Parameterized Expectations Algorithm: How to Solve for Labor Easily*

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June 8, 2004

Abstract

Euler-equation methods for solving nonlinear dynamic models involve parameterizing some policy functions. We argue that in the typical macroeconomic model with valuable leisure, labor function is particularly convenient for parameterizing. This is because under the labor-function parameterization, the intratemporal first-order condition admits a closed-form solution, while under other parameterizations, there should be a numerical solution. In the context of a simulation-based parameterized expectations algorithm, we find that using the labor-function parameterization instead of the standard consumption-function parameterization reduces computational time by more than a factor of ten.

JEL classification: C6, C63, C68

Key Words: Nonlinear models, Parameterized expectations, PEA, Monte Carlo simulation, Numerical solution

*We thank an anonymous referee and Joyce Gleason for valuable comments. All errors are ours. This research was supported by the Instituto Valenciano de Investigaciones Económicas and the Ministerio de Ciencia y Tecnología de España, the Ramón y Cajal program and BEC 2001-0535.

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1 Introduction

Euler-equation methods for solving nonlinear dynamic models involve parameterizing some optimal policy functions. There is a substantial degree of freedom in deciding which policy functions to parameterize. For example, in the typical intertemporal utility-maximization problem, one can parameterize the consumption function, investment function, asset function, etc. Depending on the model considered, some parameterizations might be more convenient for computing equilibrium than others.

In this paper, we argue that for the typical macroeconomic model where leisure is valued, it is more convenient to parameterize the labor function than the other policy functions such as e.g. the consumption function. To make the point, we consider the standard neoclassical stochastic growth model by Kydland and Prescott (1982). We show that, if the labor function is parameterized, then the intratemporal First-Order Condition (FOC) admits a closed-form solution, while if the consumption function is parameterized, there should be a numerical solution. Since the latter case requires that intratemporal decisions be computed numerically at each step, computational expense will be generally larger than in the former case, where such decisions

can be calculated analytically.[1]

In the context of the simulation-based Parameterized Expectations Algorithm (PEA) by den Haan and Marcet (1990), we find that using the labor-function parameterization instead of the standard consumption-function parameterization (see, e.g., Marcet and Lorenzoni, 1999) reduces the computational time by more than a factor of ten. We expect that the proposed modification will lead to a comparable reduction in computational expense under other Euler-equation methods, for example, under Coleman's (1990) and Christiano and Fisher's (2000) algorithms iterating on a grid of prespecified points.

2 Solving for labor easily

We consider the standard neoclassical stochastic growth model by Kydland and Prescott (1982) under the assumptions of the addilog utility function and the Cobb-Douglas production function:

$$\max_{\{c_t, n_t, k_{t+1}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \delta^t \left[\frac{c_t^{1-\gamma} - 1}{1-\gamma} + b \frac{(1-n_t)^{1-\sigma} - 1}{1-\sigma} \right], \tag{1}$$

$$\text{s.t.} \qquad c_t + k_{t+1} = (1-d)\, k_t + \theta_t k_t^{\alpha} n_t^{1-\alpha}, \tag{2}$$

[1] Neither there is no closed-form solution to the intratemporal FOC if the investment or the asset functions are parameterized.

where initial condition (k_0, θ_0) is given, $c_t, k_{t+1} \geq 0$ and $n_t \in [0,1]$. Here, c_t, k_{t+1}, n_t are consumption, capital and hours worked (labor), respectively; $\delta \in (0,1)$ is the discount factor; $\gamma, \sigma \in [0, \infty)$, $b > 0$ are the utility-function parameters; $d \in (0,1]$ is the depreciation rate of capital; $\alpha \in (0,1)$ is the capital share in production; and E_0 is the operator of conditional expectation. The time endowment is normalized to one, so that the term $(1 - n_t)$ represents leisure. The technology, θ_t, follows a first-order autoregressive process, $\ln \theta_t = \rho \ln \theta_{t-1} + \varepsilon_t$, with $\rho \in [0,1)$ and $\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$. Our objective is to compute a recursive Markov solution to the problem (1), (2) such that the optimal decision rules are functions of the current state variables, (k_t, θ_t). If the solution is interior, then it is characterized by means of FOCs.

Marcet and Lorenzoni (1999) describe how to solve the model (1), (2) by using a simulation-based variant of the PEA (see their Example 7.3).[2] This method consists in finding a time-series solution to the FOCs of the model (1), (2) by parameterizing the conditional expectation in the intertemporal

[2]Example 7.3 in Marcet and Lorenzoni (1999) also includes taxes, however, this difference between our and their setups is irrelevant for the issues studied in the present paper.

FOC (the Euler equation) as follows:

$$c_t^{-\gamma} = \delta E_t \left[c_{t+1}^{-\gamma} \left(1 - d + \theta_{t+1} k_{t+1}^{\alpha-1} n_{t+1}^{1-\alpha} \right) \right] \simeq \delta \psi \left(\beta; k_t, \theta_t \right), \tag{3}$$

$$\left(1 - n_t \right)^{-\sigma} n_t^{\alpha} = \frac{1}{b} \theta_t k_t^{\alpha} c_t^{-\gamma}, \tag{4}$$

where $\psi \left(\beta; k_t, \theta_t \right)$ is a flexible function of the current state variables with β being a vector of parameters. For given ψ, β and (k_t, θ_t), the intertemporal FOC (3) determines consumption $c_t = \left[\delta \psi \left(\beta; k_t, \theta_t \right) \right]^{-1/\gamma}$, the intratemporal FOC (4) gives n_t, and finally, budget constraint (2) yields k_{t+1}. In this way, Marcet and Lorenzoni (1999) can solve for the series $\{c_t, n_t, k_{t+1}\}_{t=0}^{T}$ for a given random draw for shock, $\{\theta_t\}_{t=0}^{T}$, where T is the simulation length. They run simulations and iterate on the vector of parameters β until the approximation becomes sufficiently accurate; see Marcet and Lorenzoni (1999) for more details.[3]

Parameterization of the intertemporal FOC (3) does not allow for easy characterization of the consumer's intratemporal choice because the intratemporal FOC (4) does not in general admit a closed-form solution for n_t. Regarding this issue, Marcet and Lorenzoni (1999, p.152) say: "This nonlinear equation [(4)] has to be solved numerically for each β and t". To see what

[3]While our subsequent discussion is built around den Haan and Marcet's (1990) PEA, our arguments are valid for any Euler-equation method, which involves parameterizing either consumption or investment or asset function.

this means in terms of computational time, let us assume that the length of simulation is $T = 10000$ and that the PEA needs $I = 300$ iterations to converge (which is close to what one has in practical applications). Then, to solve the model, one needs to find a solution to (4) by a numerical solver as many as $T \times I = 3000000$ times. It is clear that computing the labor choice numerically on each step might slow down the PEA dramatically, compared to the case when such a choice can be restored analytically.

In fact, one can reduce the computational cost by calculating the labor function outside of the iterative cycle. Specifically, let us call the right-side of equation (4) by $a_t \equiv \frac{1}{b}\theta_t k_t^{\alpha} c_t^{-\gamma}$ and construct a grid for its values $\{a_1, a_2, ..., a_M\}$. The grid should be chosen so that the value of a_t, which can effectively occur along simulations, is always within the range $[a_1, a_M]$. Define the grid function $N(a_m)$ by

$$N(a_m) = \left\{ n_m : (1 - n_m)^{-\sigma} n_m^{\alpha} = a_m \right\}, \qquad m = 1, ..., M, \tag{5}$$

and compute the value of $N(a_m)$ for each $a_m \in \{a_1, a_2, ..., a_M\}$. With such a grid function, we can compute n_t at each date t by interpolation. Computational expense is therefore reduced by the difference between the time needed to solve for n_t by this interpolation and the time needed to find n_t by solving (4) numerically. We should emphasize, however, that using interpolation

methods in this context can still be costly.[4]

The alternative we propose in this paper makes it possible to restore all intratemporal choices at a literally zero cost. Our method is extremely simple: we just re-parameterize FOCs by combining (3) and (4) so that the intratemporal FOC can be resolved analytically:

$$\frac{1}{(1-n_t)^{\sigma}} = \delta E_t \left[\frac{\left(1-d+\theta_{t+1}k_{t+1}^{\alpha-1}n_{t+1}^{1-\alpha}\right)}{(1-n_{t+1})^{\sigma}} \frac{\theta_t k_t^{\alpha} n_t^{-\alpha}}{\theta_{t+1}k_{t+1}^{\alpha}n_{t+1}^{-\alpha}} \right] \simeq \delta\phi\left(\beta; k_t, \theta_t\right), \tag{6}$$

$$c_t = \left[\frac{b n_t^{\alpha}}{\theta_t k_t^{\alpha} (1-n_t)^{\sigma}} \right]^{-1/\gamma}, \tag{7}$$

where $\phi\left(\beta; k_t, \theta_t\right)$ is the new parameterizing function. Now, the Euler equation (6) gives us $n_t = 1 - \left[\delta\phi\left(\beta; k_t, \theta_t\right)\right]^{-1/\sigma}$, the intratemporal FOC (7) determines c_t and, as before, budget constraint (2) yields k_{t+1}. Thus, we can proceed with solving the model in the same way as Marcet and Lorenzoni (1999) do. However, we have closed-form expressions for all variables, so that we need neither numerical solvers nor interpolation.

Finally, we note that the above discussion is also valid for the case when the utility function in (1) is of the constant relative risk aversion type,

[4] To be precise, doing interpolation just once is not especially costly. In particular, polynomial interpolation can be formulated as solving a Vandermonde linear system which has fast and accurate methods (see, e.g., Bjorck and Pereyra, 1970). However, if interpolation is performed a large number of times, as is in our case, then the associated computational cost is considerable.

$\left[\frac{\left(c_t^{\mu}(1-n_t)^{1-\mu}\right)^{1-\eta}-1}{1-\eta}\right]$, $\mu \in (0,1)$, $\eta \in [0,\infty)$. Here, we again cannot explicitly solve for labor from the intratemporal FOC under the standard consumption-function parameterization. However, we can explicitly solve for consumption from the intratemporal FOC under the suggested labor-function parameterization. Also, we shall emphasize that the assumption of the Cobb-Douglas production function is not essential for our results, which remain true for a general production function.

3 Numerical comparison

In this section, we compare three different versions of the PEA: one is where the labor-leisure choice is computed numerically at each step (PEA I); a second is where the choice is computed by interpolation of the previously calculated grid-function (PEA II); and the third is where the choice is restored analytically by appropriately re-parameterizing the Euler equation (PEA III). We calibrate the model as in Maliar and Maliar (2001) by fixing the parameters to reproduce the key first-moment properties of the U.S. economy such as the share of capital in production α, the capital-output ratio π_k, the consumption-output ratio π_c, and aggregate labor n (see *Table*

1).

Table 1. The model's parameters.

α	n	π_c	π_k	δ	d	ρ	σ_ε	k_0	θ_0
1/3	1/3	3/4	10	0.99	0.025	0.95	0.01	k_{ss}	1

Here, k_{ss} denotes the capital stock in the deterministic steady state. The parameter b varies with the utility function parameters (γ, σ), such that $b = (1-\alpha)\,\pi_k^{(1-\gamma)\alpha/(1-\alpha)}\pi_c^{-\gamma}\,(1-n)^{\sigma}\,n^{-\gamma}$. We consider three alternative parameterizations, $(\gamma, \sigma, b) \in \{(1,1,1.78)\,,(1,5,0.35)\,,(5,1,4.55)\}$.

In all experiments, we approximate the conditional expectation in the Euler equation by a first-order exponentiated polynomial,

$$E_t\left[x_{t+1}\right] \simeq \exp\left(\beta_0 + \beta_1 \log k_t + \beta_2 \log \theta_t\right),$$

where x_{t+1} is the expression inside the corresponding conditional expectation, and $\beta = (\beta_0, \beta_1, \beta_2)$ is a vector of coefficients to be found. For the initial iteration, we calibrate β to match the deterministic steady state of the model by setting at $\beta_0 = \log\left[x_{ss}\right]$, $\beta_1 = 0$ and $\beta_2 = 0$, where x_{ss} is the steady state value of x_{t+1}. To ensure convergence and to rule out implosive and explosive strategies, we restrict simulated solutions by bounds, as described in Maliar and Maliar (2003). In all experiments, our PEA was able to systematically converge starting from the deterministic steady state. To update the vector

of coefficients β, we use a homotopy procedure: we compute $\beta(i+1)$ for each subsequent iteration $i+1$ as a weighted average of the vector $\beta(i)$ from the previous iteration i and its currently re-estimated value $G(\beta(i))$, $\beta(i+1) = (1-v)\beta(i) + vG(\beta(i))$, with the weight $v = 0.5$. We fix the length of simulation at $T = 5000$ periods. We use the convergence criterion that the L^2 distance between vectors β obtained in two subsequent iterations is less than 10^{-5}.

Our programming language is Matlab, and all the programs used are publicly available through the internet *http://merlin.fae.ua.es/maliarl* (Lilia Maliar) or *http://merlin.fae.ua.es/maliars* (Serguei Maliar). To solve for hours worked satisfying (4) and (5) numerically, we use the procedure *"csolve"* written by Christopher Sims, which we find to be both faster and more reliable than any built-in Matlab solver. To construct the grid function N defined in (5), we perform linear interpolation on a uniformly-spaced grid of 100 points by using a built-in Matlab procedure *"interp1"*. Our simulations were carried out on Pentium IV with 2.0 Ghz processor. In *Table 2*, we report the typical computational time for finding a solution under PEA I, II and

III.

Table 2. Computational time under PEA I, II, III.

	Computational time, sec		
Method \ (γ, σ, b)	$(1, 1, 1.78)$	$(1, 5, 0.35)$	$(5, 1, 4.55)$
PEA I, sec	603	661	588
PEA II, sec	279	310	317
PEA III, sec	40	58	52

As we see, PEA II is about two times faster than PEA I, while PEA III is more than 10 times as fast as PEA I. The difference between PEA I and PEA II is not so large as expected because, as we said above, interpolation methods still involve substantial computation expense. In contrast, PEA III is very fast because there is an explicit formula for labor, so that neither numerical solver nor interpolation need be used.

4 Conclusion

Today, the standard neoclassical stochastic growth model can be solved by a variety of methods that are both fast and accurate. However, there are still severe restrictions on computational time in more complicated settings (with heterogeneous agents, different types of capital and labor, etc.); see Rust (1996) for a discussion. The problem of restoring the consumer's intratemporal choices is present in any model built around the standard neoclassical setup. For models requiring many hours to be solved, decreasing the compu-

tational expense more than ten-fold by adapting the modification proposed in this paper would be valuable indeed.

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