Coefficients of Maass Forms and the Siegel Zero

Let \( \mathcal{H} = \{ z \in \mathbb{C} | \text{Im} z > 0 \} \) and \( \Gamma = \Gamma_0(N) \).

Let \( \chi \) be an even Dirichlet character modulo \( N \) (extended to \( \Gamma \)).

We will consider \( S_0(\Gamma, \chi) \), consisting of \( f : \mathcal{H} \to \mathbb{C} \) s.t.
\[
f(\tau z) = \chi(\tau)f(z) \quad \forall \tau \in \Gamma
\]

We endow \( S_0(\Gamma, \chi) \) with the Petersson inner product
\[
\langle f, g \rangle = \frac{1}{\text{vol}(\mathcal{H} \setminus \mathcal{H})} \int_{\mathcal{H} \setminus \mathcal{H}} f(z) \overline{g(z)} \, d\mu. \quad (d\mu = y^2 \, dx \, dy)
\]

On \( S_0(\Gamma, \chi) \) act:

- the Laplacian \( \Delta = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \)
- and the Hecke operators \( T_n f = \frac{1}{n} \sum_{\substack{a \equiv n \mod d \\ 0 \leq b < d}} \chi(a) f \left( \frac{az + b}{d} \right) \).
Let \( f \in S_0(\Gamma,\chi) \) be a newform (i.e. \( \Delta \)- and \( T_n \)-eigenfunction).

If \( \Delta f = \lambda f \), then
\[
f(z) = \sum_{n \neq 0} \rho(n) \ln n^{1/2} W(nz).
\]

(\( W(z) = (y \cosh \pi z)^{1/2} K_0(2\pi y) e^{2\pi i x} \))

(\( \frac{1}{4} + z^2 = \lambda \))

If \( T_n f = a(n) f \), then \( \rho(n) = \rho(1) a(n) = \pm \rho(-n) \) \((n \geq 1)\).

We have the Hecke relations:
\[
a(1) = 1
\]
\[
a(n) = \chi(n) \overline{a(n)} \quad (\langle n, N \rangle = 1)
\]
\[
a(m) a(n) = \sum_{d \mid \langle mn \rangle} \chi(d) a\left(\frac{mn}{d^2}\right) \quad (\langle mn, N \rangle = 1)
\]
\[
a(p) a(n) = a(pn) \quad (p \mid N)
\]

Two ways to normalize: \( ||f|| = 1 \) or \( \rho(1) = 1 \).

Our goal is to translate between these.

We choose \( ||f|| = 1 \), and seek to upper bound \( \rho(1) \).

**Theorem (Goldfeld-Hoffstein-Lieman):**

If \( f \) is not a lift from \( GL(1) \), then
\[
|\rho(1)|^2 \ll \log(\sqrt{N} + 1) \quad \text{(effectively)}.
\]

If \( f \) is a lift from \( GL(1) \), then
\[
|\rho(1)|^2 \ll \max(\sqrt{N}, \log(\sqrt{N} + 1)) \quad \text{(effectively),}
\]
\[
|\rho(1)|^2 \ll \varepsilon \max(N^\varepsilon, \log(\sqrt{N} + 1)) \quad \text{(ineffectively)}.
\]
The plan:

1) Relate $\mu(1)$ to an L-value (namely $L(1, \text{Ad}^2 f)$)

2) Define the class of L-functions we will be dealing with, and state some properties.

3) Recall the proof of Siegel's Theorem.

4) Extend this proof to $L(s, \text{Ad}^2 f)$
   (and make it effective!)

5) Fill in the proofs of the properties in step 3.
Step 1: The adjoint square L-function

Suppose \( L(s, f) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \approx \prod_{p} \left(1 - \frac{\varphi(p)}{p^s}\right)^{-1} \left(1 - \frac{1}{p^s}\right)^{-1} \)
so that \( \frac{\varphi(p)}{p^s} = a(p) \) and \( \varphi(p) p^{-s} = \chi(p) \).

Consider the convolution L-function

\[
L(s, f \otimes \overline{f}) = S(2s) \sum_{n=1}^{\infty} \frac{|a(n)|^2}{n^s} \\
\approx \prod_{p} \left(1 - \frac{\varphi(p)}{p^s}\right)^{-1} \left(1 - \frac{1}{p^s}\right)^{-1} \left(1 - \frac{1}{p^s}\right)^{-2} \\
\text{where } \varphi(p) = \frac{\varphi(p)}{p^s} = \varphi(p) \chi(p).
\]

By Rankin-Selberg, \( \text{res} \ L(s, f \otimes \overline{f}) = \frac{2\pi}{3} |\rho(1)|^{-2} \).

By Gelbart-Jacquet, there is an automorphic form \( F \) on \( G \) such that

\[
L(s, f \otimes \overline{f}) \approx S(s) L(s, F) 
\]

Checking the bad primes, we have

\[
(L(1, F) \log N)^{-1} \ll |\rho(1)|^2 \ll (L(1, F) \log N)^{-1}
\]
so it suffices to lower bound \( L(1, F) \).
Step 2: L-functions

An **L-function** is a Dirichlet series \( L(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s} \) with:

(i) An Euler product \( L(s) = \prod_p (1 - \alpha_i(p)p^{-s})^{-1} \) absolutely convergent for \( \text{Re} \ s > 1 \).

(ii) A gamma factor \( G(s) = D^s \prod_{j=1}^d \Gamma \left( \frac{s+\beta_i}{2} \right) \) and a positive integer \( m \) such that

\[ \Lambda(s) = s^m(1-s)^m G(s) L(s) \]

is entire of order 1 and satisfies \( \Lambda(s) = \Lambda(1-s) \).

**Example**: \( L(s, f), \ L(s, F), \) and \( L(s, f \otimes \overline{F}) \).

**Example**: \( L(s, \phi) \) for \( \phi \) any automorphic form.

**Example**: If \( L_1(s) \) and \( L_2(s) \) are L-functions, so is \( L_1(s)L_2(s) \).

**Example**: If \( L(s, \phi) = \prod_p \prod_i (1 - \alpha_{i}(p)p^{-s})^{-1}, \ L(s, \psi) = \prod_p \prod_j (1 - \beta_{j}(p)p^{-s})^{-1} \are L-functions, then sometimes so is \( L(s, \phi \otimes \psi) \approx \prod_p \prod_{i,j} (1 - \alpha_{i}(p)\beta_{j}(p)p^{-s})^{-1} \)
Note: These last two examples form an “algebra” of L-functions.

If \( L(s, \phi) = \prod_p \prod_i (1 - \alpha_i(p) p^{-s})^{-1} \),

then \( \log L(s, \phi) = \sum_p \sum_{k=1}^\infty \frac{\alpha_i(p) k}{k p^{ks}} \).

If \( L(s, \psi) = \prod_p \prod_j (1 - \beta_j(p) p^{-s})^{-1} \),

then \( \log L(s, \phi) L(s, \psi) = \sum_p \sum_{k=1}^\infty \frac{\alpha_i(p) k + \sum_j \beta_j(p) k}{k p^{ks}} \),

and \( \log L(s, \phi \otimes \psi) = \sum_p \sum_{k=1}^\infty \left( \frac{\alpha_i(p) k}{k p^{ks}} \right) \left( \frac{\beta_j(p) k}{k p^{ks}} \right) \).

**Lemma A:** Suppose \( \log L(s) \) has positive Dirichlet series coefficients. Then, there is an effective constant \( c = c(d, m) \) such that \( L(s) \) has at most \( m \) real zeros in the range \( 1 - \frac{c}{\log M} < s < 1 \), where \( M = 1 + D \max \sum_{j=1}^d k_j j^2 \).

**Lemma B:** Suppose \( L(s) \) has a simple pole at \( s = 1 \) (i.e. \( m = 1 \)) and positive Dirichlet series coefficients.

Given \( M > 1 \), if \( |L(\frac{1}{2} + i \tau)| \leq M (|\chi| + 1)^B \), then there are effective constants \( c_1 = c_1(B), c_2 = c_2(B) \) such that

\[
\text{res}_s L(s) \frac{M c_1(1-\beta)}{1-\beta} \geq c_2, \quad \text{if } L(\beta) \leq 0, \quad \frac{1}{\delta} + \frac{1}{100} < \beta < 1.
\]
Step 3: Siegel's Theorem

Let $\chi_1 \neq \chi_2$ be real primitive characters mod $q^1 q^2$. Consider the $L$-function

$$L(s) = 5(s) L(s, \chi_1) L(s, \chi_2) L(s, \chi_1, \chi_2)$$

of degree $d=4$ with a pole of order $m=1$ at $s=1$.

Note that $\log L(s) = \sum_p \sum_{k=1}^{\infty} \frac{(1+\chi_1(p)^k)(1+\chi_2(p)^k)}{k p^{ks}}$

has positive Dirichlet series coefficients.

By Lemma A, $L(s)$ has at most one real zero in the range $1 - \frac{\varepsilon}{\log q^1 q^2} < s < 1$. Therefore, if $L(\beta_1, \chi_1) = 0$, then $\min \{ \beta_1, \beta_2 \} \leq 1 - \frac{\varepsilon}{\log q^1 q^2}$. 

Now, take any $0 < \varepsilon < \frac{1}{2} - \frac{1}{100}$. Let us show $L(1, x_2) \gg q_2^{-\varepsilon}$.

Choose $\beta \geq 1 - \varepsilon > \frac{1}{2} + \frac{1}{100}$ and $x_1$ as follows:

If all $L(s, x)$ are non-vanishing for $1 - \varepsilon < s < 1$,
then $\beta = 1 - \varepsilon$ and $x_1(n) = \left( \frac{n}{3} \right)$.

Otherwise, choose $\beta, x_1$ so that $L(\beta, x_1) = 0$.

In either case, $L(\beta) \leq 0$, which means we may apply Lemma B (with $M = c(q_1, q_2)^4$). We find

$$1 \ll L(1, x_1) L(1, x_2) L(1, x_1 x_2) \frac{(q_1 q_2)^{c(1 - \beta)}}{1 - \beta} \ll (\log q_1) L(1, x_2) (\log q_1 q_2)^{(q_1 q_2)^{c(1 - \beta)}} \ll \varepsilon L(1, x_2) \frac{q_2^{(1 + 1) \varepsilon}}{1 - \beta}.$$ 

Therefore, $L(1, x_2) \gg q_2^{-\varepsilon}$, with the implicit constant depending on $\beta$ and $x_1$. □
Step 4: Proof of the main theorem

Our goal: Show \( L(1, F) \gg \frac{1}{\log(1+\lambda N)} \) effectively.

We could repeat and get \( L(1, F) \gg (\lambda N)^{-\varepsilon} \) ineffectively.

Consider

\[
L(s) = 5(s) L(s, F)^2 L(s, F \otimes F) \quad (\text{akin to } x_1 = x_2 \text{ above})
\]

\[
= 5(s) L(s, F)^3 L(s, \text{Sym}^2 F).
\]

By Bump-Ginzburg, \( L(s, \text{Sym}^2 F) \) is entire except for a simple pole at \( s=1 \) (when \( f \) is not a lift from \( GL \)).

Thus, \( L(s) \) is an \( L \)-function of degree \( d=16 \) with a pole of order \( m=2 \) at \( s=1 \).

Crucially, if \( L(s, F) \) has a zero at \( \beta \), then \( L(s) \) has a \underline{triple} zero at \( \beta \).
Let us examine $\log L(s)$.

\[
\log L(s, f) = \sum_{p} \sum_{k=1}^{\infty} \frac{\xi_p^k + \xi_p^{-k}}{kp^{ks}}
\]

\[
\log L(s, F) = \sum_{p} \sum_{k=1}^{\infty} \frac{1 + \alpha_p^k + \alpha_p^{-k}}{kp^{ks}} \quad \text{(where } \alpha_p = \frac{\xi_p}{\xi_p} = \frac{\xi_p^2}{\xi_p} \text{)}
\]

\[
\log L(s, F\circ F) = \sum_{p} \sum_{k=1}^{\infty} \frac{(1 + \alpha_p^k + \alpha_p^{-k})^2}{kp^{ks}}
\]

\[
\log L(s) = \sum_{p} \sum_{k=1}^{\infty} \frac{(2 + \alpha_p^k + \alpha_p^{-k})^2}{kp^{ks}}
\]

Note: $\xi + \xi' = \alpha(p)$, $\xi \xi' = x(p)$, and $\alpha(p) = x(p) \bar{\alpha(p)}$.

Therefore, either $|\xi| = 1$ or $\xi = \bar{\xi} x(p)$, so either $|\alpha_p| = 1$ or $|\alpha_p| = \alpha_p$.

Either way, $\log L(s)$ has positive coefficients.

By Lemma A, $L(s)$ has at most 2 real zeros in the range $1 - \frac{c}{\log(1+\lambda N)} < s < 1$, so $L(s, F)$ cannot have any.
Let $\beta = 1 - \frac{C}{\log(1 + \lambda N)}$, so that $S(\beta)L(\beta, F) \leq 0$. By Lemma B applied to $S(s)L(s, F)$ (with $M = (1 + \lambda N)^{100}$),

$$L(1, F) \frac{M^{c(1-\beta)}}{1-\beta} \ll \log(1 + \lambda N) (1 + \lambda N)^{100c} / \log(1 + \lambda N) \ll L(1, F) \log(1 + \lambda N)$$

The moral: Find an $L$-function with positive coefficients which has more copies of $L(s, \Phi)$ than poles at $s = 1$. If you know automorphy of the factors, you can rule out Siegel zeros.

For more on this, see Hoffstein-Ramakrishnan (1995).
Step 5: Proof of Lemmas A and B

Lemma A: Suppose $\log L(s)$ has positive Dirichlet series coefficients. Then, there is an effective constant $c = c(d, m)$ such that $L(s)$ has at most $m$ real zeros in the range $1 - \frac{c}{\log M} < s < 1$, where $M = 1 + \max_{1 \leq j \leq d} \| K_j \|_2$.

Proof: Write $\Lambda(s) = s^m (1-s)^m G(s) L(s) = e^{A+B \sum \frac{1}{s-\rho}} \prod (1-\frac{s}{\rho}) e^{\delta \rho}$.

Take the logarithmic derivative to find

$$\sum_{\rho} \frac{1}{s-\rho} = \frac{m}{s} + \frac{\log M}{s-1} + \frac{G'(s)}{G(s)} + \frac{L'(s)}{L(s)}.$$ 

Each term is positive and $\ll \log M$.

If $\beta_1, \ldots, \beta_n$ are the zeros in $1 - \frac{c}{\log M} < s < 1$, then

$$\sum_{i=1}^{n} \frac{1}{s-\beta_i} \leq \frac{m}{s-1} + c \log M.$$ 

Choose $s = 1 + \frac{\delta}{\log M}$, so that

$$\frac{n \log M}{\delta + c} \leq \sum_{i=1}^{n} \frac{1}{s-\beta_i} \leq \frac{m \log M}{\delta} + c \log M,$$

and hence $n \leq m \left(1 + \frac{c}{\delta}\right) + c_1 (\delta + c)$. Choose $\delta < c_1$, and then $c$ small to conclude $n \leq m$. ☐
Suppose $L(s)$ has a simple pole at $s=1$ (i.e. $m=1$) and positive Dirichlet series coefficients.

Given $M>1$, if $|L(\frac{1}{2}+i\gamma)| \leq M (1+\gamma)^B$ then there are effective constants $c_1=c_1(B)$, $c_2=c_2(B)$ such that

$$\text{res}_{s=1} L(s) \geq c_2 \text{ if } L(\beta) \leq 0, \quad \frac{1}{a+100} < \beta < 1.$$

We will use the integral transform

$$\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{x^s ds}{s(s+1)\cdots(s+r)} = \left\{ \begin{array}{ll} \frac{1}{r!}(1-\frac{x}{r})^r & \text{if } 1<x, \\ 0 & \text{if } 0<x<1. \end{array} \right.$$ 

**Proof:**

Fix an integer $r > B$. Let $R = \text{res}_{s=1} L(s)$.

Let $L(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s}$, so that

$$I := \int_{a-i\infty}^{a+i\infty} \frac{L(s+\beta)x^s ds}{s(s+1)\cdots(s+r)} = \frac{1}{r!} \sum_{n<x} \frac{b(n)}{n^\beta} (1-\frac{x}{n})^r.$$

Since $b(1)=1$ and $b(n) \geq 0$, $I \gg 1$ for $x \geq 2$.

By hypothesis, we may shift the line of integration to $\text{Re } s = \frac{1}{2} - \beta$, picking up residues at $s=0, 1-\beta$.

We find $I = \frac{R x^{1-\beta}}{(1-\beta)(2-\beta)\cdots(r+1-\beta)} + \frac{L(\beta)}{r!} + O(M x^{\frac{1}{2}-\beta})$.

Choose $x = M$ to find $1 \ll I \ll \frac{R M^{200(1-\beta)}}{1-\beta}$. □