-fixed vectors is one-dimensional

Let ϕ° be the $GL_2(\mathbb{Z}_p)$-fixed vector in $\mathcal{K}(\pi)$ with $\phi^\circ(1)=1$ It turns out that (with $\alpha_i = \chi_i(p)$)

$$\phi^\circ(p^k) = \begin{cases} p^{-\frac{k}{2}} \frac{\alpha_1^{k+1} - \alpha_2^{k+1}}{\alpha_1 - \alpha_2} & \text{if } k \geq 0 \\ 0 & \text{if } k < 0, \end{cases}$$

so $\mathcal{Z}(s,\phi^\circ) = \sum_{k=0}^{\infty} p^{-k/2} \frac{\alpha_1^{k+1} - \alpha_2^{k+1}}{\alpha_1 - \alpha_2} = \frac{1}{(1-\alpha_1 p^{-s})(1-\alpha_2 p^{-s})} = L(s,\pi)$

Theorem (local functional equation)

Let (π, V) be an irreducible admissible representation of $GL_2(\mathbb{Q}_p)$ with central character ω

For some meromorphic function $\gamma(s,\pi)$ independent of $\phi \in \mathcal{K}(\pi)$,

$$\zeta(s,\phi) = \gamma(s,\pi)\,\zeta(1-s, \underbrace{\omega^{-1}\pi\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\phi}_{\substack{\text{an element of} \\ \mathcal{K}(\omega^{-1}\otimes\pi)=\mathcal{K}(\hat{\pi})}})$$

<u>Proof</u> For fixed s, consider the linear functionals

$$\Lambda_1(\phi) = \zeta(s,\phi) \quad \text{and} \quad \Lambda_2(\phi) = \zeta(1-s, \omega^{-1}\pi\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\phi)$$

We will show that Λ_1 and Λ_2 are (usually) proportional

For all $a \in \mathbb{Q}_p^\times$ and $\phi \in \mathcal{K}(\pi)$,

$$\Lambda_2\left(\pi\begin{pmatrix} a & \\ & 1 \end{pmatrix}\phi\right) = \int_{\mathbb{Q}_p^\times} \left[\pi\left(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} a & \\ & 1 \end{pmatrix}\right)\phi\right](x)\,\omega^{-1}(x)|x|^{-s+1/2}\,d^\times x$$

$$= \int_{\mathbb{Q}_p^\times} \left[\pi\left(\begin{pmatrix} a & \\ & a \end{pmatrix}\begin{pmatrix} a^{-1} & \\ & 1 \end{pmatrix}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\right)\phi\right](x)\,\omega^{-1}(x)|x|^{-s+\frac{1}{2}}\,d^\times x$$

$$= \omega(a)\int_{\mathbb{Q}_p^\times} \left[\pi\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\phi\right](a^{-1}x)\,\omega^{-1}(x)|x|^{-s+\frac{1}{2}}\,d^\times x$$

$(y = a^{-1}x)$

$$= |a|^{-s+\frac{1}{2}}\int_{\mathbb{Q}_p^\times} \left[\pi\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\phi\right](y)\,\omega^{-1}(y)|y|^{-s+\frac{1}{2}}\,d^\times y$$

$$= |a|^{-s+\frac{1}{2}}\,\Lambda_2(\phi)$$

Restrict Λ_1 and Λ_2 to $C_c^\infty(\mathbb{Q}_p^\times) = \ker(\mathcal{K}(\pi) \twoheadrightarrow J(\mathcal{K}(\pi)))$

We will show that for $\phi \in C_c^\infty(\mathbb{Q}_p^\times)$,

$$\Lambda_2(\phi) = \Lambda_2(\mathbb{1}_{\mathbb{Z}_p^\times})\,\Lambda_1(\phi)$$

Given $\phi \in C_c^\infty(\mathbb{Q}_p^\times)$, choose $k \gg 0$ s.t. $\phi(ax) = \phi(x)$ if $a \equiv 1\ (p^k)$

$$\phi = \sum_{j \in (\mathbb{Z}/p^k\mathbb{Z})} \sum_{m \in \mathbb{Z}} \phi(p^m j)\, \mathbb{1}_{p^m j(1+p^k\mathbb{Z}_p)}$$

$$= \sum_{j \in (\mathbb{Z}/p^k\mathbb{Z})} \sum_{m \in \mathbb{Z}} \phi(p^m j)\, \pi\begin{pmatrix} (p^m j)^{-1} & \\ & 1 \end{pmatrix} \mathbb{1}_{1+p^k\mathbb{Z}_p},$$

so $$\Lambda_2(\phi) = \sum_{j \in (\mathbb{Z}/p^k\mathbb{Z})} \sum_{m \in \mathbb{Z}} \phi(p^m j)\, p^{m(-s+\frac{1}{2})}\, \Lambda_2(\mathbb{1}_{1+p^k\mathbb{Z}_p})$$
$$= \Lambda_1(\phi)\,\Lambda_2(\mathbb{1}_{1+p^k\mathbb{Z}})$$

Letting $\phi = \mathbb{1}_{\mathbb{Z}_p^\times}$, we find $\Lambda_2(\mathbb{1}_{\mathbb{Z}_p^\times}) = \Lambda_2(\mathbb{1}_{1+p^k\mathbb{Z}_p})$, giving

Let $\Lambda = \Lambda_2 - \Lambda_2(\mathbb{1}_{\mathbb{Z}_p^\times})\Lambda_1$, which induces a map on $J(\mathcal{K}(\pi))$

In fact, Λ spans a $\begin{pmatrix} * & * \end{pmatrix}$-invariant space where $\begin{pmatrix} t_1 & t_2 \end{pmatrix}$ acts as $\omega(t_2)^{-1}\left|\frac{t_1}{t_2}\right|^{s-\frac{1}{2}}$. Since $\dim J(\mathcal{K}(\pi)) \leq 2$, $\Lambda = 0$ for all but two values of $s \pmod{\frac{2\pi i}{\log p}}$

In other words, for s outside a discrete set,

$\exists\, \gamma(s,\pi)$ s.t. $\Lambda_1(\phi) = \gamma(s,\pi)\Lambda_2(\phi)$

Since zeta functions are meromorphic, so is $\gamma(s,\pi)$ $\square$

For principal series,

$$\gamma(s, \pi(\chi_1,\chi_2)) = \gamma(s,\chi_1)\gamma(s,\chi_2) \qquad (\chi_1\chi_2^{-1} \neq |\ |^{\pm 1})$$

For special representations,

$$\gamma(s, \sigma(\chi_1,\chi_2)) = \gamma(s,\chi_1)\gamma(s,\chi_2) \qquad (\chi_1\chi_2^{-1} = |\ |^{\pm 1})$$

For supercuspidal representations, $\gamma(s,\pi) \in \mathbb{C}[p^{-s}, p^{s}]$ is nonvanishing, so $\gamma(s,\pi) = A p^{-ms}$ for some $A \in \mathbb{C}, m \in \mathbb{Z}$

<u>Proposition</u> Suppose π_1 and π_2 have the same central character. If $\gamma(s, \xi \otimes \pi_1) = \gamma(s, \xi \otimes \pi_2)$ for all characters ξ of $\mathbb{Q}_p^\times$, then $\pi_1 \cong \pi_2$

Proof

We must show $\pi_1(-1\ 1)\phi = \pi_2(-1\ 1)\phi \ \forall \phi \in \mathcal{K}(\pi_1) \cap \mathcal{K}(\pi_2)$

Replacing ϕ with $(1\ a)\phi$, it is enough to show

$$(\pi_1(-1\ 1)\phi)(1) = (\pi_2(-1\ 1)\phi)(1)$$

Let $\phi_i = \pi_i(-1\ 1)\phi$

Given a character ξ of $\mathbb{Q}_p^\times$ and $n \in \mathbb{Z}$, let

$$F_\xi(n) = \int_{|x|=p^{-n}} (\phi_1(x) - \phi_2(x))\, \xi(x)\, d^\times x$$

By Fourier inversion on $\mathbb{Z}_p^\times$,

$$\phi_1(1) - \phi_2(1) = \sum_{\xi \in \widehat{\mathbb{Z}_p^\times}} F_\xi(0),$$

so it suffices to show $F_\xi(0) = 0 \ \forall \xi$

Since $\phi_i \in C_0^\infty(\mathbb{Q}_p^\times)$, $F_\xi(n) = 0 \ \forall n \ll 0$

By hypothesis, $\zeta(s, \xi\phi_1) = \zeta(s, \xi\phi_2)$, so

$$\sum_{n \in \mathbb{Z}} F_\xi(n) p^{-ns} = \zeta(s, \xi\phi_1) - \zeta(s, \xi\phi_2) = 0$$

for $\operatorname{Re} s \gg 0$

Hence, $F_\xi(n) = 0 \ \forall n \in \mathbb{Z}$, and in particular $F_\xi(0) = 0$

What about for $GL_2(\mathbb{R})$?

Let (π, V) be an irreducible admissible $(\mathfrak{gl}_2(\mathbb{R}), O(2))$-module

A <u>Whittacker model</u> for (π, V) is a realization of (π, V) as a space of smooth functions $W\ GL_2(\mathbb{R}) \to \mathbb{C}$ s.t.

(i) $W\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) = e_\infty(x) W(g) \quad \forall g \in GL_2(\mathbb{R}),\ x \in \mathbb{R},$

(ii) $W(g) = O(y^N)$ for some N, where

$$g = \begin{pmatrix} \pm u & \\ & u \end{pmatrix} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

with $GL_2(\mathbb{R})$ acting via right multiplication

If Z and Δ act as μ and λ, and W is in the weight k space where $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ acts as $e^{ik\theta}$, then (for $\det g > 1$)

$$W(g) = u^\mu e_\infty(x) e^{ik\theta} w(y) \quad \text{where}$$

$$w'' + \left(-4\pi^2 + \frac{2\pi k}{y} + \frac{\lambda}{y^2}\right) w = 0$$

This differential equation has two linearly independent solutions, only one of which has polynomial growth

$(\mathfrak{gl}_2(\mathbb{R}), O(2))$-modules are determined by

The $SO(2)$-weights which appear
The eigenvalue μ of Z
The eigenvalue λ of Δ
The action of $\begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$

This proves uniqueness in the following.

<u>Theorem</u> If (π, V) is infinite-dimensional, then it has a unique Whittacker model $\mathcal{W}(\pi)$

The <u>Kirillov model</u> is the space

$$\mathcal{K}(\pi) = \left\{ \phi \ \mathbb{Q}_p^\times \to \mathbb{C} \;\middle|\; \phi(x) = W\left(\begin{pmatrix} x & \\ & 1 \end{pmatrix}\right) \text{ for some } W \in \mathcal{W}(\pi) \right\}$$

Note that we still have

$$\mathcal{K}(\xi \otimes \pi) = \xi \mathcal{K}(\pi) \quad \text{and}$$

$$\hat{\pi} \cong \omega^{-1} \otimes \pi$$

<u>Theorem (local functional equation)</u>.

Let (π, V) be an infinite-dimensional irreducible admissible $(\mathfrak{gl}_2(\mathbb{R}), O(2))$-module with central character ω

For some meromorphic function $\gamma(s,\pi)$ independent of $\phi \in \mathcal{K}(\pi)$,

$$\zeta(s, \phi) = \gamma(s,\pi)\, \zeta(1-s, \omega^{-1}\pi\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\phi)$$

For principal series,

$$\gamma(s, \pi(\chi_1, \chi_2)) = \gamma(s, \chi_1)\gamma(s, \chi_2) \qquad (\chi_1\chi_2^{-1} \neq \mathrm{sgn}(x)x^{\ell})$$

For discrete series,

$$\gamma(s, \sigma(\chi_1, \chi_2)) = \gamma(s, \chi_1)\gamma(s, \chi_2) \qquad (\chi_1\chi_2^{-1} = \mathrm{sgn}(x)x^{\ell})$$

<u>Proposition</u> Suppose π_1 and π_2 have the same central character

If $\gamma(s, \xi \otimes \pi_1) = \gamma(s, \xi \otimes \pi_2)$ for all characters ξ of $\mathbb{R}^\times$,

then $\pi_1 \cong \pi_2$

The Global Functional Equation for GL_2

1) Review the local theory from last time

2) Define global Whittaker models, and deduce uniqueness

3) Recall Whittaker models of automorphic representations

4) Deduce (strong) multiplicity one

5) Prove the global functional equation
   (and converse theorem)

6) See what this all means for modular/Maas forms

Last time

Let $v \leq \infty$ be a place of $\mathbb{Q}$

Let (π, V) be an infinite-dimensional irreducible admissible $\begin{cases} \text{rep of } GL_2(\mathbb{Q}_p) & \text{if } v = p < \infty \\ (\mathfrak{gl}_2(\mathbb{R}), O(2))\text{-module} & \text{if } v = \infty \end{cases}$

<u>Thm 1</u> $\pi \cong \omega \otimes \hat{\pi}$ (where ω is the central character)

<u>Thm 2</u> (π, V) has a unique Whittaker model, ie a realization of (π, V) as a space $\mathcal{W}(\pi)$ of smooth functions $W\ GL_2(\mathbb{Q}_v) \to \mathbb{C}$ st

$$W\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) = e_v(x)\, W(g)$$

and $W\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right) = O(y^N)$ for some N if $v = \infty$

The Kirillov model is $K(\pi) = \{ W|_{\begin{pmatrix} * & \\ & 1 \end{pmatrix}} \mid W \in \mathcal{W}(\pi) \}$

Note $C_c^\infty(\mathbb{Q}_p^\times) \subseteq K(\pi)$ if $v = p < \infty$

For $\phi \in K(\pi)$, we define $\zeta_v(s,\phi) = \int_{\mathbb{Q}_v^\times} \phi(x)|x|^{s-1/2} d^\times x$

If $v = p < \infty$, χ_1 and χ_2 are unramified, and ϕ° is the $GL_d(\mathbb{Z}_p)$-fixed vector in $\pi_p(\chi_1,\chi_2)$, then

$$\zeta_p(s,\phi_0) = L_p(s,\pi_p(\chi_1,\chi_2)) = \frac{1}{(1-\chi_1(p))(1-\chi_2(p))}$$

<u>Thm 3</u> For some meromorphic $\gamma_v(s,\pi)$,

$$\zeta_v(s,\phi) = \gamma_v(s,\pi)\zeta(1-s,\omega^{-1}\pi\begin{pmatrix}-1 & 1\end{pmatrix}\phi) \quad \forall \phi \in K(\pi)$$

Specifically,

$$\gamma_v(s,\pi_v(\chi_1,\chi_2)) = \gamma_v(s,\chi_1)\gamma_v(s,\chi_2)$$

$$\gamma_v(s,\sigma_v(\chi_1,\chi_2)) = \gamma_v(s,\chi_1)\gamma_v(s,\chi_2)$$

<u>Thm 4</u> If π_1 and π_2 have the same central character and $\gamma_v(s,\xi\otimes\pi_1) = \gamma_v(s,\xi\otimes\pi_2)$ for all characters ξ of $\mathbb{Q}_v^\times$, then $\pi_1 \cong \pi_2$

Let (π, V) be an irreducible admissible $(\mathfrak{gl}_2(\mathbb{R}), O(2)) \times GL_2(\mathbb{A}_f)$-module

By the tensor product theorem, $\pi \cong \hat{\bigotimes}_v \pi_v$ with each π_v irreducible admissible, and π_v spherical for almost all v

<u>Thm 1</u>: Let $\hat{\pi} = \bigotimes_v \hat{\pi}_v$

if π has central character ω Then, $\hat{\pi} \cong \omega^{-1} \otimes \pi$

If $\pi \subset \mathcal{A}_0(GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}), \omega)$,

then $\hat{\pi} \subset \mathcal{A}_0(GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}), \omega^{-1})$

A <u>Whittaker model</u> of π is a realization of π as a space of smooth functions $W\ GL_2(\mathbb{A}) \to \mathbb{C}$ st

$$W\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) = e(x)\, W(g)$$

$$W\left(\begin{pmatrix} y_\infty & \\ & 1 \end{pmatrix}\right) = O(|y_\infty|^N) \text{ for some } N$$

This is the same as a Whittaker model $\mathcal{W}(\pi_v)$ for each π_v, and taking $\mathrm{Span}\{\prod_v W_v(g_v) \mid W_v \in \mathcal{W}(\pi_v),\ W_v \text{ spherical a.e } v\}$

<u>Thm 2</u>: π has a Whittaker model iff each π_v does, (i.e each π_v is ∞-dimensional), in which case it is unique

Let $\phi \in \mathcal{A}_0(GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A}), \omega)$ be an automorphic form

For fixed $g \in GL_2(\mathbb{A})$, $\phi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right)$ is a function on $\mathbb{A}/\mathbb{Q}$,

so it has a Fourier expansion

$$\phi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) = \sum_{\alpha \in \mathbb{Q}} \hat{\phi}_\alpha(g) e(\alpha x) = \sum_{\alpha \in \mathbb{Q}^\times} \hat{\phi}_\alpha(g) e(\alpha x)$$

where $\hat{\phi}_\alpha(g) = \int_{\mathbb{A}/\mathbb{Q}} \phi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) e(-\alpha x)\, dx$

Note that
$$\begin{aligned}
\hat{\phi}_\alpha(g) &= \int_{\mathbb{A}/\mathbb{Q}} \phi\left(\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) e(-\alpha x)\, dx \\
&= \int_{\mathbb{A}/\mathbb{Q}} \phi\left(\begin{pmatrix} 1 & \alpha x \\ & 1 \end{pmatrix}\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} g\right) e(-\alpha x)\, dx \\
&= \int_{\mathbb{A}/\mathbb{Q}} \phi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} g\right) e(-x)\, dx \\
&= \hat{\phi}_1\left(\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} g\right)
\end{aligned}$$

Hence, $\phi(g) = \sum_{\alpha \in \mathbb{Q}^\times} \hat{\phi}_1\left(\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} g\right)$

Note $\hat{\phi}_1\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) = e(x)\hat{\phi}_1(g)$ and $\hat{\phi}_1\left(\begin{pmatrix} y_\infty & \\ & 1 \end{pmatrix}\right) = O(|y_\infty|^N)$,

so $\hat{\phi}_1$ is a Whittaker function!

In summary:

If (π, V) is a cuspidal automorphic representation, then $\{\hat{\Phi}_1 \mid \Phi \in V\}$ is a ready-made Whittaker model

We can recover Φ from $\hat{\Phi}_1$ via $\Phi(g) = \sum_{\alpha \in \mathbb{Q}^\times} \hat{\Phi}_1\left(\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} g\right)$

<u>Thm (Multiplicity one)</u> If (π^1, V^1) and (π^2, V^2) are cuspidal automorphic representations and $\pi^1 \cong \pi^2$, then $V^1 = V^2$ (inside $\mathcal{A}_0(GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A}))$)

<u>Thm (Strong multiplicity one)</u>: If (π^1, V^1) and (π^2, V^2) are cuspidal automorphic representations and $\pi^1_v \cong \pi^2_v$ except for finitely many <u>finite</u> v, then $V^1 = V^2$

<u>Proof</u> It suffices to show $V^1 \cap V^2 \neq \emptyset$

Define $W^1 = \prod_v W^1_v$ and $W^2 = \prod_v W^2_v$ so that

(i) $W^i_v \in \mathcal{W}(\pi^i_v)$,

(ii) $W^1_v = W^2_v$ if $\pi^1_v \cong \pi^2_v$,

(iii) $W^i_v\left(\begin{pmatrix} x & \\ & 1 \end{pmatrix}\right) = \mathbb{1}_{\mathbb{Z}_p^\times}(x)$ if $\pi^1_v \not\cong \pi^2_v$, and

(iv) W^i_v is spherical for almost all v

Let $\phi^i(g) = \sum_{\alpha \in \mathbb{Q}^\times} W^i\left(\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} g\right) \in V^i$

Then, $\phi^1(g) = \phi^2(g)$

for some open compact $K_0 \subset GL_2(\mathbb{A}_f)$

for $g \in GL_2(\mathbb{Q}) \begin{pmatrix} * & \\ & 1 \end{pmatrix} GL_2(\mathbb{R}) K_0 = GL_2(\mathbb{A})$ $\square$

by automorphy

by construction of W_v^i

since $\pi_\infty^1 \cong \pi_\infty^2$

<u>Thm (Very strong multiplicity one)</u> If (π^1, V^1) and (π^2, V^2) are automorphic cuspidal representations and $\pi_v^1 \cong \pi_v^2$ for all but finitely many v, then $V^1 = V^2$

For example, define $\alpha_1(p), \alpha_2(p) \in \mathbb{C}^\times$ for $p \notin S$ (S finite)

We can ask, "Is there a cuspidal automorphic representation π such that for $p \notin S$, π_p is unramified with parameters $\alpha_1(p), \alpha_2(p)$?"

$$\begin{tikzcd}
\text{Automorphic rep of } GL_2 \arrow[r, dashed, "?"] \arrow[d] & \text{Automorphic rep of } GL_n \arrow[d] \\
\alpha_1(p), \alpha_2(p) \text{ for } p \notin S \arrow[r, "\text{some operation}"] & \beta_1(p), \ldots, \beta_n(p) \text{ for } p \notin S
\end{tikzcd}$$

Let (π, V) be an irreducible admissible unitarizable $(\mathfrak{gl}_2(\mathbb{R}), O(2)) \times GL_2(\mathbb{A}_f)$-module with a Whittaker model

Given $W \in \mathcal{W}(\pi)$, let $\mathcal{S}(s, W) = \int_{\mathbb{A}^\times} W\left(\begin{pmatrix} x & \\ & 1 \end{pmatrix}\right) |x|^{s-\frac{1}{2}} d^\times x$

If $W = \prod W_v$, then $\mathcal{S}(s, W) = \prod_v \mathcal{S}_v(s, W_v|_{\begin{pmatrix} * & \\ & 1 \end{pmatrix}})$

$\mathcal{S}_v(s, W_v)$ converges for $\operatorname{Re} s > \frac{1}{2}$, and $\mathcal{S}_v(s, W_v) = L(s, \pi_v)$ a.e. v, so the product converges for $\operatorname{Re} s > \frac{3}{2}$

Suppose (π, V) is automorphic, so $W = \hat{\phi}_1$ for some automorphic form ϕ

Then, $\mathcal{S}(s, W) = \int_{\mathbb{A}^\times/\mathbb{Q}^\times} \sum_{\alpha \in \mathbb{Q}^\times} W\left(\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix}\begin{pmatrix} x & \\ & 1 \end{pmatrix}\right) |x|^{s-\frac{1}{2}} d^\times x$

$$= \int_{\mathbb{A}^\times/\mathbb{Q}^\times} \phi\left(\begin{pmatrix} x & \\ & 1 \end{pmatrix}\right) |x|^{s-\frac{1}{2}} d^\times x$$

$$= \int_{\mathbb{A}^\times/\mathbb{Q}^\times} \phi\left(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} x & \\ & 1 \end{pmatrix}\right) |x|^{s-\frac{1}{2}} d^\times x$$

$$= \int_{\mathbb{A}^\times/\mathbb{Q}^\times} \phi\left(\begin{pmatrix} & x \\ 1 & \end{pmatrix}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\right) |x|^{s-\frac{1}{2}} d^\times x$$

$$= \int_{\mathbb{A}^\times/\mathbb{Q}^\times} \phi\left(\begin{pmatrix} x^{-1} & \\ & 1 \end{pmatrix}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\right) \omega(x) |x|^{s-\frac{1}{2}} d^\times x$$

$x \to x^{-1}$

$$= \int_{\mathbb{A}^\times/\mathbb{Q}^\times} \phi\left(\begin{pmatrix} x & \\ & 1 \end{pmatrix}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\right) \omega^{-1}(x) |x|^{\frac{1}{2}-s} d^\times x$$

$$= \mathcal{S}\left(1-s, \omega^{-1}\pi\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} W\right)$$

Altogether, we have shown

$$\zeta(s,W) \overset{\text{global FE}}{=\!=} \zeta(1-s,\omega^{-1}\pi(\begin{smallmatrix}-1 & \\ & 1\end{smallmatrix})W)$$

$$\|$$

$$\prod_v \zeta_v(s,W_v) \overset{\text{local FE}}{=\!=} \prod_v \gamma_v(s,\pi_v)\,\zeta_v(1-s,\omega^{-1}\pi_v(\begin{smallmatrix}-1 & \\ & 1\end{smallmatrix})W_v)$$

Thus, $\prod_v \gamma_v(s,\pi_v)=1$

More palatably, if π_v is unramified and W_v is spherical for $v\notin S$, and if $L_S(s,\pi)=\prod_{v\notin S} L(s,\pi_v)$, then

$$L_S(1-s,\hat{\pi}) = \prod_{v\in S}\gamma_v(s,\pi_v)\,L_S(s,\pi)$$

We also have a converse theorem

<u>Thm</u> π is automorphic iff for every Hecke character ξ,

(i) $L_S(s,\xi\otimes\pi)$ extends to an entire function, bounded in vertical strips, and

(ii) $L_S(1-s,\xi^{-1}\otimes\hat{\pi}) = \prod_{v\in S}\gamma_v(s,\xi\otimes\pi)\,L_S(s,\xi\otimes\pi)$

<u>Proof</u> For the converse direction, we must show that

$\forall W\in\mathcal{W}(\pi)$, $\phi(g)=\sum_{\alpha\in\mathbb{Q}^\times} W(\begin{smallmatrix}\alpha & \\ & 1\end{smallmatrix})g)$ is an automorphic form

Automorphy with respect to $\begin{smallmatrix}\alpha & \\ & 1\end{smallmatrix}$, $\begin{smallmatrix}1 & \beta \\ & 1\end{smallmatrix}$, $\begin{smallmatrix}t & \\ & t\end{smallmatrix}$ are clear

It suffices to show $(\begin{smallmatrix}-1 & \\ & 1\end{smallmatrix})\phi=\phi$ This is the same as Thm 4 from last time ($\gamma_v(s,\xi_v\otimes\pi_v)$ determines π_v) $\square$

Let $C^\infty(PGL_2(\mathbb{R})^+, k)$ consist of those $F \in C^\infty(GL_2(\mathbb{R})^+)$ such that $F(ugK_\theta) = e^{ik\theta} F(g)$

for $u \in \mathbb{R}^\times$, $g \in GL_2(\mathbb{R}^+)$, $K_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in SO(2)$

Let $\mathbb{H} = \{x+iy \mid y>0\}$ $GL_2(\mathbb{R})^+ \circlearrowleft \mathbb{H}$ via $\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}$

The stabilizer of $i \in \mathbb{H}$ is $\mathbb{R}^\times SO(2)$, so $PGL_2(\mathbb{R})^+/SO(2) \xrightarrow{\sim} \mathbb{H}$,

and we can interpret $C^\infty(PGL_2(\mathbb{R})^+, k)$ as functions on $\mathbb{H}$.

$$C^\infty(PGL_2(\mathbb{R})^+, k) \longleftrightarrow C^\infty(\mathbb{H})$$

$$F \longmapsto f(x+iy) = F\left(\begin{pmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2} \end{pmatrix}\right)$$

$$F(g) = \left(\frac{ci+d}{|ci+d|}\right)^{-k} f(gi) \longleftarrow\!\shortmid f$$

$C^\infty(PGL_2(\mathbb{R})^+, k)$ has a right $GL_2(\mathbb{R})^+$-action via left multiplication On $C^\infty(\mathbb{H})$, this corresponds to

$$(f|_k g)(z) = \left(\frac{cz+d}{|cz+d|}\right)^{-k} f(gz)$$

The condition $LF=0$ corresponds to $y^{-k/2} f$ being holomorphic

On holomorphic functions, the action looks like

$$(f|'_k g)(z) = (\det g)^{k/2} (cz+d)^{-k} f(gz)$$

Suppose f is a Maass form of weight k, eigenvalue λ, and Nebentypus χ for $\Gamma_0(N)$, so

$$\Delta f = \lambda f \quad \text{and}$$

$$f|_k \gamma = \chi(d) f \text{ for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$$

Then, $F(g) = (f|_k g)(i)$ satisfies

$$\Delta F = \lambda f \text{ and}$$

$$F(u \gamma g k_\theta) = e^{ik\theta} \chi(\gamma) F(g) \text{ for } \begin{array}{l} u \in \mathbb{R}^\times \\ \gamma \in \Gamma_0(N) \\ g \in GL_2(\mathbb{R})^+ \\ k_\theta \in SO(2) \end{array}$$

We define $\tilde{f} \in \mathcal{A}_0(GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}), \tilde{\chi})$ by

$$\tilde{f}(\gamma g_\infty k_0) = (f|_k g_\infty)(i) \chi(d \pmod N)^{-1}$$

for $\gamma \in GL_2(\mathbb{Q})$, $g_\infty \in GL_2(\mathbb{R})^+$,

$$K_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(N) = \left\{ g \in GL_2(\hat{\mathbb{Z}}) \,\middle|\, g \equiv \begin{pmatrix} * & * \\ & * \end{pmatrix} \pmod N \right\}$$

Suppose $T_p f = a_p f$ $\forall p \nmid N$, and $(\pi, V) = \bigotimes_v \pi_v$ is an automorphic representation with $\tilde{f}|_V \neq 0$

If $p \nmid N$, π_p is unramified, with parameters α_1, α_2 the roots of $X^2 - p^{-1/2} a_p + \chi(p)$

By strong multiplicity one, there is only one choice of V, so $f \in V^1$

If $S = \{p \mid p \mid N\}$, then

$$L_S(s, \pi) = \prod_{p \nmid N} \frac{1}{(1-\alpha_1 p^{-s})(1-\alpha_2 p^{-s})} = \prod_{p \nmid N} \frac{1}{1 - a_p p^{-(s+1/2)} + \chi(p) p^{-2s}}$$

has an analytic continuation and functional equation

Example Let $f \in S_k(SL_2(\mathbb{Z}))$ be a Hecke eigenform

$\tilde{f}$ is in the automorphic representation $\pi = \bigotimes_v \pi_v$

where $\pi_\infty = \sigma(\mathrm{sgn}^k |\,|^{\frac{k-1}{2}}, |\,|^{\frac{1-k}{2}}) = \mathcal{D}_0(k)$

$\pi_p = \pi(\chi_1, \chi_2)$ with χ_i unramified

and if $\alpha_i = \chi_i(p)$, then α_1, α_2 are the roots

of $X^2 - p^{1/2} a_p + 1$

For $S = \{\infty\}$, $L_S(\pi) = \prod_{p<\infty} \frac{1}{1 - a_p p^{-s+1/2} + p^{-2s}} = L(s + \frac{k-1}{2}, f)$

and $\gamma_\infty(s, \pi_\infty) = i^k (2\pi)^{1-2s} \frac{\Gamma(s + \frac{k-1}{2})}{\Gamma(1 - (s + \frac{k-1}{2}))}$,

so $\Lambda(s, f) = i^k \Lambda(k-s, f)$

where $\Lambda(s, f) = (2\pi)^{-s} \Gamma(s) L(s, f)$,

as expected.