Direction Finding Using Non-coherent Measurements in Large Antenna Arrays

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***Abstract*—Direction finding based on array elements capable of acquiring only non-coherent or phaseless measurements is considered. Fundamental limits on the angle estimation error for a finite number of planar wavefronts impinging on the array as a function of the number of antennas are derived through a characterization of the Fisher information matrix. The performance of sparsity-based computationally efficient algorithms for direction finding from recent advances in the phase retrieval literature are also presented. Estimation error under both coherent and non-coherent measurements is shown to decrease as the cube of the number of antennas when this number is large, with a small gap in performance for the non-coherent case.**

I. Introduction

Multiple-antenna technology offers many performance benefits in wireless communications, and hence has been widely deployed in both cellular and WiFi systems. Coherent multiple-antenna systems typically employ an all-digital architecture where each antenna element in the antenna array is connected to a coherent radio chain or an antenna port. Cost and complexity constraints, however, often preclude having a separate coherent radio chain for each antenna element. This is especially the case when there is a large excess of antenna elements or for radio systems operating at a large carrier frequency.

In this work we investigate an all-digital architecture for large antenna arrays which do not have a phase-coherent reference, either because it is too costly or it is infeasible for the wavelength of operation. We assume that only non-coherent measurements are possible at each antenna element. With such a system, it may still be of interest to extract the directions of arrival of incoming wavefronts. In a communication system application, this direction finding may be a step in the extraction of channel parameters; whereas in a localization application, it may be related to accurate positioning. The direction finding problem in the context of coherent antenna arrays with phase measurements has been well studied in prior work (classic references include [1], [2]).

Our focus in this work is direction finding in a non-coherent system. This problem has been well studied in literature focused on imaging and optics applications, where coherent measurements are difficult if not impossible. In the imaging and optics literature, this problem is commonly referred to as the phase retrieval problem [3]. We revisit the phase retrieval problem in the context of a radio system with a large number of antenna elements, and seek to characterise its fundamental performance limits in this setting.

Specifically, we consider the problem of direction finding in a receiver with a large antenna array based on phaseless energy measurements. This is relevant not only for situations where having a coherent phase reference at each antenna element is difficult to guarantee due to a large number of antenna elements, but also where the wavelength of operation precludes any phase acquisition due to the difficulty of building systems at large carrier frequencies. Direction finding with non-coherent measurements is also related to channel estimation because, when the far-field propagation assumptions hold, the channel response is completely specified by the complex gain, and the angle of arrival of each of the wavefronts hitting the antenna element. With phaseless measurements we cannot hope to retrieve the phase factor but with a large enough number of antenna elements, we may still be able to resolve both the gain and the angles of arrival.

In order to characterize the limit of statistical error in retrieving the angles of arrival under coherent or non-coherent measurements, we characterize the leading order terms in the asymptotic expansion of the Fischer information matrix (FIM) in a uniform linear array with a large number of elements. This is a measure of the performance of all unbiased estimators for the direction of arrival. We subsequently perform a numerical comparison of the performance of different direction of arrival estimation algorithms for both coherent and non-coherent measurements. We find that even though coherent direction finding may have better performance for a certain number of antenna elements in terms of having a lower squared error, non-coherent direction finding exhibits little performance degradation compared to the coherent case. With an increasing number of antennas, the squared error decreases in both cases with the cube of the number of antennas.

The rest of the paper is organized as follows. We present the signal model in Section II, followed by an overview of prior work on direction finding both with phase-coherent and non-coherent measurements in Section III. In Section IV we present a derivation of the Fischer information matrix for the unknown angle parameters for a finite number of antenna elements. We subsequently present an asymptotic analysis of performance in the limit of a large number of antennas which gives us tractable expressions and allows us to make direct analytical observations independent of the underlying numerical algorithm chosen. In Section V we present numerical results for the performance achievable with a representative numbers of antenna elements and SNRs. Finally, we conclude in Section VI.

II. System model

We assume L planar wavefronts impinging on a uniformly linear antenna array with co-located antenna elements. These wavefronts may arise as multipath components of a single source or entirely different sources. Wavefront k has an amplitude gain $g_k \in \mathbf{R}^+$, the physical angle of arrival θ_k and a constant phase offset of ϕ_k. In this work, we also consider $\rho_k = \pi \sin(\theta_k)$, which can be thought of as the angle of arrival in the beamspace. We assume that the joint distribution on $\{\rho_k\}_{k=0}^{L-1}$ is such that

- the minimum separation between the differences between any two random angles of arrival is greater than ϵ, and
- $\rho_0 = 0, \rho_1 = \pi/2$.

The second assumption is to resolve the inherent ambiguity in the angles of arrival introduced by discarding the phase information at the receiver.

The phaseless noisy received signal at the i^{th} antenna element is given by

$$y_i = \left| \sum_{k=0}^{L-1} g_k e^{j\phi_k} e^{ji\pi\rho_k} \right|^2 + \nu_i \tag{1}$$

for $i \in [0, N-1]$, where N is the number of antenna elements. The additive noise ν_i is assumed to be $\mathcal{N}(0,1)$.

When the measurements have complete phase information at the receiver, we have the following model.

$$\tilde{y}_i = \sum_{k=0}^{L-1} g_k e^{j\phi_k} e^{ji\pi\rho_k} + \tilde{\nu}_i. \tag{2}$$

The additive noise has been assumed to be distributed as an unit energy complex Gaussian $\mathcal{CN}(0,1)$. Furthermore, for simplicity, we assume in this work that $g_k = \frac{1}{\sqrt{L}}$.

The particular form (in (1)) of the phaseless measurements considered in this work assumes that the additive noise is added after the energy detecting front-end. Since a fair comparison of the direction finding capabilities of phaseless and with-phase measurements needs to ensure comparable operating signal to noise ratios, in our formulation we have that the average detected signal energy at each antenna element is equal to 1, and that the additive noise power is the same, i.e., $\mathbb{E}[|\nu_i|^2] = \mathbb{E}[|\tilde{\nu}_i|^2]$.

III. Prior work

The study of recovering the angle parameters from measurements with phase has a rich history in the spectral estimation literature. Spectral estimation is defined as the problem of estimating frequency tones of a multi-tone signal from a noisy time series. Research in spectrum estimation has developed both fundamental theory as well as practical algorithms that are robust to noise and applicable under various modeling assumptions. Common approaches to spectrum estimation include the discrete Fourier transform, signal subspace methods, (such as Capon beamforming [4], the Pisarenko method [5], the MUSIC [6] and ESPRIT [7] algorithms), and approaches that solve the maximum likelihood parameter estimation problem in a computationally efficient manner (e.g. using algorithms such as SAGE [8]). There have also been works on characterizing the fundamental limits on direction of arrival estimation. In particular, [9] and related works study the statistical efficiency of the MUSIC algorithm in terms of the Cramer Rao bound. In a work very closely related to the results in this paper, [10] characterizes the properties of the Fischer information matrix for a large antenna array. This then yields the Cramer Rao bound, which serves as a universal bound on the performance of all unbiased estimators of the direction of arrival.

There has also been a lot of interest in phase retrieval algorithms for applications that are limited to phaseless measurements, such as crystallography, electron microscopy and optical imaging (an overview of this work can be found in [11]). Much recent interest has focused on exploiting prior knowledge about the signal structure (such as sparsity) to reduce the number of measurements needed for practical phase retrieval ([12]).

IV. Fischer information matrix characterization

The Fischer information matrix for the joint estimation of $\boldsymbol{\rho}$ from measurements $\mathbf{y}$ or $\tilde{\mathbf{y}}$ depends on the log likelihood function of the measurements given $\boldsymbol{\rho}$. The $(s,t)^{th}$ element of the Fischer information matrix $\mathcal{I}$ is defined as

$$\mathcal{I}_{s,t} = -\frac{\partial^2}{\partial\rho_s \partial\rho_t} \mathbb{E}[\log(f)|\boldsymbol{\rho}].$$

We start with the likelihood function for the signal model considered in (2). The log-likelihood function for the system with phase measurements is given by

$$\begin{aligned}\log(f(\tilde{\mathbf{y}};\boldsymbol{\rho})) &= -\sum_{i=0}^{N-1} |\tilde{y}_i - \sum_{k=0}^{L-1} g_k e^{j\phi_k} e^{ji\rho_k}|^2 - N\log(\pi) \\ &= -\sum_{i=0}^{N-1} |\mathcal{F}(\tilde{\mathbf{y}})_i - \sum_{k=0}^{L-1} g_k e^{j\phi_k} \mathtt{S}(i,\rho_k)|^2 - N\log(\pi)\end{aligned} \tag{3}$$

The second step follows from taking a Fourier transform across the antenna array. $\mathcal{F}(\tilde{\mathbf{y}})$ is the Fourier transform of $\mathbf{y}$; the i^{th} component of which is defined as

$$\mathcal{F}(\tilde{\mathbf{y}})_i = \sum_{s=0}^{N-1} \frac{1}{\sqrt{N}} \tilde{y}_s e^{-jsi2\pi/N}.$$

Since the transform is unitary, the squared norm in the second step of (3) stays the same. $\mathtt{S}$ is defined through the following:

$$\mathrm{S}(i,\rho_k) \triangleq \frac{1}{\sqrt{N}} \sum_{l=0}^{N-1} e^{jl\rho_k} e^{-jli2\pi/N} \tag{4}$$

$$= \frac{1}{\sqrt{N}} \frac{1 - e^{jN(\rho_k - 2i\pi/N)}}{1 - e^{j(\rho_k - 2i\pi/N)}}. \tag{5}$$

We now observe that:

$$|\bar{y}_i - \sum_{k=0}^{L-1} g_k e^{j\phi_k} \mathrm{S}(i,\rho_k)|^2$$
$$= |\bar{y}_i|^2 + |\sum_{k=0}^{L-1} g_k e^{j\phi_k} \mathrm{S}(i,\rho_k)|^2 - 2\mathrm{Re}(\tilde{y}_i^* (\sum_{k=0}^{L-1} g_k e^{j\phi_k} \mathrm{S}(i,\rho_k)))$$

This implies

$$\frac{\partial^2}{\partial\rho_s\partial\rho_t} \mathbb{E}[|\bar{y}_i - \sum_{k=0}^{L-1} g_k e^{j\phi_k} \mathrm{S}(i,\rho_k)|^2]$$
$$= -\frac{\partial^2}{\partial\rho_s\partial\rho_t} |\sum_{k=0}^{L-1} g_k e^{j\phi_k} \mathrm{S}(i,\rho_k)|^2, \tag{6}$$

where the last step follows by taking the expectation over the distribution of the random additive noise in the transform domain.

For large N, both $\mathrm{S}(i,\rho_k)$ and its second derivative have a significant value only in a neighborhood of size $\Theta(1/N)$ around ρ_k. This and the smoothness of the function $S(i,\rho_k)$ with respect to ρ_k makes it possible for us to focus only on values of i such that $2\pi i/N$ is near ρ_k, i.e., it is within $\Theta(1/N)$ of ρ_k. This allows us, in a limit of a large enough N, and under the assumption that the ρ_ks are separated by at least ϵ, to make the following approximate equivalence. The expressions are exact up to the highest order term as N becomes larger and larger.

$$\frac{\partial^2}{\partial\rho_s\partial\rho_t} \sum_{i=0}^{N-1} \mathbb{E}[|\tilde{y}_i - \sum_{k=0}^{L-1} g_k e^{j\phi_k} \mathrm{S}(i,\rho_k)|^2]$$
$$\approx -\frac{\partial^2}{\partial\rho_s\partial\rho_t} \sum_{k=0}^{L-1} \sum_{i\in\mathcal{N}_k} |g_k e^{j\phi_k} \mathrm{S}(i,\rho_k)|^2, \tag{7}$$

where $\mathcal{N}_k$ refers to a $\Theta(1/N)$ neighborhood of ρ_k. Observe that as $N \to \infty$, there are a countable number of terms in $\mathcal{N}_k$. From this expression we observe that the Fischer information matrix for the measurements with phase is diagonal, ignoring lower order terms. The diagonality indicates that the estimation errors of one of the angles of arrival in our problem is independent from estimation errors of other angles of arrival. Moreover we can show that

$$\frac{\partial^2}{\partial\rho_s^2} \sum_{k=0}^{L-1} \sum_{i\in\mathcal{N}_k} |g_k e^{j\phi_k} \mathrm{S}(i,\rho_k)|^2$$
$$= \sum_{i=0}^{N-1} \frac{\partial^2}{\partial\rho_s^2} \left(|g_s|^2 \frac{1}{N} \left| \frac{\sin(N(\rho_s/2 - \pi i/N))}{\sin(\rho_s/2 - \pi i/N)} \right|^2 \right) = c|g_s|^2 N^3, \tag{8}$$

where c is a constant as shown in Appendix VII-A. When $g_k = \frac{1}{\sqrt{L}}$, the Fischer information matrix $\tilde{\mathcal{I}}$ for the variables $\{\rho_k\}_{k=0}^{L-1}$ is given by the following:

$$\tilde{\mathcal{I}}_{s,t} = \begin{cases} \frac{|c|N^3}{L} \text{ if } s = t, \\ 0 \text{ otherwise.} \end{cases} \tag{9}$$

This completes the derivation of the Fischer information matrix for the measurements with phase information.

The Fischer information matrix for the phaseless measurements case has a derivation very similar to the above; with a major difference in the analog of step (8). Before we describe that, we observe that the squared energy operation is equivalent to a convolution in the beamspace domain. To see this, observe that the energy at the i^{th} antenna for the phaseless measurements can be written as

$$\left| \sum_{k=0}^{L-1} g_k e^{j\phi_k} e^{ji\rho_k} \right|^2$$
$$= \sum_{k,l} g_k g_l e^{j(\phi_k - \phi_l)} e^{ji(\rho_k - \rho_l)}. \tag{10}$$

Thus, if the actual signal in the angle of arrival domain corresponding to the coherent measurements is

$$\tilde{s}(\rho) = \sum_{k=0}^{L-1} g_k e^{j\phi_k} \delta(\rho - \rho_k),$$

then losing the phase information leads to the following signal instead:

$$s(\rho) = (\sum_{k=0}^{L-1} g_k^2)\delta(\rho) +$$
$$\sum_{(l,k):l\neq k, 0\leq l,k\leq L-1} (g_k g_l e^{j\phi_k} e^{-j\phi_l})\delta(\rho - \rho_k + \rho_l). \tag{11}$$

We can now make the following observations:

- The number of non-zero elements in the angle of arrival domain increases with phaseless measurements.
- There is a coupling between the estimation errors of the angles of arrival introduced by the loss of phase information at the antenna elements. In other words, even in the limit of a large number of antennas, the Fischer information matrix is no longer diagonal.

To characterize the Fischer information matrix for this system, we start with the likelihood function. From the system model (1), we have that the log-likelihood function is given by

$$\log(f(\mathbf{y};\boldsymbol{\rho})) = -\sum_{i=0}^{N-1} 0.5|y_i - \sum_{k,l=0}^{L-1} g_k g_l e^{j(\phi_k - \phi_l)} e^{ji(\rho_k - \rho_l)}|^2$$
$$- 0.5N\log(2\pi)$$
$$= -\sum_{i=0}^{N-1} 0.5|\mathcal{F}(\mathbf{y})_i - \sum_{k=0}^{L-1} g_k g_l e^{j(\phi_k - \phi_l)} \mathrm{S}(i,\rho_k - \rho_l)|^2$$
$$- 0.5N\log(2\pi). \tag{12}$$

The remaining steps are very similar to the corresponding steps (6) to (7) for the system with phase measurements. However, the double derivative of step (6) evaluates to a different value in this case. In particular, the double derivative of step (6) for the case of phaseless measurements is given by

$$\begin{aligned}
&0.5\frac{\partial^2}{\partial\rho_s\partial\rho_t}\mathbb{E}[|y_i-\sum_{k,l=0}^{L-1}g_kg_le^{j(\phi_k-\phi_l)}\mathrm{S}(i,\rho_k-\rho_l)|^2] \\
&=-0.5\frac{\partial^2}{\partial\rho_s\partial\rho_t}|\sum_{k,l=0}^{L-1}g_kg_le^{j(\phi_k-\phi_l)}\mathrm{S}(i,\rho_k-\rho_l)|^2 \qquad (13)\\
&\approx-0.5\frac{\partial^2}{\partial\rho_s\partial\rho_t}\sum_{l\neq k}|g_kg_l\mathrm{S}(i,\rho_k-\rho_l)|^2,
\end{aligned}$$

where the last approximation uses the assumption that the differences $\rho_k-\rho_l$ have a minimum separation independent of N. Thus we have

$$\begin{aligned}
&0.5\frac{\partial^2}{\partial\rho_s\partial\rho_t}\sum_{i=0}^{N-1}\mathbb{E}[|y_i-\sum_{k,l=0}^{L-1}g_kg_le^{j(\phi_k-\phi_l)}\mathrm{S}(i,\rho_k-\rho_l)|^2] \\
&\approx-0.5\frac{\partial^2}{\partial\rho_s\partial\rho_t}\sum_{k\neq l}\sum_{i\in\mathcal{N}_{k,l}}|g_kg_l\mathrm{S}(i,\rho_k-\rho_l)|^2 \\
&\approx-0.5\frac{\partial^2}{\partial\rho_s\partial\rho_t}\sum_{k\neq l}\sum_{i=0}^{N-1}|g_kg_l\mathrm{S}(i,\rho_k-\rho_l)|^2 \\
&=\begin{cases}\sum_{l\neq s}|g_sg_l|^2|c|N^3 & \text{when } s=t\\ -|g_sg_t|^2|c|N^3 & \text{otherwise,}\end{cases}
\end{aligned} \qquad (14)$$

where $\mathcal{N}_{k,l}$ is a collection of indices i such that $2\pi i/N$ is within $\Theta(1/N)$ of $(\rho_k-\rho_l) \mod 2\pi$, and c is the same constant that has been derived in Appendix VII-A.

When $g_k=\frac{1}{\sqrt{L}}$, the Fischer information matrix is given by

$$\mathcal{I}_{s,t}=\begin{cases}\frac{L-1}{12L^2}N^3 & \text{if } s=t,\\ -\frac{N^3}{12L^2} & \text{otherwise.}\end{cases} \qquad (15)$$

Before we go on to the numerical section, we comment here that the above analysis is not sensitive to the assumption about whether g_k or ϕ_k are known perfectly or not. Lack of perfect knowledge of these parameters will only have second order effects (in N) in terms of the error in the angle of arrival estimation.

V. Numerical results

In this section, we look at two scenarios. In the first, we corroborate the inverse Fischer matrix estimates obtained with maximum likelihood estimates. In the second, we look at practical sparsity based approaches.

A. Maximum Likelihood Estimates and the Fisher Information Matrix

From results obtained in section IV, we can obtain the leading terms of the Fischer information matrix and invert this to obtain the Cramer-Rao bound.

We compare this to maximum likelihood estimates obtained by solving the following problems in the phased and phaseless case

$$\min_{g_k,\rho_k}\sum_i|\hat{y}_i-\sum_{k=1}^{N}g_ke^{j i\rho_k}|^2$$

$$\min_{g_k,\rho_k}\sum_i\left(\sum_i\hat{y}_i-|\sum_k g_ke^{j i\rho_k}|^2\right)^2$$

These are non-convex optimization problems. As the number of sources increases in number, this amounts to performing a search over a high-dimensional grid, making it impractical.

In Fig. 1, we compare the inverse Fischer matrix and MLE estimates when we have 3 sources, only one of which is unknown. It can be verified that the behaviors of the inverse Fischer estimates and the MLE estimates follow a similar trend of decreasing as N^{-3}. Pertinently, the gap in performance between the phased and phaseless case is small and further decreases as the number of antennas increases.

B. Practical Sparsity Based Approaches

Sparsity based methods form a large grid of size m for values of ρ [13]. It is assumed that there are a small number of active sources from this large grid. Let $x\in\mathbb{C}^m$ be a vector whose i^{th} dimension holds the amplitude and phase of the wavefront from direction $\rho=2\pi i/m$.

When we have measurements with phase, we can write

$$y=Ax+n,$$

where $A_{ik}=e^{j2\pi ik/N}$. When we wish to recover sparse x from measurements $\hat{y}$, we minimize the number of non-zero components or its convex envelope, $\|x\|_1$ while ensuring that $\|\hat{y}-Ax\|_2\leq\epsilon$. This amounts to solving the following efficient lasso optimization problem for an appropriate λ:

$$\min_x \quad \|\hat{y}-Ax\|_2^2+\lambda\|x\|_1.$$

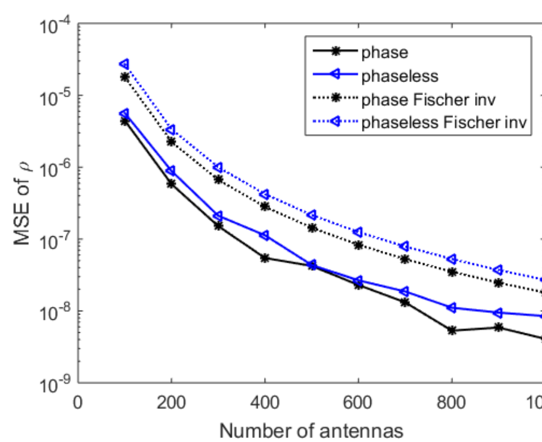

Fig. 1. Comparison of mean squared error obtained with maximum likelihood as well as the limits indicated by the inverse Fischer information matrix for the case with 3 sources (1 being unknown) as the number of antennas vary.

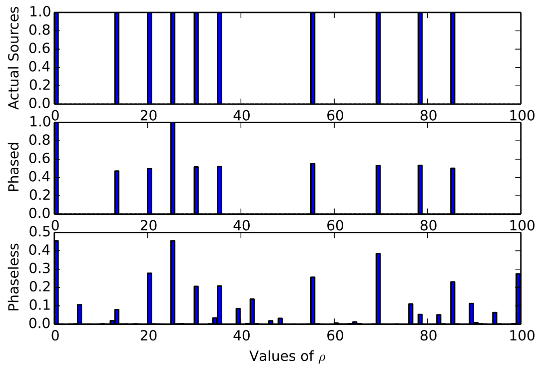

Fig. 2. A sample of the recovery of the sources with phased and phaseless measurements when we have 40 antennas and 10 sources.

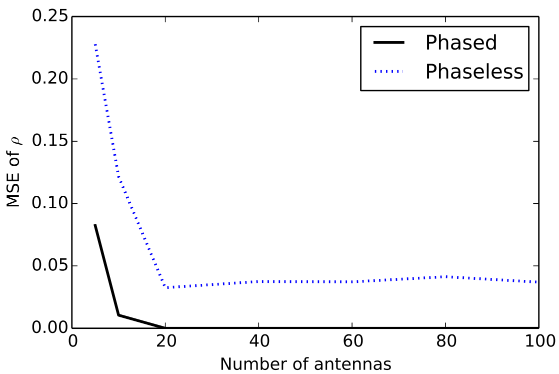

Fig. 3. The performance of sparsity based schemes as the number of antennas increases for 10 sources.

In the phaseless measurements scenario, we follow the approach of Phaselift [14]. Our measurements $\hat{y}$ are

$$\begin{aligned}\hat{y}_i &= |A_i x|^2 + n \\ &= A_i x x^T A_i^T + n = A_i X A_i^T + n,\end{aligned}$$

where A_i is the i^{th} row of matrix A. In this case, we find a sparse and single rank $X = xx^T$ where $\sum_i (\hat{y}_i - A_i X A_i^T)^2 \leq \epsilon$. Searching in the space of single rank matrices is a non-convex problem, hence we look at its convex envelope and minimize the nuclear norm of X, which amounts to minimizing its trace. This is equivalent to solving the following convex program:

$$\min_{X \succeq 0} \quad \sum_i (\hat{y}_i - A_i X A_i^T)^2 + \lambda_1 \text{Tr}(X) + \lambda_2 \sum_{i,j} X_{ij}.$$

Ideally, the solution to the program would be a single rank matrix, but in the presence of noise, we may not be guaranteed this and hence estimate x by taking the leading eigendimension. In [14], the authors show that this procedure recovers x in the noiseless case, and is stable in the presence of noise.

We assume that two components are known (i.e. $\rho = 0, \pi/2$ is known) which introduces additional equality constraints in the optimization routines. The adjustment of parameters λ, λ_1, and λ_2 plays a crucial role in obtaining good results. In our simulations, to compare the cases of measurements with and without phase, we do not optimize these parameters. In Fig. 2, we see an example of what the algorithms produce. There is a fairly close match of the support or directions of sources with both phased and phaseless measurements, however, for the case of phaseless measurements there are more components with small values. The number of non-zero values can be refined by performing the optimization over a grid of values of λ_1 and λ_2; the optimal values for these parameters would depend on the number of wavefronts and number of antennas.

From results such as those seen in Fig. 2, we obtain the directions of the sources by finding the components with largest magnitudes that are well separated. We now compare the directions obtained to the actual sources on the grid to obtain Fig. 3 which shows the impact of the number of antennas. What can be observed is that there is a gap in performance between phased and phaseless measurements and this gap narrows in the large antenna regime but does not decrease below a constant MSE.

VI. Conclusions

We have characterized the fundamental limits on the angle of arrival estimation error from phaseless energy measurements as a function of the number of antennas in a large antenna array. Our analysis characterizes the leading order term in the Fischer information matrix for the joint estimation problem; therefore it is applicable to all unbiased estimators. Our results show that, similar to known results for measurements with phase, for a large number of antenna elements, the mean squared error in the unbiased estimation of the angles of arrival from noisy energy measurements goes down as the inverse cube of the number of antenna elements. We also present numerical plots which corroborate these trends and compare them with the performance from practical sparsity-based parameter estimation algorithms from the spectral estimation and phase retrieval literature.

Our results suggest that, in wireless systems where acquiring phase information or achieving phase coherence is difficult, discarding phase information altogether may not incur much loss in optimality. We have shown in particular that the performance loss without phase information in retrieving the angle of arrival parameters of incoming wavefronts is small, especially when there is a large excess of antenna elements.

VII. Appendix

A. Derivation of the constant c

When $x \neq 2a$, we have,

$$\frac{\partial^2}{\partial x^2}\left(\frac{\sin(N(x/2-a))}{\sin(x/2-a)}\right)^2$$
$$= \frac{N^2}{2\sin(a-x/2)^2}\cos(N(x-2a))$$
$$+ \frac{1}{2}\left(\frac{\sin(N(x/2-a))}{\sin(x/2-a)}\right)^2\left(\frac{1}{\sin(a-x/2)^2}+\frac{2}{(\tan(a-x/2))^2}\right)$$
$$- \frac{N\sin(N(x-2a))}{\tan(x/2-a)\sin^2(x/2-a)}. \tag{16}$$

When $x = 2a$, the above expression simplifies to $-\frac{1}{6}N^2(N^2-1)$.

Let x be such that $\frac{x}{2\pi}$ is rational. Using (16), we can verify that

$$\sum_{i=0}^{N-1}\frac{\partial^2}{\partial x^2}\left(\frac{\sin(N(x/2-\pi i/N))}{\sin((x/2-\pi i/N))}\right)^2 = cN^4 + o(N^4), \tag{17}$$

where

$$c = -\frac{1}{6} + \frac{1}{2\pi^2}\sum_{k=1} 1/k^2 \tag{18}$$

$$= -\frac{1}{6} + \frac{1}{2\pi^2}\frac{\pi^2}{6} = \frac{-1}{12}. \tag{19}$$

When $x/2\pi$ is not rational, the same argument above can be applied to each point in a sequence of increasingly accurate rational approximations to it; thereby establishing the result for all real x.

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