Economica

Economica (2011) **78**, 240–259
doi:10.1111/j.1468-0335.2009.00817.x

Capital–Skill Complementarity and Balanced Growth

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Final version received 16 March 2009.

We construct a general equilibrium version of the Krusell *et al. Econometrica* **68**, 1029, 2000 model with capital–skill complementarity. We assume several sources of growth simultaneously: exogenous growth of skilled and unskilled labour, equipment-specific technological progress, skilled and unskilled labour-augmenting technological progress and Hicks-neutral technological progress. We derive restrictions that make our model consistent with balanced growth. A calibrated version of our model can account for the key growth patterns in the US data, including those for capital equipment and structures, skilled and unskilled labour and output, but it fails to explain the long-run behaviour of skilled-labour wages and, consequently, the skill premium.

Introduction

Krusell *et al*. (2000) show that a constant elasticity of substitution (CES) production function with four production inputs—capital structures, capital equipment, skilled and unskilled labour—is consistent with the key features of the US economy data.[1] In the data, the growth patterns over the 1963–92 period appear to be highly unbalanced: output and the stock of structures increased by a factor of two; the stock of equipment increased by more than seven times; the number of unskilled workers slightly decreased, whereas the number of skilled workers nearly doubled; the price of equipment relative to consumption (structures) went down by more than four times; and the skill premium was roughly stationary. All the above regularities are matched in Krusell *et al*. (2000), by construction, under the appropriate degrees of capital–skill complementarity.[2]

In this paper, we attempt to account for the above growth patterns in the context of a general equilibrium version of the Krusell *et al*. (2000) model. We restrict our attention to the standard class of models that are consistent with balanced (steady-state) growth in which all variables grow at constant (possibly differing) rates. A convenient property of such models is that they can be converted into stationary ones, so that their equilibria can be studied with standard numerical methods. We ask: Is a general equilibrium balanced growth model parameterized by the Krusell *et al*. (2000) CES production function still consistent with the US data?

The standard way to introduce balanced growth in macroeconomic models is to assume labour-augmenting technological progress (see, for example, King *et al*. 1988).[3] However, this assumption is not sufficient for our purpose since it implies that all variables (except labour) grow at the same rate, which does not agree with the empirical facts listed above. As shown in Greenwood *et al*. (1997), it is possible to account for the empirical observation that equipment grows at a higher rate than output by introducing two other kinds of technological progress, such as equipment-specific and Hicks-neutral ones. However, these two kinds of progress alone are consistent with balanced growth only under the assumption of the Cobb–Douglas production function (see Greenwood *et al*. 1997, p. 347) and not under our assumption of the CES production function.

The main theoretical result of the present paper is that we can make the CES production function consistent with balanced growth by combining the standard labour-augmenting technological progress with two kinds of progress introduced in Greenwood *et al.* (1997). To be specific, we simultaneously introduce equipment-specific technological progress, skilled and unskilled labour-augmenting technological progress, and Hicks-neutral technological progress, as well as exogenous growth of skilled and unskilled population. We impose the assumption of complete markets, which allows us to analyse equilibrium by considering the corresponding planner's problem. A distinctive feature of our setup is that skilled and unskilled populations grow at different rates. We show that in spite of this feature, welfare weights assigned by the planner to the two subpopulations depend not on their growth rates but only on their initial sizes. With this result and with some additional restrictions on preferences and the rates of progress, there exists a stationary economy associated with our growing economy.

We calibrate the model to match a set of relevant observations about the US economy. We find that the calibrated version of our model can account remarkably well for the key growth patterns in the data, including those for capital equipment and structures, skilled and unskilled labour, and output. Specifically, the above variables in our model grow at different rates, which are close to those in the data. Nonetheless, our model has an important drawback: it understates the growth rate of skilled-labour wages, and as a result it dramatically fails on the growth pattern of the skill premium predicting that the skill premium falls, while in the data, the skill premium exhibits a roughly stationary behaviour. The reason for this drawback is the following. Our assumption of balanced growth implies that the relative importance of all production inputs in output remains constant along the balanced growth path. If the number of skilled workers grows more than that of unskilled workers, then to maintain the same relative importance of the two types of labour, the productivity of skilled labour must grow proportionally less than that of unskilled labour, and hence the skill premium must decrease over time. We argue that the above drawback is a generic feature of our model, and it is difficult to correct it without relaxing our restriction of balanced growth.

As far as the business cycle properties of our model are concerned, it turns out that the stationary version of our model is virtually identical to the one considered in Lindquist (2004), where there is no growth, by construction. Lindquist (2004) performs an extensive study of the business cycle predictions of a stochastic general equilibrium version of the Krusell *et al.* (2000) model. The implications of our model are very similar and hence are not reported.

The rest of the paper is organized as follows. Section I describes a competitive equilibrium economy, presents the associated social planner's economy, introduces growth and derives the corresponding stationary model. Section II describes the calibration and the solution procedures. Section III presents the results from simulations, and Section IV concludes.

I. The Economy

In this section, we construct a general equilibrium model with the production function considered in Krusell *et al.* (2000). We first describe the environment, then introduce technological progress, and finally provide analytical results on the existence of a stationary equilibrium in our economy.

The environment

Time is discrete and the horizon is infinite, $t = 1, 2, \ldots, \infty$. There are two types of agents: skilled and unskilled; their variables are denoted by superscripts s and u, respectively.

There are two types of capital stocks: capital structures and capital equipment. The economy has two sectors: one sector produces consumption goods and capital structures, and the other sector produces capital equipment. Both sectors use the same technology; however, there is a technology factor specific to the capital equipment sector. We aggregate the production of the two sectors by introducing an exogenous relative price between consumption (structures) and equipment, q_t.

Let us denote by B_t a collection of all possible exogenous states in period t. We assume that B_t follows a stationary first-order Markov process. Specifically, let $\Re$ be the Borel σ-algebra on $\Im$. Define a transition function for the distribution of skills $\Pi : \Im \times \Re \to [0,1]$ on the measurable space $(\Im, \Re)$ such that for each $z \in \Im$, $\Pi(z, \cdot)$ is a probability measure on $(\Im, \Re)$, and for each $Z \in \Re$, $\Pi(\cdot, Z)$ is an $\Re$-measurable function. We shall interpret the function $\Pi(z, Z)$ as the probability that the next period's distribution of skills lies in the set Z given that the current distribution of skills is z, i.e. $\Pi(z, Z) = \Pr\{B_{t+1} \in Z | B_t = z\}$. The initial state $B_0 \in \Im$ is given. We assume that there is a complete set of markets, i.e. that the agents can trade state-contingent Arrow securities. The agent's $i \in \{s, u\}$ portfolio of securities is denoted by $\{M_t^i(B)\}_{B \in \Re}$. The claim of type $B \in \Re$ pays one unit of $t+1$ consumption good in the state B, and nothing otherwise. The price of such a claim is $p_t(B)$.

In the presence of population growth, the problem of skilled and unskilled groups of agents, $i \in \{s, u\}$, can be written as

$$(1) \qquad \max_{\{C_t^i, n_t^i, K_{b,t+1}^i, K_{e,t+1}^i, \{M_{t+1}^i(Z)\}_{Z \in \Re}\}} E_0 \sum_{t=0}^{\infty} \beta^t N_t^i U^i(C_t^i, 1 - n_t^i),$$

$$(2) \qquad \begin{aligned} &N_t^i C_t^i + N_{t+1}^i \left[K_{b,t+1}^i + \frac{K_{e,t+1}^i}{q_t} + \int_{\Re} p_t(Z)\, M_{t+1}^i(Z)\, \mathrm{d}Z \right] \\ &= N_t^i \left[w_t^i n_t^i + (1 - \delta_b + r_{bt}) K_{bt}^i + (1 - \delta_e + r_{et}) \frac{K_{et}^i}{q_t} + M_t^i(B_t) \right], \end{aligned}$$

where initial endowments of capital structures (buildings) and equipment, K_{b0}^i and K_{e0}^i, and Arrow securities $M_0^i(B_0)$ are given. Here, $\beta \in (0,1)$ is the subjective discount factor, E_t is the operator of expectation conditional on information set in period t, and N_t^i is an exogenously given number of agents of group $i \in \{s, u\}$. The variables C_t^i, n_t^i, w_t^i, K_{bt}^i and K_{et}^i are, respectively, consumption, labour, the wage per unit of labour, the capital stock of structures and equipment of an agent of group $i \in \{s, u\}$. The time endowment is normalized to 1, so the term $1 - n_t^i$ represents leisure; r_{bt} and r_{et} are the interest rates paid on capital invested in structures and equipment, respectively; and $\delta_b \in (0,1)$ and $\delta_e \in (0,1)$ are the depreciation rates of capital structures and capital equipment, respectively. The period utility function U^i is continuously differentiable, strictly increasing in both arguments and concave.

The production function is of the constant elasticity of substitution (CES) type:

$$(3) \qquad \begin{aligned} Y_t &= A_t G(K_{bt}, K_{et}, L_{st}, L_{ut}) \\ &= A_t K_{bt}^{\alpha} \left[\mu L_{ut}^{\sigma} + (1 - \mu) \left(\lambda K_{et}^{\rho} + (1 - \lambda) L_{st}^{\rho} \right)^{\sigma/\rho} \right]^{(1-\alpha)/\sigma}. \end{aligned}$$

Here, Y_t is output, A_t is an exogenously given level of technology (common to both sectors), and K_{bt} and K_{et} are the inputs of capital structures and capital equipment, respectively. Functions $L_{st} \equiv L_{st}(N_t^s n_t^s)$ and $L_{ut} \equiv L_{ut}(N_t^u n_t^u)$ give the efficiency labour inputs of skilled and unskilled agents, respectively, and will be specified in the next section. The parameters $\alpha \in (0,1)$, $\mu \in (0,1)$, $\lambda \in (0,1)$, ρ and σ govern the elasticities of substitution between structures, equipment, skilled labour and unskilled labour.

The firm maximizes period-by-period profits by hiring capital and labour

$$\max_{\{K_{bt},K_{et},N_t^s n_t^s,N_t^u n_t^u\}} \left[A_t G(K_{bt}, K_{et}, L_{st}, L_{ut}) - r_{bt}K_{bt} - r_{et}K_{et} - w_t^s N_t^s n_t^s - w_t^u N_t^u n_t^u\right], \tag{4}$$

taking the market prices as given.

Labour growth and technological progress

Krusell *et al.* (2000) provide time series data for the US economy over the 1963–92 period, including those for output, the stocks of structure and equipment, the numbers of skilled and unskilled workers, and the relative price between consumption (structures) and equipment. In the data, the growth patterns appear to be highly unbalanced. To be specific, over the sample period, the output and the stock of structures increased roughly by about a factor of two, while the stock of equipment increased by more than seven times; furthermore, the number of skilled workers nearly doubled, while the number of unskilled workers slightly decreased; and finally, the price of equipment relative to consumption (structures) went down by more than four times.

To make our model consistent with the above unbalanced growth patterns, we introduce several sources of exogenous growth simultaneously. First, we assume that skilled and unskilled populations can grow at differing rates, i.e.

$$N_t^s = N_0^s(\gamma^s)^t \quad \text{and} \quad N_t^u = N_0^u(\gamma^u)^t, \tag{5}$$

where γ^s and γ^u are the growth rates of the skilled and unskilled labour, respectively. Furthermore, we assume three different kinds of technological progress: the first increases efficiency of both skilled and unskilled labour at possibly different rates (labour-augmenting technological progress); the second increases the level of technology A_t (Hicks-neutral technological progress); the third improves the technology of the equipment sector relative to that of the consumption and structure sector or, equivalently, decreases the relative price of equipment $1/q_t$ (equipment-specific technological progress). We specifically assume that the aggregate labour input of skilled and unskilled agents evolves according to

$$L_{st} = N_t^s n_t^s(\Gamma^s)^t \quad \text{and} \quad L_{ut} = N_t^u n_t^u(\Gamma^u)^t, \tag{6}$$

where Γ^s and Γ^u are deterministic labour-augmenting technological progress of skilled and unskilled labour, respectively. The remaining two kinds of progress have an identical structure: they include a deterministic time trend and a stochastic stationary component. In particular, the level of technology is given by

$$A_t = A_0(\Gamma^A)^t z_t, \tag{7}$$

where Γ^A is a deterministic growth rate, and z_t is a stationary process. Similarly, the relative price is given by

$$\frac{1}{q_t} = \frac{\kappa_t}{q_0(\Gamma^q)^t}, \tag{8}$$

where Γ^q is a deterministic growth rate of q_t, and κ_t is a stationary process.

Competitive equilibrium

A competitive equilibrium in the economy (1)–(8) is a sequence of contingency plans for the agents' allocation $\{C_t^i, n_t^i, K_{b,t+1}^i, K_{e,t+1}^i, M_{t+1}^i(Z)\}_{Z\in\Re, t\in T}^{i\in\{s,u\}}$, for the firm's allocation $\{K_{bt}, K_{et}, L_{st}, L_{ut}\}_{t\in T}$ and for the prices $\{r_{bt}, r_{et}, w_t^s, w_t^u, p_t(Z)\}_{Z\in\Re, t\in T}$ such that given the prices:

- the sequence of plans for the agents' allocation solves the utility-maximization problem (1), (2), (8) for $i\in\{s,u\}$;
- the sequence of plans for the firm's allocation solves the profit-maximization problem of the firm (3)–(8);
- all markets clear and the economy's resource constraint is satisfied.

Moreover, the equilibrium plans are to be such that $C_t^i > 0$ and $0 < n_t^i < 1$ for $i\in\{s,u\}$, $K_{bt}, K_{et} > 0$ and $p_t(Z) > 0$ for all $Z\in\Re$.

Pareto optimum

To simplify the analysis of equilibrium in our decentralized economy (1)–(8), we construct the associated planner's economy. The planner solves

$$\max_{\{C_t^s, C_t^u, n_t^s, n_t^u, K_{b,t+1}, K_{e,t+1}\}} \left\{ E_0 \sum_{t=0}^{\infty} \beta^t \left[\theta N_0^s U^s(C_t^s, 1-n_t^s) + (1-\theta) N_0^u U^u(C_t^u, 1-n_t^u)\right] \right\}, \tag{9}$$

subject to the economy's resource constraint

$$\begin{aligned} & N_t^s C_t^s + N_t^u C_t^u + K_{b,t+1} + \frac{K_{e,t+1}}{q_t} \\ & = A_t G(K_{bt}, K_{et}, L_{st}, L_{ut}) + (1-\delta_b) K_{bt} + (1-\delta_e)\frac{K_{et}}{q_t}, \end{aligned} \tag{10}$$

where initial endowments of capital structures and equipment, K_{b0} and K_{e0}, are given. The production function $G(K_{bt}, K_{et}, L_{st}, L_{ut})$ is given by (3); skilled and unskilled labour grow according to (5); and the exogenous shocks are given by (6)–(8). In (9), θ and $(1-\theta)$ are the welfare weights of skilled and unskilled agents, respectively, with $\theta\in(0,1)$.

With the following proposition, we establish the connection between the decentralized and the planner's economies.

Proposition 1. For any distribution of initial endowments in the decentralized economy (1)–(8), there exist welfare weights θ and $(1-\theta)$ in the planner's economy (9), (10) such that a competitive equilibrium is a solution to the planner's problem.

Proof. See Appendix A.

The result of Proposition 1 might seem surprising. By assumption, the two heterogeneous groups of skilled and unskilled agents can grow at different rates. At first glance, this feature could make the planner's objective function non-stationary because the planner is to maximize the weighted sum of individual utilities where the weights, in particular, depend on the groups' sizes. As follows from Proposition 1, this first-glance intuition is, however, not correct: the appropriate weights for the planner's problem are those that depend on the initial sizes of the two groups; the growth rates of the skilled and unskilled groups do not enter the planner's objective function.

Stationary economy

As described in the subsection entitled 'Labour growth and technological progress' above, our economy contains several sources of growth. To be able to apply standard dynamic programming methods, we should convert the growing economy into a stationary one. With the following proposition, we show the restriction that is necessary for the existence of a stationary (balanced growth) equilibrium.[4]

Proposition 2. In order for the economy (3)–(10) to have a stationary (balanced growth) equilibrium, it is necessary that $\Gamma^s\gamma^s = \Gamma^u\gamma^u = \Gamma^q\gamma = (\Gamma^A)^{1/(\alpha-1)}\gamma$, where γ is a long-run growth rate of output.

Proof. This proposition follows from an inspection of the production function (3). Given formulas (5), (6) and (8), constant marginal rates of substitution between L_{st} and L_{ut} and between L_{st} and K_{et} require that

$$\Gamma^s\gamma^s = \Gamma^u\gamma^u = \Gamma^q\gamma. \tag{11}$$

Given formula (3), a constant K_b/Y ratio requires that

$$\gamma = \gamma^A\gamma^\alpha(\Gamma^q\gamma)^{1-\alpha}, \tag{12}$$

or, equivalently, $\Gamma^q = (\Gamma^A)^{1/(\alpha-1)}$.[5]

We next provide sufficiency results under the restrictions of Proposition 2. With the following proposition, we derive a stationary version of the resource constraint (10).

Proposition 3. The stationary resource constraint that corresponds to (3), (10) is given by

$$\begin{aligned} N_0^s c_t^s + N_0^u c_t^u + \gamma k_{b,t+1} + \gamma\Gamma^q \frac{\kappa_t}{q_0} k_{e,t+1} &= (1-\delta_b)k_{bt} + (1-\delta_e)\frac{\kappa_t}{q_0}k_{e,t} + A_0 z_t k_{bt}^\alpha \\ &\times \left[\mu(N_0^u n_t^u)^\sigma + (1-\mu)\left(\lambda k_{et}^\rho + (1-\lambda)(N_0^s n_t^s)^\rho\right)^{\sigma/\rho}\right]^{(1-\alpha)/\sigma}, \end{aligned} \tag{13}$$

where

$$c_t^s = \frac{(\gamma^s)^t C_t^s}{\gamma^t}, \quad c_t^u = \frac{(\gamma^u)^t C_t^u}{\gamma^t}, \quad k_{bt} = \frac{K_{bt}}{\gamma^t} \quad \text{and} \quad k_{et} = \frac{K_{et}}{(\Gamma^q)^t\gamma^t}.$$

Proof. See Appendix A.

Thus C_t^s and C_t^u grow at the rates γ/γ^s and γ/γ^u, respectively; K_{bt} and Y_t grow at the rate γ; and L_{st}, L_{ut} and K_{et} grow at the same rate $\Gamma^s\gamma^s = \Gamma^u\gamma^u = \Gamma^q\gamma$.

We now turn to preferences. In terms of new variables c_t^s and c_t^u, we can rewrite the objective function in (9) as follows:

$$(14)\qquad \left\{ E_0 \sum_{t=0}^{\infty} \beta^t \left[\theta N_0^s U^s \left(\frac{\gamma^t c_t^s}{(\gamma^s)^t}, 1 - n_t^s \right) + (1-\theta) N_0^u U^u \left(\frac{\gamma^t c_t^u}{(\gamma^u)^t}, 1 - n_t^u \right) \right] \right\}.$$

King *et al.* (1988) show that the standard Kydland and Prescott (1982) model is consistent with balanced growth only under the following two classes of preferences:

$$(15)\qquad U(C, 1-n) = \ln(C) + V(1-n),$$

$$(16)\qquad U(C, 1-n) = \frac{C^{1-\varrho}}{1-\varrho} V(1-n), \quad 0 < \varrho < 1 \text{ or } \varrho > 1,$$

where under the additively separable utility function (15), $V(1-n)$ is increasing and concave, and under the multiplicatively separable utility function (16), $V(1-n)$ is increasing and concave if $0 < \varrho < 1$, and decreasing and convex if $\varrho > 1$.

With the following proposition, we show that the above two utility functions are also consistent with balanced growth in our heterogeneous-agent setup. However, under (16), we should impose additional restrictions on the inverse of intertemporal elasticity of substitution in consumption for skilled and unskilled agents if these two groups grow at different rates.

Proposition 4. Preferences (14) are stationary if and only if the momentary utility function for $i \in \{s, u\}$ is given by:

1. $U^i(C, 1-n) = \ln(C) + V^i(1-n)$;
2. $U^i(C, 1-n) = [C^{1-\varrho^i}/(1-\varrho^i)] V^i(1-n)$, with ϱ^s and ϱ^u satisfying $(\gamma/\gamma^s)^{1-\varrho^s} = (\gamma/\gamma^u)^{1-\varrho^u}$.

Proof. See Appendix A.

We finally mention two properties of the model that are useful for our future analysis.

Proposition 5. In the economy that is consistent with balanced growth:

1. if $\gamma^s \gtrless \gamma^u$, then $\Gamma^s \lessgtr \Gamma^u$;
2. if $\Gamma^q \gtrless 1$, then $\Gamma^A \lessgtr 1$.

Proof. The results (1) and (2) follow, respectively, from the restrictions $\Gamma^s \gamma^s = \Gamma^u \gamma^u$ and $\Gamma^q = (\Gamma^A)^{1/(\alpha-1)}$ of Proposition 2.

That is, the assumption of balanced growth requires that: (1) whenever skilled labour grows at a higher (lower) rate than unskilled labour, efficiency of high-skilled labour should grow at a proportionally lower (higher) rate than efficiency of low-skilled labour; (2) whenever the efficiency of producing equipment relative to structures increases (decreases), Hicks-neutral technological progress is negative (positive).

II. Calibration and Solution Procedures

In this section, we describe the methodology of our numerical study. For the numerical part, we restrict attention to the additively separable utility function of the addilog type, with the sub-function $V^i(1-n)$ being identical for two types of agents,

$$U(C, 1-n) = \ln(C) + D\frac{(1-n)^{1-v} - 1}{1-v}. \tag{17}$$

Consequently, a stationary version of the planner's problem can be written as

$$\max_{\{c_t^s, c_t^u, n_t^s, n_t^u, k_{b,t+1}, k_{e,t+1}\}} E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \theta N_0^s \left[\log(c_t^s) + D\frac{(1-n_t^s)^{1-v} - 1}{1-v} \right] + (1-\theta) N_0^u \left[\log(c_t^u) + D\frac{(1-n_t^u)^{1-v} - 1}{1-v} \right] \right\}, \tag{18}$$

subject to (13). The first-order conditions (FOCs) of the problem (18) are derived in Appendix B.

Krusell *et al.* (2000) estimate the parameters in the production function (3) as well as the parameters for the stochastic shocks for the US economy data over the 1963–92 period. Since we assume the same production function, and we use the same dataset, we follow the parameter choice in Krusell *et al.* (2000) as close as possible. However, we cannot use all their estimates because there is an important difference between our framework and theirs: Krusell *et al.* (2000) impose no restrictions on the growth and cyclical patterns, while we assume balanced growth and a first-order recursive stationary Markov equilibrium. We outline the main steps of the calibration procedure below; further details are provided in Appendix C.

To estimate the growth rates of skilled and unskilled labour, γ^s and γ^u, respectively, and the initial numbers of skilled and unskilled workers, N_0^s and N_0^u, respectively, we use the data provided by Krusell *et al.* (2000). To calibrate time worked by skilled and unskilled agents in the steady state, we use the results of Maliar and Maliar (2001), who report time worked by eight educational groups in the US economy. For our purpose, from the given eight groups, we construct two weighted representative groups. The resulting shares of discretionary time worked by skilled and unskilled agents are $n^s = 0.3685$ and $n^u = 0.2459$, respectively.

In the benchmark case, we assume the depreciation rates of capital structures and capital equipment, $\delta_b = 0.05$ and $\delta_e = 0.125$, and the parameters of the production function, $\alpha = 0.117$, $\sigma = 0.401$, $\rho = -0.495$, as estimated in Krusell *et al.* (2000). We estimate the process for q_t in (8) by assuming that the error term follows a first-order autoregressive process $\log(\kappa_t) = b^q \log(\kappa_{t-1}) + \varepsilon_t^q$ with $\varepsilon_t^q \sim N(0, \sigma^q)$. (The estimate of Krusell *et al.* (2000) for q_t is not applicable to us since they assume an ARIMA (autoregressive integrated moving-average) process, which is not consistent with our assumption of a first-order recursive Markov equilibrium.) To estimate the parameters of the production function, λ and μ, the parameters for shock A_t and the sizes of labour-augmenting technological progress, Γ^s and Γ^u, we employ the following iterative procedure.

1. Fix some initial value of Γ^s and compute the corresponding value of $\Gamma^u = \gamma^s \Gamma^s / \gamma^u$, given γ^s and γ^u computed from the data.
2. Find the parameters λ and μ to reproduce two statistics in the data: the average (total) labour share of income over the period, and the average ratio of skilled labour's share of income to unskilled labour's share of income.

3. Use the data and the obtained parameters Γ^s, Γ^u, λ and μ to restore the process A_t according to (3), and estimate the parameter Γ^A in (7) by assuming a first-order autoregressive process for the error term, $\log(z_t) = b^A \log(z_{t-1}) + \varepsilon_t^A$ with $\varepsilon_t^A \sim N(0, \sigma^A)$.
4. Given the obtained value of Γ^A, update the value of Γ^s for the next iteration by

$$\Gamma^s = \frac{\left(0.5\Gamma^A + 0.5(\Gamma^q)^{\alpha-1}\right)^{1/(\alpha-1)}\gamma}{\gamma^s}.$$

5. Repeat iterations until convergence so that the value of Γ^s assumed initially is the same as the one obtained at the end of computations. Notice that the above iterative scheme simultaneously ensures that $(\Gamma^A)^{1/(\alpha-1)} = \Gamma^q$, which is another restriction necessary for balanced growth. At the end, we have that $\Gamma^s\gamma^s = \Gamma^u\gamma^u = \Gamma^q\gamma = (\Gamma^A)^{1/(\alpha-1)}\gamma$, as required in Proposition 2.

We have to resort to this iterative procedure because our model has labour-augmenting technological progress for skilled and unskilled labour, whose sizes cannot be directly estimated from the data. (This problem does not arise in the analysis of Krusell *et al.* (2000), since they assume no labour-augmenting technological progress.) We then calibrate the discount factor β, the welfare weight θ and the utility function parameter D by using the FOCs of problem (18), evaluated in steady state (see Appendix C). The obtained values of the parameters are summarized in Tables 1 and 2.

To solve the model, we use a simulation-based variant of the parameterized expectations algorithm (PEA) by den Haan and Marcet (1990). To ensure the convergence of the PEA, we bound the simulated series on initial iterations, as described in Maliar and Maliar (2003b). The model has two features that complicate the computation procedure. First, there are two intertemporal FOCs, so we must parameterize two conditional expectations. Second, there are two intratemporal conditions that cannot be resolved analytically with respect to skilled and unskilled labour. Solving numerically the two intratemporal conditions on each date within the iterative cycle is costly, so we find it easier to parameterize the intratemporal conditions in the same way as we do the intertemporal FOCs. We then solve for equilibrium by iterating on the parameters of the resulting four decision rules simultaneously. The details of the solution procedure are described in Appendix D. Once the solution to the

TABLE 1
THE PARAMETERS OF THE UTILITY AND PRODUCTION FUNCTIONS

Parameter	N_0^s	γ^s	n^s	N_0^u	γ^u	n^u	β	λ	μ	θ	B
Value	4.4558	1.0224	0.3685	17.990	0.9945	0.2459	0.9823	0.9979	0.9197	0.3530	1.8843

TABLE 2
THE TECHNOLOGY GROWTH RATES AND THE SHOCK PARAMETERS

Parameter	Γ^s	Γ^u	q_0	γ^q	$(\sigma^q)^2$	b^q	A_0	γ^A	$(\sigma^A)^2$	b^A
Value	1.0562	1.0856	0.9664	1.0491	0.0306	0.9352	10.213	0.9586	0.0326	0.7143

stationary model is computed, we restore the growing variables by incorporating the corresponding deterministic trends.

III. Results

In Figure 1, we plot the key variables (in logarithms) of the benchmark version of our model with the elasticity of substitution of labour $1/v = 1$ under the actual sequence of relative prices, $1/q_t$, and under the fitted sequence of technology levels, A_t. As we see, the model is overall successful in explaining the growth patterns observed in the data. First, by construction, it generates appropriate labour growth patterns, namely, an increasing pattern for skilled labour and a decreasing pattern for unskilled labour. Second, it produces series for capital structures and equipment growing at different rates, which are comparable to those observed in the data. Finally, the model predicts increasing patterns for output and wages of unskilled agents, which also agrees with the data.

A striking but not surprising implication of our model is that the rate of Hicks-neutral technological progress is, on average, negative, $\Gamma^A < 1$. Indeed, given that in the data, equipment becomes cheaper than structures over time in relative terms (i.e. $\Gamma^q > 1$), by Proposition 5, we should necessarily have that $\Gamma^A < 1$. In the calibrated version of the model, this effect proved to be very large, $\Gamma^A = 0.9586$, as Table 1 shows. Our finding that Hicks-neutral technological progress is, on average, negative is the same as that of Greenwood *et al.* (1997), who also report a dramatic downturn in total factor productivity since the early 1970s. To explain their result, Greenwood *et al.* (1997) make

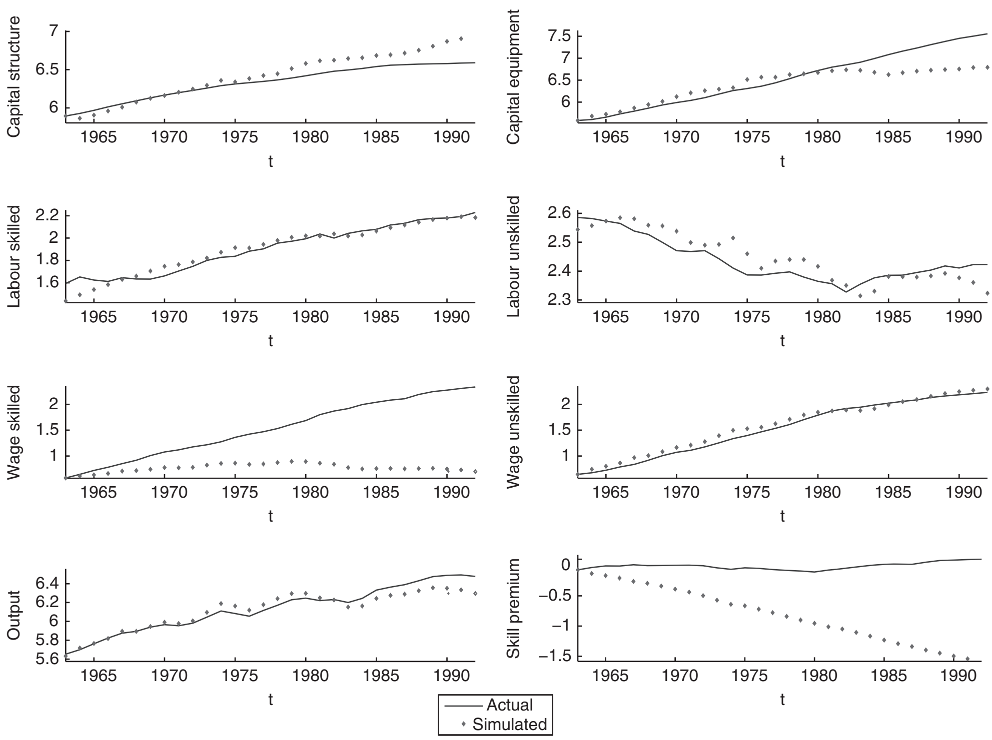

FIGURE 1. The actual and simulated paths for the US economy 1963–92.

a growth accounting exercise and demonstrate that the average growth rate of total factor productivity depends on how capital is incorporated in the model. Specifically, they show that once total capital is split between equipment and structures, the productivity downturn increases.

There is one undesirable growth feature of our model that is difficult to correct given our assumption of balanced growth. Specifically, the model significantly underpredicts the growth rate of skilled-labour wages. As a result, the model fails to explain the time series behaviour of the skill premium, $\pi_t \equiv w_t^s/w_t^u$: in the model, the skill premium has a strong downward trend, while in the data, such a trend is absent.

In fact, the above undesirable feature has been already anticipated in Proposition 5. Specifically, the skill premium is given by

$$\begin{aligned}\pi_t &= \left\{\frac{G_3(K_{bt},K_{et},L_{st},L_{ut})}{G_4(K_{bt},K_{et},L_{st},L_{ut})}\right\}\times\left[\frac{\Gamma^s}{\Gamma^u}\right]^t \\ &= \left\{\frac{(1-\mu)(1-\lambda)\left[\lambda\left(k_{et}/(N_0^s n_t^s)\right)^\rho+(1-\lambda)\right]^{(\sigma/\rho)-1}}{\mu}\frac{(N_0^s n_t^s)^{\sigma-1}}{(N_0^u n_t^u)^{\sigma-1}}\right\}\times\left[\frac{\Gamma^s}{\Gamma^u}\right]^t.\end{aligned} \tag{19}$$

Since, in the data, skilled labour grows at a higher rate than unskilled labour, $\gamma^s>\gamma^u$, the assumption of balanced growth implies that labour-augmenting technological progress is larger for unskilled agents than for skilled agents, $\Gamma^u>\Gamma^s$. As follows from Table 1, the difference between Γ^s and Γ^u in the calibrated version of the model is very large, i.e. $\Gamma^s=1.0562$ and $\Gamma^u=1.0856$. Given that the term $(G_3(K_{bt},K_{et},L_{st},L_{ut}))/(G_4(K_{bt},K_{et},L_{st},L_{ut}))$ in expression (19) is stationary, and that the term $[\Gamma^s/\Gamma^u]^t$ has a downward trend, we have a strong decreasing pattern in the skill premium. In general, formula (19) implies that as long as $\gamma^s>\gamma^u$, we have $\Gamma^u>\Gamma^s$, and consequently our model will generate the skill premium with a downward trend no matter how the parameters are calibrated.[6] Therefore the failure of the model to account for the risk premium pattern is a generic feature of the model that is difficult to correct within our framework of balanced growth.[7] The analysis of Krusell *et al.* (2000) does not suffer from this shortcoming because they do not impose the restriction of balanced growth, and hence the skill premium in their model does not have a downward growth component $[\Gamma^s/\Gamma^u]^t$.

To quantify the regularities discussed above, in Table 3 we provide the growth rates for the US and model economies over the 1963–92 period. In column I, referred to as the benchmark model (BM), we report the statistics generated by the model under our benchmark parameterization. In columns II–VII, we provide the results of the sensitivity experiments where we vary one of the model's parameters, holding the rest of the parameters equal to their benchmark values. Finally, in column VIII, we report the corresponding statistics for the US economy computed from the Krusell *et al.* (2000) dataset. The comparison of the first and last columns shows that our benchmark model is able to account for the growth rates of all considered variables, except of the skilled-labour wage and skill premium.

As far as our sensitivity experiments are concerned, we first study the robustness of our results to modifications in the value of the elasticity of substitution of labour, $1/v$, the only parameter that is not identified by our calibration procedure; namely, we set $v=5$ (see column II). Furthermore, we perform three sensitivity experiments in which we consider a 100% increase in the production function parameters α, σ and ρ relative to their benchmark values (see columns III–V). Recall that the values of these parameters were borrowed from Krusell *et al.* (2000) and therefore it is of interest to see how their variations can affect the long-run properties of the solution. Finally, we implement two

TABLE 3
GROWTH RATES FOR THE US AND ARTIFICIAL ECONOMIES

	Artificial economy							US economy
Column[a]	BM	$\nu = 5.0$	$\alpha = 0.234$	$\sigma = 0.802$	$\rho = -0.99$	$\delta_s = 0.1$	$\delta_e = 0.25$	
Statistic[b]	I	II	III	IV	V	VI	VII	VIII
$\gamma(y_t)$	1.0349	1.0318	1.0338	1.0354	1.0344	1.0349	1.0349	1.0294
	(0.0032)	(0.0025)	(0.0027)	(0.0035)	(0.0031)	(0.0032)	(0.0033)	
$\gamma(k_{bt})$	1.0254	1.0256	1.0258	1.0255	1.0255	1.0256	1.0253	1.0244
	(0.0029)	(0.0025)	(0.0023)	(0.0030)	(0.0028)	(0.0028)	(0.0031)	
$\gamma(k_{et})$	1.0301	1.0306	1.027	1.0292	1.0305	1.0299	1.0303	1.0707
	(0.0059)	(0.0053)	(0.0050)	(0.0079)	(0.0055)	(0.0059)	(0.0061)	
$\gamma(N_t^s n_t^s)$	1.0246	1.0241	1.0246	1.0231	1.0238	1.0247	1.0245	1.0224
	(0.0017)	(0.0007)	(0.0012)	(0.0033)	(0.0021)	(0.0016)	(0.0020)	
$\gamma(N_t^u n_t^u)$	0.9975	0.99429	0.99646	0.99954	0.99743	0.99743	0.99758	0.9945
	(0.0015)	(0.0004)	(0.0011)	(0.0022)	(0.0014)	(0.0015)	(0.0014)	
$\gamma(w_{st})$	1.0001	1.0004	1.0033	1.0008	1.0022	0.99977	1.0006	1.0628
	(0.0024)	(0.0029)	(0.0021)	(0.0031)	(0.0025)	(0.0023)	(0.0026)	
$\gamma(w_{ut})$	1.0273	1.0286	1.0350	1.0270	1.0268	1.0273	1.0273	1.0564
	(0.0021)	(0.0021)	(0.0019)	(0.0020)	(0.0020)	(0.0020)	(0.0021)	
$\gamma(\pi_t)$	0.97435	0.97404	0.96996	0.97647	0.97702	0.9740	0.97495	1.0063
	(0.0009)	(0.0017)	(0.0006)	(0.0022)	(0.0011)	(0.0008)	(0.0010)	

Notes
[a]In column I, we report the statistics generated by our benchmark model (BM); in columns II–VII, we provide the results of the experiments where we vary one of the parameters, holding the rest of the parameters equal to their benchmark values; in column VIII, we compute the corresponding statistics for the US economy from the Krusell *et al.* (2000) dataset.
[b]$\gamma(x_t)$ denotes the growth rate of variable x_t. The growth rates in the model are sample averages computed across 500 simulations. Each simulated series has a length of 30 periods, as do time series for the US economy. The numbers in brackets are sample standard deviations of the corresponding growth rates.

experiments where we assume a 100% increase in the depreciation rates δ_b and δ_e relative to their benchmark values (see columns VI and VII). The last two experiments are of interest as Hornstein (2004) found that the depreciation rates of capital have been increasing considerably over time. As is seen from the table and as was anticipated in our previous discussion, all the above modifications do not help us to improve the model's prediction about the growth rates of the skilled-labour wage and skill premium.

We should draw attention to the fact that we do not report the business cycle predictions of the model such as standard deviations and correlation coefficients. Our predictions are very similar to those obtained in Lindquist (2004). This is because the stationary version of our model is identical to the one considered in Lindquist (2004), up to a different choice of the utility function (he uses the Cobb–Douglas function while we use the addilog one) and up to some differences in the calibration procedure (in particular, he uses quarterly US data while we use yearly US data). Hence the results of Lindquist (2004) are also valid for our model.

IV. CONCLUSION

Most macroeconomic models, which explicitly incorporate economic growth, rely on the assumption of balanced growth. This assumption is technically convenient since balanced growth models can be converted into stationary models, which can be solved by standard methods of dynamic programming. Balanced growth has been justified in the literature on the basis that the observed long-run behaviour of the key macroeconomic variables is broadly consistent with this specification. In particular, the stationarity of output and the capital–output ratio in the data is a part of the Kaldor (1961) stylized facts.

In this paper, we develop a general equilibrium version of the Krusell *et al.* (2000) model that is capable of generating a balanced growth path on which different variables grow at different rates. In particular, the stock of capital equipment grows at a higher rate than do consumption and output, and skilled labour grows at a higher rate than does unskilled labour. A calibrated version of our model proved to be successful in matching the long-run properties of the US economy data on capital equipment, structures, skilled and unskilled labour, and output. Nonetheless, the model has an important shortcoming, namely, it considerably understates an increase in wages of skilled labour and, as a result, fails to explain the long-run behaviour of the skill premium. Therefore the answer to the question posed in the introduction is as follows: In stark contrast to the Krusell *et al.* (2000) setup, our general equilibrium balanced growth model parameterized by the CES production function cannot explain all the growth features of the US data.

We argue that the shortcoming of our analysis is the consequence of the assumption of balanced growth. A mechanism that helps Krusell *et al.* (2000) to account for the skill premium behaviour is the capital–skill complementarity: equipment is a complement with skilled labour and a substitute with unskilled labour, so that an increase in equipment increases productivity of skilled labour and decreases productivity of unskilled labour. This mechanism is not consistent with the assumption of balanced growth, which lies at the basis of our analysis. Under this assumption, the share of each input in production remains constant even though different variables grow at different rates. Therefore it cannot happen in our model that one production input substitutes another production input over time, which is the key insight of the Krusell *et al.* (2000) analysis.

Developing models with non-balanced growth would restore the importance of the Krusell *et al.* (2000) capital–skill complementarity mechanism for the long-run behaviour of the economy. This direction is worth exploring, as recent work calls into question the assumption of balanced growth; see, for example, Ramey and Francis (2006) for evidence on a temporal rise in leisure, and Mulligan (2002) and Hornstein (2004) for evidence on changes in the capital–output ratio, as well as Ngai and Pissarides (2007), Kylymnyuk *et al.* (2007a, b) and Acemoglu and Guerrieri (2008) for evidence on and theoretical explanations of non-balanced systematic changes in the relative importance of different production sectors.

APPENDIX A: PROOFS OF PROPOSITIONS 1, 3 AND 4

Proof of Proposition 1

Consider the problem of a representative agent of type $i \in \{s, u\}$, given by (1) and (2). Dividing by the number of agents N_t^i, we get

$$\text{(A1)} \qquad \max_{\{C_t^i, n_t^i, K_{b,t+1}^i, K_{e,t+1}^i, \{M_{t+1}^i(Z)\}_{Z\in\Re}} E_0 \sum_{t=0}^{\infty} \beta^t U^i(C_t^i, 1 - n_t^i),$$

subject to

$$C_t^i + \gamma^i K_{b,t+1} + \frac{\gamma^i K_{e,t+1}}{q_t} + \gamma^i \int_{\Re} p_t(Z)\, M_{t+1}^i(Z)\, \mathrm{d}Z = w_t^i n_t^i + (1-\delta_b + r_{bt})K_{bt} + (1-\delta_e + r_{et})\frac{K_{et}^i}{q_t} + M_t^i(B_t). \tag{A2}$$

The first-order condition (FOC) of the problem (A1), (A2) with respect to holdings of Arrow securities is

$$\phi_t^i p_t(B)\gamma^i = \beta\phi_{t+1}^i(B')\cdot \Pi\{B_{t+1}=B'|B_t=B\}_{B',B\in\Re}, \tag{A3}$$

where ϕ_t^i is the Lagrange multiplier associated with the budget constraint (A2). By taking the ratio of FOC (A3) of a skilled agent s to that of an unskilled agent u, we obtain

$$\frac{\phi_0^s}{\phi_0^u} = \frac{\phi_1^s/\gamma^s}{\phi_1^u/\gamma^u} = \ldots = \frac{\phi_t^s/(\gamma^s)^t}{\phi_t^u/(\gamma^u)^t} = \ldots \equiv \frac{\phi^u}{\phi^s}, \tag{A4}$$

where ϕ^s and ϕ^u are some constants. Given that $\phi_t^i = U_1^i(C_t^i, 1-n_1^i)$, we have that the ratio of marginal utilities of consumption of two heterogeneous consumers, adjusted to the corresponding growth rates of population, is constant across time and states of nature

$$\frac{U_1^i(C_0^s,1-n_0^s)}{U_1^i(C_0^u,1-n_0^u)} = \frac{U_1^i(C_1^s,1-n_1^s)/\gamma^s}{U_1^i(C_1^u,1-n_1^u)/\gamma^u} = \ldots = \frac{U_1^i(C_t^s,1-n_t^s)/(\gamma^s)^t}{U_1^i(C_t^u,1-n_t^u)/(\gamma^u)^t} = \frac{\phi^u}{\phi^s}. \tag{A5}$$

This is a consequence of the assumption of complete markets. The FOCs with respect to physical hours worked, capital structures and equipment of a representative agent of type i, respectively, are

$$U_2^i(C_t^i, 1-n_t^i) = U_1^i(C_t^i, 1-n_t^i)w_t^i(\Gamma^i)^t, \tag{A6}$$

$$\gamma^i U_1^i(C_t^i, 1-n_t^i) = \beta E_t\left[U_1^i(C_{t+1}^i, 1-n_{t+1}^i)(1-\delta_b + r_{b,t+1})\right], \tag{A7}$$

$$\gamma^i U_1^i(C_t^i, 1-n_t^i)/q_t = \beta E_t\left[\frac{U_1^i(C_{t+1}^i, 1-n_{t+1}^i)}{q_{t+1}}(1-\delta_e + r_{e,t+1})\right]. \tag{A8}$$

Thus (A5)–(A8) are the FOCs of the competitive equilibrium economy.

Let us consider now the planner's problem (9), (10). The FOCs with respect to consumption of the skilled and the unskilled agents, respectively, are

$$\theta U_1^s(C_t^s, 1-n_t^s) = \eta_t(\gamma^s)^t, \tag{A9}$$

$$(1-\theta)U_1^u(C_t^u, 1-n_t^u) = \eta_t(\gamma^u)^t, \tag{A10}$$

where η_t is the Lagrange multiplier associated with the economy's resource constraint (10). Dividing (A9) by (A10) and setting the value of θ so that $\phi^u/\phi^s = (1-\theta)/\theta$, we obtain condition (A5) of the competitive equilibrium economy. The FOC with respect to capital structures is

$$\eta_t = \beta E_t\left[\eta_{t+1}(1-\delta_b + r_{b,t+1})\right]. \tag{A11}$$

Combining (A9) and (A10) with (A11), we get condition (A7) of the competitive equilibrium economy. Similarly, the FOC with respect to equipment is

$$\eta_t/q_t = \beta E_t\left[\frac{\eta_{t+1}}{q_{t+1}}(1-\delta_e + r_{e,t+1})\right]. \tag{A12}$$

After substituting conditions (A9) and (A10) into (A12), we obtain condition (A8) of the competitive equilibrium economy. From the firm's problem (4), equilibrium wages are given by

$$w_t^s = A_t G_3(K_{bt}, K_{et}, L_{st}, L_{ut})(\Gamma^s)^t \quad \text{and} \quad w_t^u = A_t G_4(K_{bt}, K_{et}, L_{st}, L_{ut})(\Gamma^u)^t.$$

By substituting these wages into the FOC with respect to physical hours worked of the planner's problem, we get (A6). Finally, the resource constraint (10) should be satisfied in competitive equilibrium by definition. The fact that the optimality conditions of the planner's problem are necessary for competitive equilibrium proves the statement of Proposition 1.[8]

Proof of Proposition 3

Let us introduce a new variable $\tilde{K}_{et} \equiv K_{et}/(\Gamma^q)^t$. In terms of this new variable, the budget constraint (10) combined with the production function (3) becomes

$$\begin{aligned}
&N_t^s C_t^s + N_t^u C_t^u + K_{b,t+1} + \Gamma^q \frac{\kappa_t}{q_0}\tilde{K}_{e,t+1} \\
&= (1-\delta_b)K_{bt} + (1-\delta_e)\frac{\kappa_t}{q_0}\tilde{K}_{et} + A_0(\Gamma^A)^t z_t K_{bt}^{\alpha} \\
&\times \{\mu\left[N_t^u n_t^u (\Gamma^u)^t\right]^{\sigma} + (1-\mu)\left(\lambda\tilde{K}_{et}^{\rho}\left[(\Gamma^q)^t\right]^{\rho} + (1-\lambda)\left[N_t^s n_t^s (\Gamma^s)^t\right]^{\rho}\right)^{\sigma/\rho}\}^{(1-\alpha)/\sigma}.
\end{aligned} \tag{A13}$$

Let us introduce γ, which is defined as a common long-run growth rate of output, Y_t, structures K_{bt} and adjusted equipment $\tilde{K}_{et}$. We divide (A13) by γ^t to obtain

$$\begin{aligned}
&\frac{N_0^s(\gamma^s)^t C_t^s}{\gamma^t} + \frac{N_0^u(\gamma^u)^t C_t^u}{\gamma^t} + \gamma\frac{K_{b,t+1}}{\gamma^{t+1}} + \gamma\Gamma^q\frac{\kappa_t}{q_0}\frac{\tilde{K}_{e,t+1}}{\gamma^{t+1}} \\
&= (1-\delta_b)\frac{K_{bt}}{\gamma^t} + (1-\delta_e)\frac{\kappa_t}{q_0}\frac{\tilde{K}_{et}}{\gamma^t} + A_0 z_t\left(\frac{K_{bt}}{\gamma^t}\right)^{\alpha} \\
&\times \left\{\frac{\mu\left[N_0^u n_t^u (\Gamma^u\gamma^u)^t\right]^{\sigma} + (1-\mu)\left[\lambda\tilde{K}_{et}^{\rho}\left[(\Gamma^q)^t\right]^{\rho} + (1-\lambda)\left[N_0^s n_t^s(\Gamma^s\gamma^s)^t\right]^{\rho}\right]^{\sigma/\rho}}{\left[(\Gamma^A)^t\right]^{\sigma/(\alpha-1)}(\gamma^t)^{\sigma}}\right\}^{(1-\alpha)/\sigma},
\end{aligned} \tag{A14}$$

where we take into account that skilled and unskilled labour grow at constant rates γ^s and γ^u, as is assumed in (5). By imposing the restrictions $\Gamma^s\gamma^s = \Gamma^u\gamma^u = \Gamma^q\gamma = (\Gamma^A)^{1/(\alpha-1)}\gamma$ and by introducing notation c_t^s, c_t^u, k_{bt} and k_{et}, as shown in Proposition 3, we get the budget constraint (13).

Proof of Proposition 4

The necessity part can be shown following the steps outlined in King *et al.* (1988). The sufficiency part can be shown as follows. Under the additively-separable addilog preferences of type (1), the stationary version of the planner's preferences is

$$\begin{aligned}
&E_0\sum_{t=0}^{\infty}\beta^t\left\{\theta N_0^s\left[\log\left(\frac{(\gamma^s)^t c_t^s}{\gamma^t}\right) + V^s(1-n_t^s)\right] + (1-\theta)N_0^u\left[\log\left(\frac{(\gamma^u)^t c_t^u}{\gamma^t}\right) + V^u(1-n_t^u)\right]\right\} \\
&= E_0\sum_{t=0}^{\infty}\beta^t\left\{\theta N_0^s\left[\log(c_t^s) + V^s(1-n_t^s)\right] + (1-\theta)N_0^u\left[\log(c_t^u) + V^u(1-n_t^u)\right]\right\} + \Upsilon,
\end{aligned} \tag{A15}$$

where

$$\Upsilon \equiv E_0\sum_{t=0}^{\infty}\beta^t\left\{\theta N_0^s\left[\log\left(\frac{(\gamma^s)^t}{\gamma^t}\right)\right] + (1-\theta)N_0^u\left[\log\left(\frac{(\gamma^u)^t}{\gamma^t}\right)\right]\right\}$$

is a finite additive term, which has no effect on equilibrium allocation.

Under the multiplicatively-separable Cobb–Douglas preferences (type 2), restricted to satisfy

$$\left(\frac{\gamma}{\gamma^s}\right)^{1-\varrho^s} = \left(\frac{\gamma}{\gamma^u}\right)^{1-\varrho^u},$$

the stationary planner's preferences are given by

(A16)
$$E_0\sum_{t=0}^{\infty}\beta^t\left[\theta N_0^s\frac{\left((\gamma^s)^t c_t^s/\gamma^t\right)^{1-\varrho^s}}{1-\varrho^s}V^s(1-n_t^s)+(1-\theta)N_0^u\frac{\left((\gamma^u)^t c_t^u/\gamma^t\right)^{1-\varrho^u}}{1-\varrho^u}V^u(1-n_t^u)\right]$$
$$=E_0\sum_{t=0}^{\infty}\widehat{\beta}^t\left[\theta N_0^s\frac{(c_t^s)^{1-\varrho^s}}{1-\varrho^s}V^s(1-n_t^s)+(1-\theta)N_0^u\frac{(c_t^u)^{1-\varrho^u}}{1-\varrho^u}V^u(1-n_t^u)\right],$$

where

$$\hat{\beta}\equiv\beta\left(\frac{\gamma}{\gamma^s}\right)^{1-\varrho^s}=\beta\left(\frac{\gamma}{\gamma^u}\right)^{1-\varrho^u}.$$

APPENDIX B: FIRST-ORDER CONDITIONS FOR PROBLEM (18)

Let us denote $l_{st}=N_0^s n_t^s$ and $l_{ut}=N_0^u n_t^u$. Optimality conditions of the problem (18), (13) with respect to c_t^s, c_t^u, n_t^s, n_t^u, $k_{b,t+1}$ and $k_{e,t+1}$, respectively, are

(A17) $$\theta(c_t^s)^{-1}=\eta_t,$$

(A18) $$(1-\theta)(c_t^u)^{-1}=\eta_t,$$

(A19) $$\theta D(1-n_t^s)^{-\nu}=\eta_t A_0 z_t G_3(k_{bt},k_{et},l_{st},l_{ut}),$$

(A20) $$(1-\theta)D(1-n_t^u)^{-\nu}=\eta_t A_0 z_t G_4(k_{bt},k_{et},l_{st},l_{ut}),$$

(A21) $$\gamma\eta_t=\beta E_t\{\eta_{t+1}[1-\delta_b+A_0 z_{t+1}G_1(k_{b,t+1},k_{e,t+1},l_{s,t+1},l_{u,t+1})]\},$$

(A22) $$\frac{\gamma\Gamma^q\kappa_t}{q_0}\eta_t=\beta E_t\left\{\eta_{t+1}\left[\frac{(1-\delta_e)\kappa_{t-1}}{q_0}+A_0 z_{t+1}G_2(k_{b,t+1},k_{e,t+1},l_{s,t+1},l_{u,t+1})\right]\right\},$$

where G_i is a first-order partial derivative of the function G with respect to the ith argument, $i=1,\ldots,4$. These derivatives are given by

(A23) $$G_1(k_{bt},k_{et},l_{st},l_{ut})=\alpha A_0 z_t k_{bt}^{\alpha-1}\left[\mu l_{ut}^{\sigma}+(1-\mu)\left(\lambda k_{et}^{\rho}+(1-\lambda)l_{st}^{\rho}\right)^{\sigma/\rho}\right]^{(1-\alpha/)\sigma},$$

(A24) $$G_2(k_{bt},k_{et},l_{st},l_{ut})=A_0 z_t k_{bt}^{\alpha}(1-\alpha)(1-\mu)\lambda\left(\lambda k_{et}^{\rho}+(1-\lambda)l_{st}^{\rho}\right)^{(\sigma/\rho)-1}k_{et}^{\rho-1}$$
$$\times\left[\mu l_{ut}^{\sigma}+(1-\mu)\left(\lambda k_{et}^{\rho}+(1-\lambda)l_{st}^{\rho}\right)^{(\sigma/\rho)}\right]^{(1-\alpha)/\sigma-1},$$

(A25) $$G_3(k_{bt},k_{et},l_{st},l_{ut})=A_0 z_t k_{bt}^{\alpha}(1-\alpha)(1-\mu)(1-\lambda)\left(\lambda k_{et}^{\rho}+(1-\lambda)l_{st}^{\rho}\right)^{(\sigma/\rho)-1}l_{st}^{\rho-1}$$
$$\times\left[\mu l_{ut}^{\sigma}+(1-\mu)\left(\lambda k_{et}^{\rho}+(1-\lambda)l_{st}^{\rho}\right)^{(\sigma/\rho)}\right]^{(1-\alpha/\sigma)-1},$$

(A26) $$G_4(k_{bt},k_{et},l_{st},l_{ut})=A_0 z_t k_{bt}^{\alpha}(1-\alpha)\mu$$
$$\times\left[\mu l_{ut}^{\sigma}+(1-\mu)\left(\lambda k_{et}^{\rho}+(1-\lambda)l_{st}^{\rho}\right)^{(\sigma/\rho)}\right]^{(1-\alpha/\sigma)-1}l_{ut}^{\sigma-1}.$$

After some algebra, conditions (A17)–(A22) can be rewritten as follows:

(A27) $$\gamma c_t^{-1}=\beta E_t\left[c_{t+1}^{-1}\left(1-\delta_b+A_0 z_t G_1(k_{b,t+1},k_{e,t+1},l_{s,t+1},l_{u,t+1})\right)\right],$$

(A28) $$\frac{\gamma\Gamma^{q}\kappa_t}{q_0}c_t^{-1}=\beta E_t\left[c_{t+1}^{-1}\left(\frac{(1-\delta_e)\kappa_{t-1}}{q_0}+A_0z_tG_2(k_{b,t+1},k_{e,t+1},l_{s,t+1},l_{u,t+1})\right)\right],$$

(A29) $$l_{st}=N_0^s\left[1-\frac{c_t^{1/v}\theta^{1/v}D^{1/v}\left(A_0z_tG_3(k_{bt},k_{et},l_{st},l_{ut})\right)^{-(1/v)}}{\left[N_0^s\theta+N_0^u(1-\theta)\right]^{1/v}}\right],$$

(A30) $$l_{ut}=N_0^u\left[1-\frac{c_t^{1/v}(1-\theta)^{1/v}D^{1/v}\left(A_0z_tG_4(k_{bt},k_{et},l_{st},l_{ut})\right)^{-(1/v)}}{\left[N_0^s\theta+N_0^u(1-\theta)\right]^{1/v}}\right].$$

Optimality conditions (A27)–(A30) together with the resource constraint (13) characterize the equilibrium.

APPENDIX C: CALIBRATION PROCEDURE

To compute the values of λ and μ in step 2 of the iterative procedure described in Section II, we use the derivatives (A25) and (A26) of the production function to get

(A31) $$\mu=\left[1+\frac{((G_{3t}L_{st})/(G_{4t}L_{ut}))\,(L_{ut}/K_{et})^{\sigma}}{(L_{st}/K_{et})^{\rho}(1-\lambda)\,(\lambda+(1-\lambda)\,(L_{st}/K_{et})^{\rho})^{(\sigma/\rho)-1}}\right]^{-1},$$

(A32) $$\frac{G_{3t}L_{st}+G_{4t}L_{ut}}{Y_t}=(1-\alpha)\times\frac{\left[(1-\mu)(1-\lambda)\,(\lambda+(1-\lambda)\,(L_{st}/K_{et})^{\rho})^{(\sigma/\rho)-1}\,(L_{st}/K_{et})^{\rho}+\mu\,(L_{ut}/K_{et})^{\sigma}\right]}{(1-\mu)\,(\lambda+(1-\lambda)\,(L_{st}/K_{et})^{\rho})^{\sigma/\rho}\,(L_{st}/K_{et})^{\rho}+\mu\,(L_{ut}/K_{et})^{\sigma}},$$

where $G_{it}\equiv G_i(K_{bt},K_{et},L_{st},L_{ut})$. We compute the ratios

$$\frac{G_{3t}L_{st}}{G_{4t}L_{ut}},\quad\frac{L_{st}}{K_{et}},\quad\frac{L_{ut}}{K_{et}}\quad\text{and}\quad\frac{G_{3t}L_{st}+G_{4t}L_{ut}}{Y_t}$$

as time series averages of variables

$$\frac{w_t^sN_t^s(\Gamma^s)^t}{w_t^uN_t^u(\Gamma^u)^t},\quad\frac{N_t^s(\Gamma^s)^t}{K_{et}},\quad\frac{N_t^u(\Gamma^u)^t}{K_{et}}\quad\text{and}\quad\frac{w_t^sN_t^s(\Gamma^s)^t+w_t^uN_t^u(\Gamma^u)^t}{Y_t},$$

respectively, where the last four variables are constructed from the data in Krusell *et al.* (2000) under the assumed values of Γ^s and Γ^u. We then solve numerically equations (A31) and (A32) with respect to λ and μ.

In step 3, we use the obtained parameters to restore the process for A_t from (3), i.e.

(A33) $$A_t=\frac{Y_t}{K_{bt}^{\alpha}\left\{\mu\left[N_t^u(\Gamma^u)^t\right]^{\sigma}+(1-\mu)\left[\lambda K_{et}^{\rho}+(1-\lambda)\left[N_t^s(\Gamma^s)^t\right]^{\rho}\right]^{\sigma/\rho}\right\}^{(1-\alpha)/\sigma}},$$

where Y_t, N_t^u, N_t^s, K_{bt} and K_{et} are the corresponding time series taken from the data in Krusell *et al.* (2000).

We calibrate the discount factor β by using FOC (A27) evaluated in the steady state

$$\text{(A34)} \qquad \beta = \gamma / \left(1 - \delta_b + \alpha \frac{Y}{K_b}\right),$$

where Y/K_b is the time series average of output to structures ratio in the Krusell *et al.* (2000) data.[9]

Given $l_s = N_0^s n^s$ and $l_u = N_0^u n^u$, we compute steady-state values of capital equipment, k_e, and structures, k_b, by solving FOCs (A27) and (A28) numerically. Combining equations (A29) and (A30), and evaluating the resulting condition in the steady state, we obtain a formula for calibrating the welfare weight θ:

$$\text{(A35)} \qquad \theta = \frac{(1-\mu)(1-\lambda)\left(\lambda k_e^{\rho} + (1-\lambda) l_s^{\rho}\right)^{(\sigma/\rho)-1} l_s^{\rho-1}}{(1-\mu)(1-\lambda)\left(\lambda k_e^{\rho} + (1-\lambda) l_s^{\rho}\right)^{(\sigma/\rho)-1} l_s^{\rho-1} + \mu l_u^{\sigma-1}}.$$

Finally, to calibrate the utility function parameter D, we use (A29) evaluated in the steady state

$$\text{(A36)} \qquad D = c^{-\gamma}(1-n^s)^{v} G_3(k_{bt}, k_{et}, l_{st}, l_{ut}) \frac{\left[N_0^s \theta + N_0^u (1-\theta)\right]}{\theta},$$

where the balanced consumption $c \equiv N_0^s c^s + N_0^u c^u$ is obtained from the budget constraint (13) evaluated in the steady state.

APPENDIX D: SOLUTION PROCEDURE

We first notice that if the expectations were parameterized in both intertemporal FOCs (A27) and (A28), then both conditions would identify consumption. As a consequence, consumption would be overidentified, while the rest of variables would be not identified. We therefore rewrite the FOCs in a way that is more suitable for parameterization, by premultiplying (A27) by $k_{b,t+1}$ and premultiplying (A28) by $k_{e,t+1}$. In this way, we obtain two equations that identify two capital stocks,

$$\text{(A37)} \qquad k_{b,t+1} = \frac{\beta E_t[1]}{\gamma c_t^{-\gamma}} k_{b,t+1} \quad \text{and} \quad k_{e,t+1} = \frac{q_0 \beta E_t[2]}{\gamma \gamma^{q} \kappa_t c_t^{-\gamma}} k_{e,t+1},$$

where $E_t[1]$ and $E_t[2]$ denote the expectation terms within the brackets in FOCs (A27) and (A28), respectively.

As far as the intratemporal conditions (A29) and (A30) are concerned, they do not allow for analytical solution with respect to l_{st} and l_{ut}. Finding a numerical solution to the intratemporal conditions on each date within the iterative cycle is costly, so, as we mentioned in the main text, we find it easier to parameterize the intratemporal conditions in the same way as we parameterize the intertemporal FOCs. To be specific, we parameterize the total hours worked by skilled and unskilled agents,

$$\text{(A38)} \qquad l_{st} = N_0^s[3] \quad \text{and} \quad l_{ut} = N_0^u[4],$$

where [3] and [4] are the expressions within the brackets of FOCs (A29) and (A30), respectively. Each of the four variables $k_{b,t+1}, k_{e,t+1}, l_{st}, l_{ut}$ is parameterized by a first-order exponentiated polynomial of the type

$$\text{(A39)} \qquad \exp(\xi_0 + \xi_1 \ln k_{bt} + \xi_2 \ln k_{et} + \xi_3 \ln z_t + \xi_4 \ln \kappa_t).$$

We therefore have to identify 20 unknown coefficients—five coefficients for each of the four variables parameterized. We do this by using the following iterative procedure.

1. Fix the initial ξ terms. Fix initial condition $(k_{b0}, k_{e0}, z_0, \kappa_0)$. Draw and fix a random series for exogenous shocks $\{z_t, \kappa_t\}_{t=0}^{T}$.
2. Use the assumed decision rules (A37), (A38) and the budget constraint (13) to calculate recursively $\{k_{b,t+1}, k_{e,t+1}, l_{st}, l_{ut}, c_t\}_{t=0}^{T}$.
3. Run the nonlinear least squares regressions of the corresponding variables on the functional form (A39). Use the re-estimated coefficients $\Phi(\xi(j))$ obtained on iteration j to update each of

the 20 coefficients for the next iteration $(j+1)$ according to $\xi(j+1) = (1-\varpi)\xi(j) + \varpi\Phi(\xi(j))$, $\varpi \in (0,1)$.

Iterate on the ξ terms until a fixed point is found.

As an initial guess, we set the values of the ξ equal to the deterministic steady state. The algorithm is able to systematically converge to the true solution if the coefficients are updated slowly, $\varpi \leqslant 0.01$, and if the simulated series are bounded to rule out implosive (explosive) strategies as described in Maliar and Maliar (2003b). The program was written in Matlab and the simulation was carried out a laptop with a Dual Core 2.5 GHz processor. The computational time was around a half an hour when the length of simulations was $T = 10{,}000$.

ACKNOWLEDGMENTS

We thank two anonymous referees and editor Alex Michaelides for many valuable suggestions. We also benefited from comments of Gianluca Violante. This research was partially supported by the Instituto Valenciano de Investigaciones Económicas, the Hoover Institution at Stanford University and the Spanish Ministry of Science and Technology under the grant SEJ-62656, the José Castillejo program (Lilia Maliar) and the Salvador Madariaga program (Serguei Maliar).

NOTES

1. The CES specification of the production function is extensively used in the literature for studying growth issues; see Papageorgiou and Saam (2008) for a review.
2. Lindquist (2005) and Domeij and Ljungqvist (2007) use the Krusell *et al.* (2000) model to study long-run trends in the skill premium in Sweden. Also, Domeij and Ljungqvist (2007) extend the analysis that Krusell *et al.* (2000) perform for the US economy over the 1963–92 period to include another decade, and find that the predicted skill premium remains remarkably close to its empirical counterpart.
3. A one-sector neoclassical growth model is consistent with steady-state growth only if technical change is labour-augmenting; see Uzawa (1961) and Jones and Scrimgeour (2008) for the proof. Recently, in the context of an endogenous growth model, Acemoglu (2003) argues that technical change can be also capital-augmenting.
4. The case of a zero-growth steady state is analysed in Papageorgiou and Saam (2008). This paper establishes the conditions for the existence of a zero-growth steady state in the Solow and Diamond growth models under a CES production function with capital–skill complementarity.
5. Ngai and Pissarides (2007) also construct a model with differences in technology growth rates across sectors and derive restrictions on the utility and production functions that are sufficient for balanced aggregate growth (as well as for sectorial labour reallocation).
6. In fact, we could have generated a stationary pattern of risk premium in the model by assuming $\Gamma^u = \Gamma^s$. However, according to Proposition 2, the assumption of balanced growth will therefore imply that $\gamma^s = \gamma^u$, so the model will not match the differing growth rates of the skilled and unskilled populations.
7. A possible reason for this failure of the model might be that the particular time period considered by Krusell *et al.* (2000) reflects a temporary off steady state behaviour. Indeed, Greenwood *et al.* (1997) argue that a major technology change occurred in the mid-1970s, and under this hypothesis, the data analysed in Krusell *et al.* (2000) represent a large departure from the steady state.
8. Strictly speaking, we also need to show that the individual transversality conditions in the decentralized economy imply the aggregate transversality condition in the planner's economy. This can be shown as in Maliar and Maliar (2001, 2003a).
9. Here and elsewhere, we use variables without time subscripts to denote the corresponding steady-state values.

REFERENCES

ACEMOGLU, D. (2003). Labor- and capital-augmenting technical change. *Journal of European Economic Association*, **1**, 1–37.

——— and GUERRIERI, V. (2008). Capital deepening and non-balanced endogenous growth. *Journal of Political Economy*, **116**, 467–98.

DEN HAAN, W. and MARCET, A. (1990). Solving the stochastic growth model by parameterizing expectations. *Journal of Business and Economic Statistics*, **8**, 31–4.

DOMEIJ, D. and LJUNGQVIST, L. (2007). The missing Swedish skill premium: Sweden versus the United States 1970–2002. Manuscript, Stockholm School of Economics.

GREENWOOD, J., HERCOWITZ, Z. and KRUSELL, P. (1997). Long-run implications of investment-specific technological change. *American Economic Review*, **87**, 342–62.

HORNSTEIN, A. (2004). (Un)balanced growth. *Federal Reserve Bank of Richmond Economic Quarterly*, **90**, 25–45.

JONES, C. and SCRIMGEOUR, D. (2008). A new proof of Uzawa's steady state growth theorem. *Review of Economics and Statistics*, **90**, 180–2.

KALDOR, N. (1961). Capital accumulation and economic growth. In F. A. Lutz and D. C. Hague (eds), *The Theory of Capital: Proceedings of a Conference of the International Economic Association*. London: Macmillan.

KING, R., PLOSSER, C. and REBELO, S. (1988). Production, growth and business cycles. *Journal of Monetary Economics*, **21**, 195–232.

KRUSELL, P., OHANUIAN, L., RIOS-RULL, V. and VIOLANTE, G. (2000). Capital–skill complementarity and inequality. *Econometrica*, **68**, 1029–53.

KYDLAND, F. and PRESCOTT, E. (1982). Time to build and aggregate fluctuations. *Econometrica*, **50**, 1345–70.

KYLYMNYUK, D., MALIAR, L. and MALIAR, S. (2007a). Rich, poor and growth-miracle nations: multiple equilibria revisited. *Topics in Macroeconomics*, **7**(1), article 20.

———, ——— and ——— (2007b). A model of unbalanced sectorial growth with application to transition economies. *Economic Change and Restructuring*, **40**(4), 309–25.

LINDQUIST, M. (2004). Capital–skill complementarity and inequality over the business cycle. *Review of Economic Dynamics*, **7**, 519–40.

——— (2005). Capital–skill complementarity and inequality in Sweden. *Scandinavian Journal of Economics*, **107**, 711–35.

MALIAR, L. and MALIAR, S. (2001). Heterogeneity in capital and skills in a neoclassical stochastic growth model. *Journal of Economic Dynamics and Control*, **25**, 1367–97.

——— and ——— (2003a). The representative consumer in the neoclassical growth model with idiosyncratic shocks. *Review of Economic Dynamics*, **6**, 362–80.

——— and ——— (2003b). Parameterized expectations algorithm and the moving bounds. *Journal of Business and Economic Statistics*, **21**, 88–92.

MULLIGAN, C. (2002). Capital, interest, and aggregate intertemporal substitution. NBER working paper no. 9373.

NGAI, R. and PISSARIDES, C. (2007). Structural change in a multi-sector model growth. *American Economic Review*, **97**, 429–43.

PAPAGEORGIOU, C. and SAAM, M. (2008). Two-level CES production technology in the Solow and Diamond growth models. *Scandinavian Journal of Economics*, **110**, 119–43.

RAMEY, V. and FRANCIS, N. (2006). A century of work and leisure. NBER working paper no. 12264.

UZAWA, H. (1961). Neutral inventions and the stability of growth equilibrium. *Review of Economic Studies*, **28**, 117–24.