# Supply Disruptions and Optimal Network Structures

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This paper studies multi-tier supply chain networks in the presence of disruption risk. Firms decide how much to source from their upstream suppliers so as to maximize their expected profits, and prices of intermediate goods are set so that markets clear. We provide an explicit characterization of (expected) equilibrium profits, which allows us to derive insights into how the network structure, i.e., the number of firms in each tier, production costs, and disruption risk, affect firms' profits. Furthermore, we establish that networks that maximize profits for firms that operate in different stages of the production process, i.e., for upstream suppliers and downstream retailers, are structurally different. In particular, the latter have relatively less diversified downstream tiers and generate more variable output than the former. Finally, we consider supply chains that are formed endogenously. Specifically, we study a setting where firms decide whether to engage in production by considering their (expected) post-entry profits. We argue that endogenous entry may lead to chains that are inefficient in terms of the number of firms that engage in production.

*Key words*: Multi-sourcing; Competition; Disruption risk; Supply chain networks.

## 1. Introduction

Recent trends in the globalization of trade have led to the emergence of multi-tier supply chains that involve firms competing at different stages of the production process. Importantly, firms are subject to disruption risk that may be attributed to a variety of sources ranging from natural disasters to changes in the regulatory and political environment. Given the complex structure of supply chain networks, dealing with such risks and ensuring the resilience of their procurement channels has become a key objective for firms.

Undoubtedly, supply disruptions have a severe effect on a firm's operations and can result in significant losses in profits and market share. As an illustrative example, just months after suffering major disruptions in the aftermath of the devastating 2011 earthquake in the Tohoku region in

Japan, the automotive industry was hit again by an explosion at the manufacturing plant of Evonik, the prime supplier of a specialty resin. Evonik was forced to halt production for several months, creating serious issues for several automakers.[1] In another incident, an explosion at the Tianjin port in China in 2015 disrupted one of the primary export centers for a vast number of goods produced in China causing delays and higher transportation costs for a number of industries. Finally, Boeing's widely documented struggles during the development and initial service stages of its 787 Dreamliner jets point to the pitfalls associated with outsourcing and increasingly complex supply networks. Boeing's revolutionary plane entered service only after multiple delays attributed to poor quality control from some of its suppliers and shortages of key components.[2]

These observations, i.e., the ever-increasing complexity of supply chains along with the adverse effects on a firm's bottom line that disruptive events may entail, motivate studying how the interplay of supply chain structure, disruption risk, and production costs impacts firms in a complex supply chain. To this end, the present paper aims to provide an understanding of the following closely related questions:

- How do firms' profits depend on the structure of their supply chain networks, the risks they (as well as their suppliers) are exposed to, and the costs they incur from engaging in production? Relatedly, how do shifts in the underlying markets, e.g., an increase in the likelihood of disruptive events or a change in the number of available suppliers (e.g., due to bankruptcy or consolidation), affect the entire supply chain?

- Firms' profits may be intimately tied to their position in the chain, e.g., being an upstream supplier or a downstream retailer. Are there any structural properties that can be used to predict a firm's profitability for a given supply chain network and disruption risk profile?

- Finally, complex chains are typically formed as the outcome of the endogenous entry of profit-maximizing firms. What are the implications of this process on aggregate welfare? How do equilibrium networks compare to networks that are optimal for aggregate welfare in the presence of disruptions?

As a starting point for addressing these questions, we develop a stylized model of multi-tier supply chains, where production is subject to disruption risk. In the model, firms source their

[1] The Financial Times (2012) reported on the ripple effects experienced by most auto makers in the aftermath of the explosion at Evonik's plant. Simchi-Levi et al. (2014) use this as a motivating example to develop metrics that assess the importance of a supplier for a downstream manufacturer as a function of not only total spending but also its position in the supply network.

[2] There are several articles reporting on a variety of problems that Boeing had to face during the development, production, and first months of service of their 787 jets. Anupindi (2009) describes in detail the industry and the background behind the Dreamliner and focuses on the problems Boeing experienced with the plane's fasteners, i.e., the bolts that hold the structure together. One of the issues that led to the shortages of their supply was identified as the post 9/11 consolidation in the fastener industry, which, in turn, led to a sharp decline in production output (see also Reuters (2007)).

inputs and produce their outputs in anticipation of disruptive events that may affect them and their same-tier competitors, with the goal of maximizing their expected profits. Firms in different tiers determine their sourcing quantities sequentially, after observing the realized production output of their upstream suppliers. Prices for intermediate goods are determined endogenously and are such that the corresponding markets clear.[3] Consequently, firms' profits depend not only on the risk profiles of the firms in the same tier but also on those of firms upstream and downstream. This dependence is nontrivial, as it is affected by the underlying supply network structure. Finally, in addition to risk considerations, firms compete with one another to procure the inputs necessary for production and to secure the business of downstream firms.

We start our exposition by providing a characterization of the equilibrium, which allows us to relate firms' (expected) profits to their position in the supply chain network, as well as to their disruption risk and production costs. Then, we perform comparative statics that aim to relate changes in the underlying market conditions, e.g., a decrease in the number of available suppliers, to firms' profits. Our analysis complements the conventional wisdom built from two-tier supply chains and may be useful for quantifying the supply chain-wide effects of consolidation among suppliers (e.g., as in the case of the Boeing fasteners) or an increase in the uncertainty associated with a tier's production as a result of changes in the economic environment firms operate in.

On the descriptive side, we establish a close connection between the supply chain's structure and the coefficient of variation of the chain's resulting output. In particular, we focus on networks with a fixed number of firms, and we show that the coefficient of variation is low when the networks are "balanced," i.e., when they have a similar number of firms in all tiers, whereas it is high when the networks have tiers with a relatively small number of firms. Thus, our analysis identifies two sources that contribute to a chain's output volatility: first, the inherent risk associated with the production of intermediate inputs and the final good and, second, the supply chain structure. Furthermore, we relate the coefficient of variation (and, consequently, the extent to which the supply chain is balanced) to how the aggregate surplus generated by the chain is allocated between upstream and downstream firms. Specifically, we show that the networks that maximize the profits for the retailers are less balanced and induce a higher coefficient of variation for the chain's output than those that maximize the profits of raw materials suppliers. Interestingly, this finding is attributed solely to the presence of disruption risk: in the absence of supply uncertainty, the same set of networks leads to maximum profits for both sets of firms. On the prescriptive side, our equilibrium

[3] In fact, the price formation process for the output of each intermediate tier can be interpreted as a spot market. Spot market mechanisms are employed in a number of real-world supply chains, such as those for semiconductors and microelectronics (see, e.g., http://www.dramexchange.com), albeit, not in all tiers of production from raw materials to final goods.

characterization may be useful in evaluating the benefits/costs associated with a firm's strategic initiatives or serve as a building block for assessing the effectiveness of other operational measures.

Finally, we consider the process of endogenous supply chain formation. Firms decide whether to engage in production by taking into account their expected post-entry profits and fixed cost of entry. Our analysis indicates that equilibrium supply chain structures take the form of an inverted pyramid with a relatively larger number of firms in the upstream tiers of the manufacturing process. We also illustrate that equilibrium supply chains may be inefficient in terms of the number of firms that engage in production. Notably, we highlight that in the presence of disruption risk, firms fail to internalize the positive externality that engaging in production may exert on tiers other than their own. Thus, unlike in models of entry that restrict attention to a single market (tier), we illustrate that the equilibrium network may feature too few firms. In addition, the associated aggregate welfare loss may be significant. As a result, targeted interventions that may incentivize entry in given tiers may lead to substantial benefits for firms and consumers.

### 1.1. Related Literature

Our paper builds on the recent but growing literature that studies equilibrium outcomes in multi-tier supply chains. Corbett and Karmarkar (2001) and Carr and Karmarkar (2005) study entry decisions and post-entry Cournot competition in multi-tier serial supply chains, and establish the existence of an entry equilibrium. Federgruen and Hu (2016) analyze a model of price competition where at each level of the manufacturing process an arbitrary number of firms offer a set of differentiated products. They provide insights into how changes in the cost rates and demand intercept values affect prices and profits at equilibrium. Relatedly, Federgruen and Hu (2015) study a similar model in which each of a number of firms offers subsets of a given line of $N$ products. Their goal is to show that a price equilibrium exists and to provide a characterization of product assortments and sales volumes that arise at equilibrium. Unlike our paper, which focuses on the interplay between disruption risk, competition, and network structure, these papers involve deterministic models and mostly stress how the firms' cost structure affects equilibrium outcomes.

A central feature of our modeling framework is that sourcing decisions are taken in the presence of disruption risk. The literature that explores the impact of supply uncertainty on multi-tier chains is relatively sparse. Osadchiy et al. (2016) and Carvalho et al. (2017) provide novel empirical evidence that the structure of supply chains may contribute to the amplification of shocks to the firms' production output. Closer to our work, Acemoglu et al. (2012) study the propagation of productivity shocks in a multi-sector economy while assuming that each sector is competitive. In their modeling framework, unlike ours, both prices and sourcing decisions are determined after productivity shocks to *all* sectors are realized. By contrast, we assume that sourcing decisions in a tier are taken *before* disruption shocks further downstream are realized. In addition, the questions our paper investigates are substantially different. In particular, we explore endogenous supply network formation and provide a characterization of structures that maximize profits for suppliers of raw materials and downstream retailers to illustrate that they view different structures as optimal.

Bimpikis et al. (2017) study sourcing decisions in a three-tier supply chain in the presence of disruption risk that takes the form of yield uncertainty. They establish conditions for the production technology as well as the profit functions that lead to an inefficient structure at equilibrium from the point of view of the manufacturer. Their analysis highlights that mitigating risk at the individual firm level may increase the overlap in the downstream manufacturer's procurement channels and consequently amplify the aggregate risk for the chain. Ang et al. (2017) compare several different three-tier chains in terms of their profits for the manufacturer. Their focus is on environments where firms may be different in terms of their cost and disruption risk profiles and the manufacturer's objective is to provide contracts that incentivize suppliers to source optimally.[4] Both papers study three-tier chains with a single downstream manufacturer and two firms in each of the top tiers. By contrast, the present paper considers multi-tier supply chains, incorporates competition between firms, and derives several insights into the effect of the chain's structure on equilibrium profits and sourcing decisions. Furthermore, we highlight that endogenous supply chain formation leads to inefficient structures in terms of the aggregate number of firms that engage in production.[5]

Nguyen (2017) studies a local bargaining model for firms in a multi-tier supply chain and analyzes the impact of transaction costs on convergence to an equilibrium. Nakkas and Xu (2017) consider bargaining in two-sided supply chain networks, and explore the impact that both the heterogeneity in the valuations of the suppliers' goods and the network structure that determines the set of potential supply relationships have on prices and efficiency at equilibrium. These papers consider deterministic models and assume that bargaining between firms is the primary way to determine the terms of trade. In concurrent and independent work from ours, Kotowski and Leister (2018) consider a multi-tier intermediation network where intermediaries compete in a series of auctions

[4] The modeling assumptions in the three papers differ significantly in accordance with the focus of each work. Unlike Bimpikis et al. (2017), we consider "all-or-nothing" disruption risk and a simple production technology that maps one unit of input to one unit of output. Similarly, Ang et al. (2017) employ "all-or-nothing" disruption risk and a simple production technology but feature a perfectly reliable supplier and an unreliable one in the second tier of production.

[5] Also of relevance is Adida and DeMiguel (2011) who consider a setting in which multiple manufacturers compete in quantities to supply a set of products to a set of retailers. Furthermore, Bernstein and Federgruen (2005) study a two-echelon supply chain in which a single manufacturer supplies to a set of competing retailers that face stochastic demand, and provide insights on contractual agreements between the parties that ensure that the decentralized chain performs as well as the centralized one. Finally, Bakshi and Mohan (2017) consider supply chain networks in the presence of disruption risk when firms produce complementary products. They assume that there is one supplier for each of the products and, thus, a single disruptive event implies that there is no production of the final good.

and facilitate the trade of a single indivisible good from a supplier to consumers. Traders in intermediate tiers can consume the product or resell it to lower tiers. Each intermediary is inactive with some probability, in which case she does not participate at all in trading. Their main result that an inefficiently low number of intermediaries participate in the equilibrium trading network (when entry is costly) is similar to our discussion in Section 5. By contrast with these papers, we study a different trading model, and provide a series of comparative statics results that shed light on how disruption risk and network structure affect firms depending on the stage of the production process they participate in. We also provide structural insights into how different network structures may yield higher profits for upstream suppliers and downstream retailers.

Finally, several papers study interesting questions arising in the management of disruptions in supply chains, mostly focusing on models that involve two-tier supply chains (e.g., Tomlin (2006), Babich et al. (2007), Yang et al. (2009), Yang et al. (2012), and for a comprehensive survey see Aydin et al. (2011)). For example, Tomlin (2006) considers a retailer sourcing from two suppliers that differ in their disruption risk and cost. He provides guidelines on when dual sourcing or carrying extra inventory (or a combination thereof) is more effective in dealing with the suppliers' risks. Babich et al. (2007) study a model where two firms prone to default compete in supplying their output to a retailer. The main question involves the interplay between default correlations and the profits of the retailer or the chain on aggregate. Although simpler in some dimensions, our model involves a multi-tier chain and several competing firms. This leads to a different set of questions and insights (e.g., the impact of the network structure on equilibrium profits) that two-tier models do not capture.

## 2. Model

We consider a multi-tier supply chain for a single final good consisting of $N$ risk-neutral firms organized in $K+1$ tiers of production. We denote the set of firms in tier $k$ by $Tier(k)$, and denote the number of firms in this tier by $n(k)$. The output of a firm in tier $k>1$ can only be used by firms in tier $k-1$. Furthermore, we assume that each firm in tier $k-1$ can source inputs from any firm in tier $k$ (see Figure 1 for an illustration). Thus, network structure in the context of our model refers to the number of firms in each of the $K+1$ tiers of the chain.

We refer to firms in tier 1, i.e., the most downstream tier, as "retailers" in the remainder of the paper. Retailers sell the final good to the end consumer market, which is modeled with a linear demand curve. In particular, if the price of the final good is equal to $p$, the consumers' demand is given by

$$D(p) = \max\left\{\frac{\alpha}{\beta} - \frac{1}{\beta}p, 0\right\}, \tag{1}$$

where $\alpha, \beta > 0$ are known constants.

The most upstream tier (tier $K+1$) processes "raw materials" that are procured at price $p_c \geq 0$ per unit (e.g., in a commodity market). The firms in tier $k \notin \{1, K+1\}$ convert the output of tier $k+1$ to the input of tier $k-1$. The production technology of each firm is such that, in the absence of a disruption, one unit of input generates one unit of output.

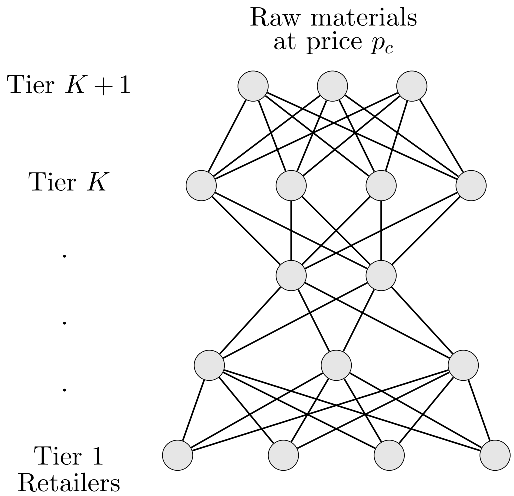


**Figure 1 Multi-tier supply chain with $K+1$ tiers of production. Production output flows from upstream to downstream tiers.**

*Sourcing in the presence of disruption risk.* Production is subject to disruption risk. In particular, firm $i$'s output is equal to its input with probability $q_i > 0$. Otherwise, the firm suffers a disruption and fails to produce anything (while wasting any inputs it has procured). We assume that disruptions affecting different firms are independent and refer to $\bar{q}_i \triangleq 1 - q_i$ as firm $i$'s *disruption probability.*

We denote by $Z_{k,i} \in \{0, 1\}$ the Bernoulli random variable associated with the event of successful production of firm $i$ in tier $k$ (i.e., $Z_{k,i}$ is equal to one when firm $i$ does not experience a disruptive event and $\mathbb{P}(Z_{k,i} = 1) = q_i$). Note that although the disruptive events affecting two firms in a tier are assumed to be independent, the outputs of these firms are still correlated, since disruptive events in upstream tiers affect the access to input materials for both.[6]

[6] Although our benchmark model assumes that disruptive events affecting firms are independent, Appendix B.2 extends our equilibrium characterization to settings where disruptive events in a tier may be correlated (the corresponding expressions remain essentially the same with the addition of a single parameter that captures the covariance between the disruptive events affecting firms in the same tier).

*Supply equilibrium.* Production proceeds sequentially from raw materials, i.e., the input of tier $K+1$, to final goods, i.e., the output of tier 1. Firms in tier $k$ determine their order quantities (demand for input materials) after the output of their upstream suppliers is realized. Furthermore, the price of tier $k+1$'s output is determined so that the total demand of tier $k$ is equal to tier $k+1$'s *realized* supply. We proceed by describing the notation, firms' payoffs, and introducing our equilibrium concept.[7]

Let $\omega_k$ denote the realized state after firms in tiers $K+1,\dots,k+1$ complete production. This state consists of the realization of disruptions in tiers $\ell > k$. We use the convention $\omega_{K+1} = \emptyset$, and define $\omega_k$ recursively as $\omega_k = \{\omega_{k+1}, \{z_{k+1,i}\}_i\}$, where $\{z_{k+1,i}\}_i$ is the set of disruptions realized in tier $k+1$, i.e., the realizations of the Bernoulli variables $\{Z_{k+1,i}\}$ corresponding to the firms in tier $k+1$. Similarly, we denote by $\hat{\omega}_k$ the corresponding random variable, which is also defined recursively as $\hat{\omega}_k = \{\hat{\omega}_{k+1}, \{Z_{k+1,i}\}_i\}$ with $\hat{\omega}_{K+1} = \emptyset$. The set of all possible states that can be realized with nonzero probability at tier $k$ is denoted by $\Omega_k$. Before they finalize their procurement decisions, firms in tier $k$ observe $\omega_k \in \Omega_k$.

The market price of the output of tier $k \in \{1,\dots,K+1\}$ is a function of the state $\omega_{k-1}$ realized after the tier completes production, and is denoted by $p_k(\omega_{k-1})$. [8] We use the convention $p_{K+2}(\omega_{K+1}) = p_c$ to denote the price of raw materials available to firms in tier $K+1$. Firms take these prices (which will be determined at equilibrium) as given and decide on their procurement quantities. The expected profit of firm $i$ in tier $k$ (for $1 \le k \le K+1$), conditional on the realization of the upstream uncertainty $\omega_k$ and for procuring $y_{k,i}$ units of input is given as follows:

$$\bar{\pi}(i,\omega_k,y_{k,i}) = \mathbb{E}\left[ p_k(\hat{\omega}_{k-1}) Z_{k,i} y_{k,i} - p_{k+1}(\hat{\omega}_k) y_{k,i} - c_i y_{k,i}^2 \middle| \hat{\omega}_k = \omega_k \right]. \tag{2}$$

Note that when agents in tier $k$ determine their procurement quantities, the uncertainty in tier $k$ has not been realized and only information about the disruptions in upstream tiers, i.e., $\omega_k$, is available to the firms (thus, the expectation is taken over the Bernoulli variables $\{Z_{k,i}\}_i$ associated with production in tier $k$). In other words, when choosing their order quantities to maximize their expected profits, firms in tier $k$ take into account possible disruptions in their tier and the impact of these disruptions on the realization of the market price $p_k(\omega_{k-1})$ of their output.

[7] In real-world supply chains, the interaction between firms engaged in consecutive stages of the production process may involve a combination of ex-ante contractual arrangements with ex-post negotiations (when disruptive events occur). The price formation process we assume, which essentially posits that prices are set so that the corresponding markets clear after the realization of any supply uncertainty, could serve as a reasonable starting point for exploring richer ways to model how firms transact with one another in the presence of disruption risk.

[8] Our equilibrium concept is akin to models of general equilibrium under uncertainty, e.g., refer to Mas-Colell et al. (1995) (Chapter 19), where a market-clearing price is associated with each potential realization of the state of the world.

The first term in Expression (2) captures the revenues of firm $i$ for the output it successfully produces and sells to downstream firms (recall that one unit of input is converted into one unit of output in the absence of disruptions; hence, the output of firm $i$ is given by $Z_{k,i}y_{k,i}$). The second term captures the payments made by firm $i$ for procuring its input. Finally, the last term is equal to the firm's cost of production, which is assumed to be quadratic in the input firm $i$ procures and processes and $c_i > 0$ denotes the corresponding production cost coefficient.[9]

The equilibrium procurement quantity of firm $i \in Tier(k)$ depends on the realization of the state $\omega_k$. We denote by $x_{k,i}(\omega_k)$ the equilibrium procurement quantity of firm $i \in Tier(k)$, at state $\omega_k$. Definition 1 states the conditions that these equilibrium quantities satisfy and formalizes our equilibrium concept.

Definition 1 (Supply equilibrium). A supply equilibrium is defined as a tuple of nonnegative, state-contingent prices and procurement quantities $\{p_k(\omega_{k-1}), x_{k,i}(\omega_k)\}$ for every tier $k \in \{1, \ldots, K+1\}$, firm $i \in Tier(k)$, and states $\omega_{k-1} \in \Omega_{k-1}$ and $\omega_k \in \Omega_k$ that satisfy the following conditions:

(i) $x_{k,i}(\omega_k) \in \arg\max_{y_{k,i} \geq 0} \bar{\pi}(i, \omega_k, y_{k,i})$ for all $k \in \{1, \ldots, K+1\}$, $i \in Tier(k)$, and $\omega_k \in \Omega_k$.

(ii) $\sum_{i \in Tier(k)} x_{k,i}(\omega_k) \leq \sum_{i \in Tier(k+1)} z_{k+1,i} x_{k+1,i}(\omega_{k+1})$, for all $k \in \{1, \ldots, K\}$, $\omega_{k+1} \in \Omega_{k+1}$, $\omega_k = \{\omega_{k+1}, \{z_{k+1,i}\}_i\} \in \Omega_k$, and $z_{k+1,i} \in \{0,1\}$, where the inequality holds with equality if $p_{k+1}(\omega_k) > 0$.

(iii) $D(p_1(\omega_0)) \leq \sum_{i \in Tier(1)} z_{1,i} x_{1,i}(\omega_1)$, for all $\omega_1 \in \Omega_1$, $\omega_0 = \{\omega_1, \{z_{1,i}\}_i\} \in \Omega_0$, and $z_{1,i} \in \{0,1\}$, where the inequality holds with equality if $p_1(\omega_0) > 0$.

In this definition, the first condition implies that given the disruption realizations $\omega_k \in \Omega_k$ in tiers upstream of $k$, and market price $p_{k+1}(\omega_k)$ for the output of tier $k+1$, each firm $i \in Tier(k)$ procures the quantity that maximizes its (expected) profits given as in Expression (2). The second condition ensures that the market for the output of each tier $k > 1$ clears. In particular, given a positive price $p_{k+1}(\omega_k)$, when firms in tier $k$ procure their optimal quantities, the entire (realized) output of tier $k+1$ is sold. In principle, the output of tier $k+1$ may be large enough that even when it is made available at a zero price, the total demand of tier $k$ is lower than the available supply. In such cases, the price of the output of tier $k+1$ is equal to zero. Similarly, the third condition ensures that the price for the output of tier 1, i.e., the retailers, is set so that the corresponding market clears when demand is given as in Expression (1).

Note that at a supply equilibrium prices endogenously reflect the availability of supply at each tier. For instance, when disruptive events limit the aggregate output of tier $k+1$ (that serves as input for production in tier $k$), equilibrium conditions (i) and (ii) imply that the resulting market

[9] This assumption captures production in industries with diseconomies of scale (which is a fairly common modeling assumption in both operations management and economics, e.g., Eichenbaum (1989), Anand and Mendelson (1997), Porteus (2002), and Ha et al. (2011)).

price for the output of tier $k+1$ would be higher than in realizations where upstream tiers were not affected by any disruptive events.

Intuitively, the model can be operationalized and interpreted as a sequence of spot markets. Given the aggregate output of their direct suppliers, firms in tier $k$ procure their inputs in order to maximize their expected profits. Then, production takes place in tier $k$ and the total output of the tier is realized, which, in turn, is traded at a (spot) market. The resulting market price is such that the demand of the firms in tier $k-1$ matches with the available supply in the spot market (unless the realized supply is too large for the market to clear, in which case the corresponding price is set to zero).

We emphasize that procurement quantities of firms in tier $K+1$ are determined prior to the realization of any disruptive events, unlike those of firms in tiers $k<K+1$. In order to make this distinction clear, we often employ a different notation for the procurement quantities of firms in tier $K+1$, i.e., we let $s_i$ denote the amount procured by firm $i \in Tier(K+1)$. We also denote by $s=\sum_{i\in Tier(K+1)} s_i$ the total procurement of raw materials by tier $K+1$ firms. This quantity plays a key role in our subsequent analysis, and we alternatively refer to it as the *aggregate supply of raw materials* for the supply chain network. Finally, in the presence of disruptions, the profit of each firm is ex-ante random. We denote by

$$\pi(i)=\mathbb{E}[\bar{\pi}(i,\hat{\omega}_k,x_{k,i}(\hat{\omega}_k))],$$

the expected equilibrium profits of firm $i \in Tier(k)$ before any disruptions are realized. In the analysis that follows, we study how this quantity depends on the network structure and the likelihood of disruptions in the supply network.

## 3. Competition in the Presence of Disruption Risk

Our goal in this section is to provide an equilibrium characterization as a function of the supply chain structure and firms' disruption risk profiles. For the remainder of the paper, we assume that firms in each tier are symmetric in terms of their production costs and disruption risk profiles. We let $n(k)$ denote the number of firms in tier $k$ and, for any firm in tier $k$, we denote by $c(k)$ and $q(k)$ the common production cost coefficient and the probability that the firm successfully engages in production, respectively. Apart from simplifying the analysis and exposition of our results, this assumption allows us to isolate the effect of the network structure, i.e., the allocation of firms to different tiers, on equilibrium quantities.

The results in this section can be briefly summarized as follows. First, we provide a characterization of the supply equilibrium, and the expected equilibrium profits of firms at different stages of the production process. Then, we perform a set of comparative statics that shed light on the relation between the structure of the supply chain and various quantities of interest at equilibrium such as (i) the aggregate supply of raw materials, (ii) profits, and (iii) output variability.

### 3.1. Equilibrium Characterization

Before we provide our equilibrium characterization, we introduce the following notation:

$$\alpha(k) = \alpha \prod_{\ell=1}^{k-1} q(\ell), \text{ and} \tag{3}$$

$$\beta(k) = \begin{cases} \beta & \text{if } k = 1, \\ \beta(k-1)\dfrac{q(k-1)^2}{n(k-1)}\left(n(k-1) - 1 + \dfrac{1}{q(k-1)}\right) + \dfrac{2c(k-1)}{n(k-1)} & \text{if } 1 < k \le K+2. \end{cases} \tag{4}$$

Furthermore, we state an assumption relating $\alpha(k)$'s, $\beta(k)$'s, and $p_c$.

Assumption 1. *The supply chain network is such that:*

*(i)* $\alpha(K+2) > p_c$,

*(ii)* $\dfrac{\alpha(k)}{\beta(k)} > \dfrac{\alpha(K+2) - p_c}{\beta(K+2)}$, *for* $k \in \{1, \dots, K+1\}$.

Expression (1) directly implies that when the price of the final goods is larger than $\alpha$, the consumers' demand is zero. In turn, this implies that $\alpha$ can be viewed as an upper bound on the price that can be charged for the final goods. Moreover, the quantity $\prod_{\ell=1}^{K+1} q(\ell)$ can be interpreted as the probability that a unit of raw materials leads to the successful production of a unit of the final good. Thus, $\alpha(K+2) = \alpha \prod_{\ell=1}^{K+1} q(\ell)$ can be seen as the highest expected revenue that the supply chain can generate from one unit of raw materials. The first condition in Assumption 1 implies that the cost of raw materials is low enough to ensure that firms find it profitable to engage in production. It can be readily seen that if this condition does not hold, then the chain does not generate any output at any supply equilibrium. As we subsequently establish in Theorem 1, the second condition rules out settings, where the price for the output of at least one of the $K+1$ tiers may be zero under some realizations of the supply uncertainty.

We refer to a state $\omega_k \in \Omega_k$ as *valid* if it is such that in each of the upstream tiers $\ell > k$, there is at least one firm that does not suffer a disruptive event. We say that the supply equilibrium is *essentially unique* if at any valid state $\omega_k$, all equilibria feature the same procurement decisions $x_{k,i}(\omega_k)$ and prices $p_{k+1}(\omega_k)$.[10] Our first result establishes that the supply equilibrium is essentially unique, and provides a closed form characterization under Assumption 1.

Theorem 1. *Suppose that Assumption 1(i) holds. Then, the supply equilibrium is essentially unique. In addition, if Assumption 1(ii) also holds, the (essentially unique) equilibrium can be characterized as follows:*

[10] Note that if all firms in a tier suffer a disruptive event, then there is no available supply for any of the subsequent tiers. In this case, setting the price for the output of any of the subsequent tiers to any large enough positive value yields an equilibrium; thus, there may exist multiple equilibria. However, all these equilibria feature the same production levels and firms' profits for any realization of the disruptions. Essential uniqueness does not differentiate among such equilibria.

*(i) The aggregate supply $s$ of raw materials is given by*

$$s = \frac{\alpha(K+2) - p_c}{\beta(K+2)}. \tag{5}$$

*(ii) The price for the output of tier $k$ when the state is $\omega_{k-1}$ is given by*

$$p_k(\omega_{k-1}) = \alpha(k) - \beta(k) s \prod_{\ell=k}^{K+1} \frac{n(\ell, \omega_{k-1})}{n(\ell)} > 0, \tag{6}$$

*for all $k \in \{1, \dots, K+1\}$ and $\omega_{k-1} \in \Omega_{k-1}$. Here, we let $n(\ell, \omega_{k-1})$ denote the number of firms in tier $\ell \geq k$ that did not experience a disruption at state $\omega_{k-1}$.*

*(iii) The procurement quantity of firm $i$ in tier $k$ when the state is $\omega_k$ is given by*

$$x_{k,i}(\omega_k) = \frac{s}{n(k)} \prod_{\ell=k+1}^{K+1} \frac{n(\ell, \omega_k)}{n(\ell)}, \tag{7}$$

*for all $k \in \{1, \dots, K+1\}$ and $\omega_k \in \Omega_k$.*

*Finally, if Assumption 1(ii) does not hold, then at any supply equilibrium, there exists at least one tier $k' \in \{1, \dots, K+1\}$, such that $p_{k'}(\bar{\omega}_{k'-1}) = 0$, where $\bar{\omega}_{k'-1} \in \Omega_{k'-1}$ is the state where no firm experiences a disruption in tiers $\{k', \dots, K+1\}$.*

As can be seen from Theorem 1, the aggregate supply of raw materials at equilibrium as well as the prices and procurement quantities in all tiers, reflect both the network structure and the firms' disruption profiles through the $\alpha(k)$ and $\beta(k)$ terms defined in expressions (3) and (4), respectively.[11] Furthermore, it is evident from the equilibrium characterization that Assumption 1(ii) rules out (arguably unrealistic) settings, where the equilibrium price for the output of at least one tier is equal to zero in the absence of any disruptive events upstream of the tier. Note that this imposes a mild restriction on the modeling primitives, which is satisfied, e.g., when the price of raw materials $p_c$ is sufficiently high (but lower than $\alpha(K+2)$).[12] For the rest of the paper, we restrict attention to settings where Assumption 1 holds. We use the shorthand notation $\mathcal{A}$ to denote the set of all modeling primitives $\{\alpha, \beta, p_c, n(k), q(k), c(k)\}_k$ for which Assumption 1 holds.

When firms in a tier are symmetric (in terms of their disruption probabilities and cost coefficients) for a given market price of the output of their direct suppliers, the order quantities for all firms in the same tier are equal, and the equilibrium is symmetric. Exploiting this symmetry allows for the closed-form characterization of the equilibrium in Theorem 1 (under Assumption 1). In addition,

[11] Parameters $\alpha(k)$ and $\beta(k)$ can be viewed as the intercept and the slope associated with the equilibrium price curve corresponding to tier $k$, respectively. Note that $\alpha(1) = \alpha$ and $\beta(1) = \beta$, i.e., the intercept and the slope of the consumer demand, respectively.

[12] Lemma 3 in Section 4 provides a sufficient condition on $p_c$, $\alpha$, and the $q(k)$'s that guarantees that Assumption 1 holds for any network and any vector of cost parameters.

due to symmetry any two firms $i, j \in Tier(k)$ have the same expected profits (before any of the supply uncertainty is realized), i.e., $\pi(i) = \pi(j)$. In the remainder of the paper (with some abuse of notation), we denote the expected (equilibrium) profits of a firm in tier $k$ by $\pi(k)$.

Using Theorem 1, we next obtain the firms' expected profits at equilibrium as a function of the modeling primitives as summarized in the following corollary.

COROLLARY 1. *Suppose that Assumption 1 holds. Then, at equilibrium, the expected profits of a firm in tier $k$ are given by*

$$\pi(k) = s^2 \frac{c(k)}{n(k)^2} \prod_{\ell=k+1}^{K+1} \frac{q(\ell)^2}{n(\ell)} \left( n(\ell) - 1 + \frac{1}{q(\ell)} \right), \tag{8}$$

*where $k \in \{1, \ldots, K+1\}$ and $s$ is given by Expression (5).*

Corollary 1 provides an explicit characterization of how disruption risk, production costs, and network structure affect a firm's expected profits. Interestingly, this result readily allows for assessing the impact of mitigating the disruption risk in a tier (or investing in reducing the cost of production) on the profits of any given firm in the supply chain. Notably such interventions both affect the payoffs of firms in the tier where the intervention takes place (direct effect), and the payoffs of firms in other tiers (indirect effect), which is a feature that Corollary 1 clearly captures. Our result may be useful for prioritizing operational interventions e.g., reducing the risk of disruptive events in a tier by closely monitoring their production process, that could benefit a particular firm or the supply chain on aggregate (a point we revisit in Appendix B.3).

Furthermore, shifts in market conditions have a profound impact on the performance of complex supply chains. Such shifts are increasingly common nowadays and understanding their profit implications for firms is an important practical concern. For example, as Anupindi (2009) illustrates, Boeing was largely affected by changing demand conditions, rising costs for raw materials, and consolidation in the market for intermediate goods essential for its production process (fasteners). Our equilibrium characterization (Theorem 1 and Corollary 1), which provides an explicit mapping between the environment's primitives and the firms' expected profits, may find use in assessing the impact of such shifts on firms at different parts of the supply chain.

### 3.2. Comparative Statics

In this subsection, we restrict attention to settings where Assumption 1 holds and we leverage Theorem 1 and Corollary 1 to obtain several comparative statics results that shed light on how changes in supply uncertainty or in the number of firms engaged in production affect the chain's expected output and ultimately firms' profits.[13] Contrary to what might be expected, we establish that the relationship among these quantities is not always monotonic.

[13] Specifically, the propositions that follow state whether the aggregate supply $s$ of raw materials and firms' profits in tier $k \in \{1, \cdots, K+1\}$ increase or decrease as a function of changes in the cost coefficient, the number of firms, or

*Aggregate supply of raw materials.* First, we consider quantity $s = \sum_{i \in Tier(K+1)} s_i$ that firms in tier $K+1$ procure for their production, which, given that in the absence of a disruptive event one unit of input generates one unit of output in each of the different stages of production, sets an upper bound on the total output of the supply chain network. We are interested in exploring how this quantity changes as a function of the disruption probabilities and the structure of the chain. Proposition 1 follows directly from Theorem 1.

PROPOSITION 1. *Consider modeling primitives that belong to set $\mathcal{A}$. At equilibrium, the aggregate supply $s$ of raw materials that firms in tier $K+1$ procure for their production is:*

*(i) decreasing in the production cost coefficient $c(\ell)$;*

*(ii) increasing in the number of firms $n(\ell)$;*

*(iii) possibly nonmonotonic in $q(\ell)$, for any tier $\ell$.*

Parts (i) and (ii) are fairly intuitive. In particular, if the production costs in a tier $\ell$ increase (or the number of firms decreases), then for a given price for the output of tier $\ell+1$, the total demand of firms in tier $\ell$ decreases. Equivalently, since the prices of intermediate goods are set so that their supply matches the demand, it follows that such a change would lead to a decrease in the price associated with any output level of tier $\ell+1$. In turn, the decrease in the price of their outputs leads firms in tier $\ell+1$ to produce less and demand less inputs from tier $\ell+2$. This effect propagates upstream and results in firms in tier $K+1$ procuring a lower quantity of raw materials.

The intuition behind part (iii) can be best understood in light of Theorem 1. In particular, Expressions (3) and (4) imply that for $\ell < K+1$, as $q(\ell)$ increases, both $\alpha(K+1)$ and $\beta(K+1)$ increase. According to the theorem these quantities determine the price of the output of tier $K+1$ as a function of the realization of the supply uncertainty. The increase in $\alpha(K+1)$ captures the fact that each unit produced by tier $K+1$ has a higher probability of turning into a final good when $q(\ell)$ is larger. Hence, for low output levels of tier $K+1$, the price downstream firms are willing to pay for a unit of its output is higher. On the other hand, the increase in $\beta(K+1)$ captures the fact that a higher $q(\ell)$ results in a higher expected production output for the chain (for the same aggregate supply $s$), and a lower price for the final good at the end consumer market. In turn, the willingness to pay of firms downstream of tier $K+1$ for their inputs decreases. When $q(\ell)$ is small, the first effect (increase in $\alpha(K+1)$) dominates, and incentivizes firms in tier $K+1$ to procure and produce more. When $q(\ell)$ is large, the second effect may dominate and result in a decrease

the probability of successful production in tier $\ell$. To state our results, we consider modeling primitives that belong to set $\mathcal{A}$. That is, we consider modeling primitives that belong to $\mathcal{A}$ both before and after the change in the relevant parameter. This enables us to leverage Theorem 1 and Corollary 1 for our comparative statics.

in the equilibrium procurement quantities of these firms.[14] The top panel in Figure 2 illustrates Proposition 1(iii).[15]

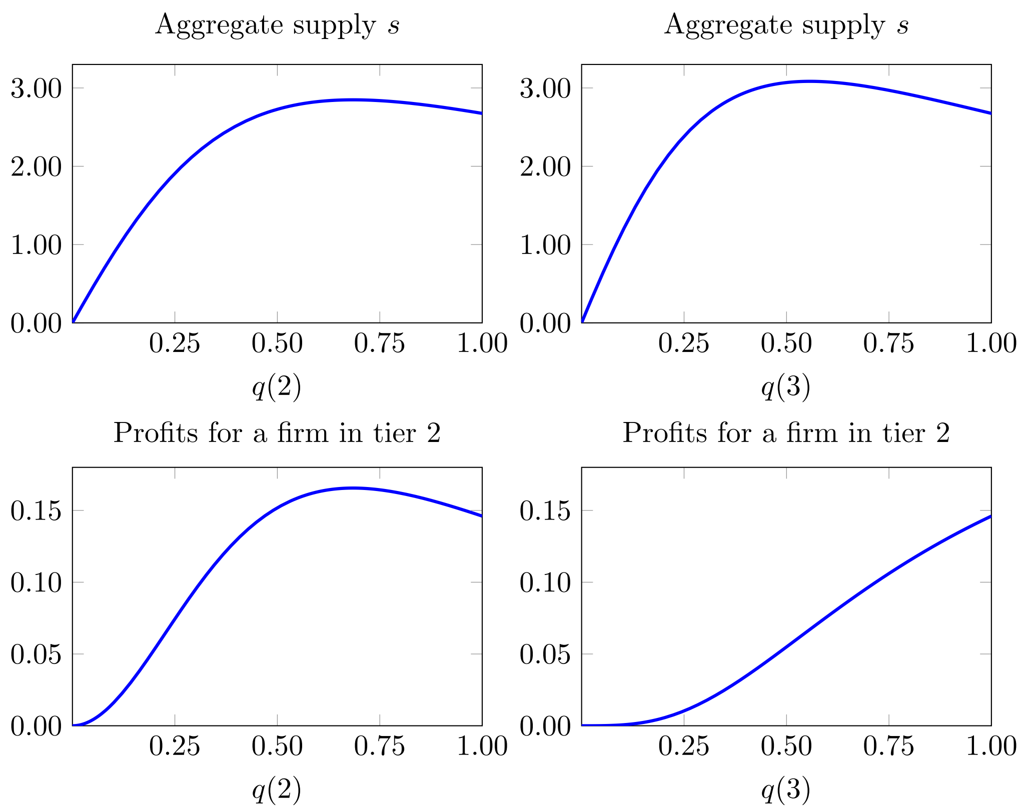


**Figure 2** **Consider a network with four tiers and $n(4)=4$, $n(3)=7$, $n(2)=7$, and $n(1)=3$. The demand parameters for the downstream market are $\alpha=10, \beta=2$, the cost coefficient for all firms is equal to $c=1$, and $p_c=0$. The two plots on the left illustrate the aggregate supply $s$ of raw materials and the expected profits for a firm in tier 2, respectively, as a function of the probability of successful production in tier 2, i.e., $q(2)$, when $q(1)=q(3)=q(4)=1$. On the other hand, the ones on the right illustrate $s$ and expected profits for a firm in tier 2 as a function of $q(3)$ when $q(1)=q(2)=q(4)=1$. It can be readily verified that Assumption 1 holds for the set of parameters we use for the figure.**

*Equilibrium profits.* Next, we consider the firms' profits at equilibrium and study how they relate to the firms' disruption risk profiles and the structure of the supply chain.

Proposition 2. *Consider modeling primitives that belong to set $\mathcal{A}$. Firms' profits in tiers downstream of $k$, i.e., $\{\pi(\ell)\}_{\ell<k}$, are increasing in the probability $q(k)$ that production is successful in*

[14] Finally, when $\ell = K+1$, the price of the output of tier $K+1$ does not depend on $q(\ell)$. However, nonmonotonicity of $s$ in $q(\ell)$ can still be established using Theorem 1.

[15] In the proof of Proposition 1, we also establish that when $s$ is nonmonotonic in $q(\ell)$ for some tier $\ell$, it is first increasing and then decreasing in $q(\ell)$ (after restricting attention to modeling parameters in $\mathcal{A}$). This fact, i.e., that $s$ is single-peaked, is also illustrated in Figure 2.

*tier $k$. On the other hand, firms' profits in upstream tiers including tier $k$, i.e., $\{\pi(\ell)\}_{\ell \geq k}$, can be nonmonotonic in $q(k)$.*

The fact that the profits of some firms can be nonmonotonic in $q(k)$ may seem counterintuitive. To understand this result, note that Corollary 1 implies that profits of firms in tiers $\ell \geq k$ depend on $q(k)$ only through the aggregate supply of raw materials $s$. However, as established in Proposition 1, supply $s$ can be nonmonotonic in $q(k)$, readily implying the nonmonotonicity of profits. The bottom panel of Figure 2 illustrates that a lower probability of disruption, i.e., higher $q(k)$, may actually lead to lower profits at equilibrium for firms in tier $\ell \geq k$. Corollary 1 shows that for firms in tier $\ell < k$, the dependence of profits on $q(k)$ is more intricate, and it is through term $s^2 \left(q(k)^2 \left(n(k) - 1 + \frac{1}{q(k)}\right)\right)$. Note that while $s$ can be nonmonotonic in $q(k)$ the remaining terms in this expression are increasing in $q(k)$. In the proof of the proposition we establish that the combined effect is such that for firms in tier $\ell < k$ profits are increasing in $q(k)$.

We next study how profits are affected by changes in the network structure. In particular, we characterize the change in profits when the number of firms in some tier $k$ increases; i.e., the competition in tier $k$ is intensified.

PROPOSITION 3. *Consider modeling primitives that belong to set $\mathcal{A}$. Increasing the number of firms in tier $k$ results in a decrease in the profits of firms in tier $k$ and in an increase in the profits of firms in tiers upstream of $k$. On the other hand, the profits of firms in any tier $\ell < k$ can be increasing or decreasing in $n(k)$. However, there exists $\hat{q} < 1$ such that if $q(k) \geq \hat{q}$, profits in every tier $\ell < k$ are increasing in $n(k)$.*

To understand why Proposition 3 holds, first note that increasing the number of firms in tier $k$ intensifies competition and naturally leads to a decrease in the profits of the firms in this tier, which is consistent with the result for $\ell = k$. Next, consider tier $\ell < k$. On the one hand, Proposition 1 indicates that the supply $s$ of raw materials increases at equilibrium when more firms engage in production in tier $k$. On the other hand, it can be seen from Expression (8) that the last term in the profits of firms in tier $\ell$ (i.e., $\pi(\ell)$) decreases with $n(k)$. When $q(k)$ is large enough the increase in $s$ dominates and results in an increase in the profits. However, in general, the combined effect of the decrease in the aforementioned term and the increase in $s$ is indeterminate. Thus, profits of firms in tier $\ell < k$ can increase or decrease as a result of an increase in the number of firms in tier $k$. Finally, for $\ell > k$, Corollary 1 implies that profits in tier $\ell$ depend on $n(k)$ only through the $s$ term. As mentioned above, $s$ increases at equilibrium when more firms engage in production in tier $k$; thus, profits in tier $\ell > k$ are also increasing with $n(k)$.

*Output variability.* In the presence of disruptions, the quantity of the final good that reaches the downstream consumer market is random. Moreover, its realization directly impacts the market price as well as the profits of the downstream retailers. Thus, it is of interest to understand the extent to which the final output of the production process varies due to the presence of disruptive events. In what follows, we focus on the *coefficient of variation* ($CV$) of the realized output as our measure of output variability. Apart from being a natural measure of how volatile the production process is, the coefficient of variation of the final output is also useful for comparing the networks that are preferred by different tiers in the supply chain as we show in a subsequent section.

In particular, we let $U$ denote the (random) output that reaches the downstream consumer market. At equilibrium, the coefficient of variation of $U$ for a network $\mathcal{N}$ can be explicitly expressed in terms of the disruption risk profile and the number of firms in each tier, as we establish next.

LEMMA 1. *Suppose that Assumption 1 holds. At equilibrium, the expected value of quantity $U$ that reaches the downstream consumer market is equal to*

$$\mathbb{E}[U] = s \prod_{\ell=1}^{K+1} q(\ell),$$

*whereas its coefficient of variation is given as follows:*

$$CV(U) = \sqrt{\frac{\text{Var}(U)}{\mathbb{E}[U]^2}} = \left( \prod_{\ell=1}^{K+1} \left( \frac{n(\ell) - 1 + 1/q(\ell)}{n(\ell)} \right) - 1 \right)^{1/2}. \tag{9}$$

Note that the first part of the lemma follows from the fact that $\prod_{\ell=1}^{K+1} q(\ell)$ is equal to the expected output of the final good that corresponds to one unit of supply of raw materials. The second part provides a characterization of the coefficient of variation, which, as one would expect, establishes that $CV$ decreases with the number of firms in any tier $\ell$ whereas it increases with the disruption probability $1 - q(\ell)$.

It is possible to provide an explicit characterization of the retailers' expected profits in terms of the disruption probabilities, the number of firms in different tiers, and the coefficient of variation of output $U$. Such a characterization, which we provide in the following lemma, allows for relating the retailers' profits to the variability of the supply chain's output.

LEMMA 2. *Suppose that Assumption 1 holds. At equilibrium, the expected profits for a retailer are given by*

$$\pi(1) = s^2 \frac{c(1)}{n(1)} \frac{1}{n(1) - 1 + 1/q(1)} \left( CV(U)^2 + 1 \right) \prod_{\ell=2}^{K+1} q(\ell)^2.$$

Lemma 2 implies that among networks with the same expected output, i.e., $s\prod_{\ell=1}^{K+1} q(\ell)$, and the same modeling primitives for the retailers $(c(1), n(1), q(1))$, the latter's profits are maximized for the networks that induce the highest coefficient of variation for the output of the supply chain. Thus, in a sense, $CV(U)$ may be a useful tool for characterizing the retailers' profitability as a function of the network structure.

Furthermore, for a fixed disruption risk profile and number of firms, the coefficient of variation is also related to how "balanced" the supply chain is, i.e., how evenly distributed firms are in different tiers. For instance, if there are tiers in the network with only a few firms, then Expression (9) implies that $CV$ takes larger values. We discuss the relation between $CV$ and balanced network structures in detail in Appendix B. In that appendix, we formally define a preorder in the space of supply chain networks that captures how balanced they are (in terms of the number of firms in different tiers). We establish that more balanced networks induce a smaller coefficient of variation of the output; in addition, we show that networks with the same number of firms in all tiers are the ones that minimize the coefficient of variation (among all networks with the same total number of firms). Our results suggest that among networks with the same expected output the less balanced ones yield higher profits for the retailers.

## 4. Structural Properties of Optimal Networks

In this section, we illustrate that firms may view different networks as optimal depending on the stage of the production process they participate in. In particular, we consider the case where $n(1)$ and $n(K+1)$ firms engage in production in tiers 1 and $K+1$ respectively, and we compare the networks that maximize their respective aggregate profits. As shown in Proposition 3, when the likelihood of disruptive events is low, the profits of tier 1 and tier $K+1$ firms increase in the number of firms in each of the intermediate tiers. However, fixing the total number of firms that participate in the production process, it is not clear if the networks that yield the largest profits for firms in tiers 1 and $K+1$ are identical. This is precisely the question we ask in this section: what is the best way to organize $N - n(1) - n(K+1)$ firms in the $K-1$ intermediate stages of production so that the corresponding equilibrium profits of raw materials suppliers (firms in tier $K+1$) and/or downstream retailers are maximized? What are the structural differences between these networks?

We begin our analysis by providing a sufficient condition for Assumption 1 to hold for any network and vector of cost coefficients. In turn, restricting attention to settings where this condition holds, allows us to leverage Corollary 1 when we compare firms' profits for different network structures.

Lemma 3. *Assumption 1 holds for any vector of positive cost coefficients* $\{c(k)\}_{k=1}^{K+1}$ *and any network structure* $\{n(k)\}_{k=1}^{K+1}$ *as long as we have*

$$\alpha\left(\prod_{\ell=1}^{K+1} q(\ell) - \prod_{\ell=1}^{K+1} q(\ell)^2\right) < p_c < \alpha \prod_{\ell=1}^{K+1} q(\ell). \tag{10}$$

In the remainder of this section we assume that the modeling primitives satisfy (10) and consider the set of all supply chains with $n(1)$ retailers, $n(k+1)$ raw materials suppliers, and $N$ firms in total over $K+1$ tiers. In general, there may exist multiple networks that maximize the retailers' and the raw materials suppliers' expected profits. We denote the set of networks that maximize the retailers' expected profits by $\mathcal{V}_R$ and, similarly, we denote by $\mathcal{V}_S$ the set of networks that maximize the expected profits of suppliers in tier $K+1$. In order to obtain sharper structural insights and a clear comparison of the optimal networks $\mathcal{V}_R$ and $\mathcal{V}_S$, in this section we assume that all tiers have the same disruption probability, i.e., we let $q(k) = q$ for all $k$.

First, we establish that the set of networks that maximize the aggregate supply $s$ of raw materials, hereafter $\mathcal{V}_{supply}$, coincide with the set of networks that maximize the raw materials suppliers' profits.

Proposition 4. *Any network that maximizes the raw materials suppliers' profits also maximizes the aggregate supply* $s$ *of raw materials, i.e.,* $\mathcal{V}_S = \mathcal{V}_{supply}$.

Networks for which aggregate supply is maximized coincide with those to which $Tier(K)$ pays the highest (expected) unit price for procuring its inputs. This can be seen by noting that due to the large supply of inputs, the (marginal) production cost at $Tier(K+1)$ at equilibrium is maximized for those networks. Since at equilibrium production levels, firms' (expected) marginal profits are zero, it follows that for such networks the firms in the top tier charge the largest unit price and receive the largest revenues (in expectation). Since the cost function in the top tier is quadratic (and it is the same regardless of the network structure), this also implies that the corresponding profits are also the largest for the aforementioned networks.

We next turn our attention to characterizing the optimal network structure for retailers. The characterization in the absence of any disruption risk is straightforward: in this case, the network structures that maximize the raw materials suppliers' profits (as well as aggregate supply $s$) also maximize the retailers' profits.

Proposition 5. *When there is no disruption risk, networks that maximize the aggregate supply of raw materials, also maximize the raw materials suppliers' and the retailers' profits, i.e.,*

$$\text{if } q = 1, \text{ then } \mathcal{V}_{supply} = \mathcal{V}_S = \mathcal{V}_R.$$

*Moreover, if the production cost coefficients are the same in all tiers, i.e.,* $c(k) = c$ *for all* $k$*, then every network in* $\mathcal{V}_{supply}$ *has a "box" structure; i.e., the number of firms in any two tiers* $k_1, k_2 \neq 1, K+1$ *differ by at most one.*

Proposition 5 implies that when there is no disruption risk, the raw materials suppliers' and the retailers' incentives regarding the structure of the supply chain are aligned. Figure 3 provides an illustration of optimal networks in this case. As the proposition suggests, they have a "box" structure; i.e., firms are evenly distributed in the intermediate stages of the production process.

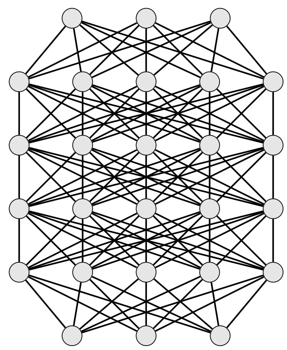

**Figure 3 The network structure that maximizes the aggregate supply of raw materials, the raw materials suppliers' profits, and the downstream retailers' profits when there is no disruption risk and all tiers have the same cost structure (for a network with $N = 26$ firms with 3 retailers and 3 raw materials suppliers).**

The situation is quite different in the presence of disruption risk. In this case, the optimal networks may not coincide.

Theorem 2. *Let the probability of a disruptive event be equal to* $q < 1$ *for all tiers and consider networks* $\mathcal{N}_R \in \mathcal{V}_R$ *and* $\mathcal{N}_S \in \mathcal{V}_S$*. Denote by* $s_{\mathcal{N}_R}$ *and* $s_{\mathcal{N}_S}$ *the equilibrium supply of raw materials induced by* $\mathcal{N}_R$ *and* $\mathcal{N}_S$*, respectively. Similarly, denote by* $U_{\mathcal{N}_R}$ *and* $U_{\mathcal{N}_S}$ *the output of networks* $\mathcal{N}_R$ *and* $\mathcal{N}_S$*, respectively.*

*(i) We have* $s_{\mathcal{N}_R} \leq s_{\mathcal{N}_S}$ *and* $CV(U_{\mathcal{N}_S}) \leq CV(U_{\mathcal{N}_R})$*.*

*(ii) In addition, if the cost coefficients for all tiers are equal, i.e.,* $c(k) = c$ *for all* $k$*, both optimal networks* $\mathcal{N}_S$ *and* $\mathcal{N}_R$ *take the form of an inverted pyramid, i.e., the number of firms in any two tiers* $k_1, k_2$ *with* $1 < k_1 < k_2 < K+1$ *satisfy* $n(k_2) \geq n(k_1)$*.*

The inequalities provided in Theorem 2(i) are in general strict; i.e., retailer-optimal networks may induce a strictly lower aggregate supply of raw materials and a higher coefficient of variation for the output than the networks that maximize profits for the raw materials suppliers, even when different tiers have the same cost structure.

Proposition 4 implies that networks that maximize the raw materials suppliers' profits also maximize the total expected output that reaches the downstream market. This is generally not the case for networks that maximize the retailers' profits: retailers not only benefit from a large expected output in the downstream market but they also benefit from variability (and, hence, prefer a relatively larger coefficient of variation).

The last observation can be seen from Lemma 2. Also, it can be perhaps best understood by noting that the profits of a retailer are convex (quadratic) in the realization of the output of tier 2 (which follows by Expression (2) and the equilibrium market clearing conditions, i.e., conditions (ii) and (iii) in Definition 1). In turn, this convexity implies that between two networks that generate the same expected supply for the retailers, the one that leads to higher supply variance generates higher profits for them. Such networks also yield higher output variability for the supply chain as a whole, implying that for the same expected output level, higher output variance is associated with higher profits for retailers.

As we discussed earlier, the coefficient of variation of the quantity that is sold to the downstream market is related to how evenly distributed firms are in different tiers of the network. Thus, Theorem 2 suggests that networks that maximize the retailers' profits are structurally different from those that maximize the raw materials suppliers' profits. Proposition 6 below builds on this observation and establishes that networks that maximize the retailers' profits have less-diversified downstream tiers compared to those that maximize the raw materials suppliers' profits.

Proposition 6. *Let the probability of a disruptive event be equal to $q<1$ for all tiers, and suppose that the production cost coefficients are the same for all tiers. Consider networks $\mathcal{N}_R \in \mathcal{V}_R$ and $\mathcal{N}_S \in \mathcal{V}_S$, and let $n_R(k)$ and $n_S(k)$ respectively denote the number of firms in tier $k$ of these networks. Then, there exists a tier $k>1$ such that $n_R(\ell) \leq n_S(\ell)$ for every $\ell \leq k$ and if $\mathcal{N}_R \neq \mathcal{N}_S$ at least one inequality is strict.*

Finally, if we further restrict attention to the case of four-tier networks, i.e., networks that consist of raw materials suppliers, retailers, and two intermediate tiers of production, we can complete the intuition developed in Proposition 6 by showing that retailers prefer an over-diversified tier 3 (upstream) and an under-diversified tier 2 (downstream) relative to the networks that maximize the raw materials suppliers' profits.

Corollary 2. *Consider the setting of Proposition 6, and suppose that $K=3$. Then,*

$$n_S(3) \leq n_R(3) \text{ and } n_S(2) \geq n_R(2).$$

## 5. Endogenous Entry and Equilibrium Supply Chains

In this section, we consider the endogenous process of supply chain formation, where a set of firms decide on whether to pay a fixed entry cost and engage in production. We derive sufficient conditions for a nonempty equilibrium supply chain to exist and provide a characterization of the structure of equilibrium supply chains. In particular, we establish that when all tiers have the same cost structure (and disruption risk) equilibrium again takes the form of an inverted pyramid, i.e., the number of firms in each tier increases toward the upstream tiers of the supply chain. Moreover, we show through examples that equilibrium networks are inefficient in terms of the number of firms that engage in the production process. Throughout this section for ease of exposition we let $q(k) = q$ for all tiers $k$. We also assume that the modeling primitives, i.e.,$\alpha$, $p_c$ and $q$ satisfy (10); thus, Assumption 1 holds irrespective of the network structure.

The game of entry we study follows Corbett and Karmarkar (2001) who consider a similar problem albeit in the absence of any disruption risk. As we show below, the presence of such risk has first-order implications on the structure of equilibrium chains. For the remainder of the section, we assume that any number of firms can enter production in tier $1 \leq k \leq K+1$ (assuming that it is profitable for them to do so). Specifically, we denote by $\mathcal{M}_k$ a large set of potential entrants for tier $k \in \{1, \ldots, K+1\}$, and let $\kappa > 0$ be the fixed cost of entry, which we assume to be identical for all tiers. We say that a particular network structure constitutes an equilibrium of the entry game if (i) firms that are part of the network (and engage in the production process) make at least $\kappa$ in expected profits at the supply equilibrium associated with the network that is formed after firms decide whether to enter, and (ii) no additional firm can unilaterally enter the supply chain and make nonnegative profits in expectation (net of the entry cost).

Similar to Corbett and Karmarkar (2001), we let $\mathcal{W}_\kappa$ denote the set of all network structures where $\pi(k) \geq \kappa$ for $k \in \{1, \cdots, K+1\}$. In other words, $\mathcal{W}_\kappa$ is the set of all network structures for which firms' participation constraints are satisfied; i.e., firms' profits in expectation are larger than their entry cost $\kappa$.[16] Our first result provides sufficient conditions for the existence of an equilibrium of the entry game that results in a nonempty supply chain network, i.e., a network that has at least one firm in each tier and generates a strictly positive expected output. In addition, this result establishes that the equilibrium set admits a *maximal element* under these conditions, i.e., there exists an equilibrium network that in every tier has a (weakly) larger number of firms than any other equilibrium network.

Theorem 3. *Assume that the entry cost $\kappa$ is sufficiently low so that $\mathcal{W}_\kappa$ is nonempty. Then, there exists $\hat{q} < 1$ such that if $q \geq \hat{q}$, then a nonempty equilibrium supply chain network exists. Moreover, the set of equilibria admits a maximal element.*

[16] One can interpret $\kappa$ as the annualized fixed cost associated with a firm's participation in the production process (e.g., annualized cost of owning and maintaining production facilities). Furthermore, $\pi(k)$ can be interpreted as the firms' expected annual operating profits (ignoring the fixed costs). Then, a firm will only enter the supply chain if its fixed costs can be covered by the operating profits.

In this theorem, the assumption that $\mathcal{W}_\kappa$ is nonempty guarantees that the entry cost is not prohibitively high for firms to engage in the production process.[17]

We next provide a structural characterization of equilibrium networks and, under some assumptions, we show that the number of firms in a tier increases exponentially as we move further upstream from the end consumer market.

Proposition 7. *Let $\mathcal{N}_\mathcal{E}$ be any nonempty equilibrium network of the entry game. Then, if we let $n_\mathcal{E}$ denote the number of firms in tier $k$ in equilibrium network $\mathcal{N}_\mathcal{E}$, we have*

$$n_\mathcal{E}(k) > q n_\mathcal{E}(k+1)\sqrt{\frac{c(k)}{c(k+1)}}\sqrt{\frac{n_\mathcal{E}(k+1)-1+1/q}{n_\mathcal{E}(k+1)}}-1, \text{ and}$$
$$n_\mathcal{E}(k) < q\Big(n_\mathcal{E}(k+1)+1\Big)\sqrt{\frac{c(k)}{c(k+1)}}\sqrt{\frac{n_\mathcal{E}(k+1)-1+1/q}{n_\mathcal{E}(k+1)}}.$$

*In addition, when all tiers have the same cost structure, i.e., $c(k)=c>0$ for $1\leq k\leq K+1$, we have*

$$\Big\lfloor q\cdot n_\mathcal{E}(k+1)\Big\rfloor \leq n_\mathcal{E}(k) \leq \Big\lceil q\cdot n_\mathcal{E}(k+1)\Big\rceil.$$

Thus, relatively more firms enter the upstream stages of the production process and this effect is more pronounced when the disruption probability is higher (when $q$ is lower). It is worthwhile to note that in the absence of disruptive events ($q=1$), provided that all tiers have the same positive cost coefficient $c$, the resulting equilibrium networks again take the form of a box; i.e., the number of firms in each tier is the same.

We close this section by focusing on the maximal equilibrium network and investigating its efficiency properties. We say that a network is efficient if it maximizes aggregate welfare, i.e., the sum of aggregate profits of all firms (net of entry costs) and consumer surplus over all networks (possibly with a different number of firms). Table 1 illustrates that the number of firms that engage in the production process at equilibrium can be lower than the number of firms in the efficient network for different values of the entry cost (the values in Table 1 were obtained numerically through exhaustive search). Interestingly, in competition models that involve costly entry, equilibrium behavior typically leads to entry by an inefficiently large number of firms (e.g., Mankiw and Whinston (1986); see also Tirole (1988), pp. 299-300 for a related discussion). This relies on the fact that a new entrant imposes a negative externality on the rest of the firms, which, in the presence of an entry cost, tends to drive efficiency down. However, this intuition is incomplete when firms operate in the presence of disruption risk. Although a new entrant imposes a negative

[17] Observe that in general given a network in $\mathcal{W}_\kappa$ additional firms may find it optimal to enter to some tier $k$. Moreover, such entry may result in a network structure that is no longer in $\mathcal{W}_\kappa$. In the proof of the theorem we establish equilibrium existence by showing that this never is the case when $q$ is large enough.

| Entry Cost $\kappa$ | Firms at Equilibrium | Firms in Efficient Network | Efficiency Loss at Equilibrium |
|---|---|---|---|
| 2 | 33 | 53 | 6.8% |
| 2.5 | 26 | 46 | 10.3% |
| 3 | 25 | 41 | 8.3% |
| 3.5 | 20 | 37 | 14.5% |
| 4 | 16 | 35 | 20.1% |

**Table 1 We compare the number of firms that enter the supply chain at equilibrium with the number of firms in the efficient network as a function of the fixed cost of entry $\kappa$. As the table illustrates, equilibrium entry can be inefficiently low. In addition, we report the percentage loss in aggregate welfare in the equilibrium network compared to the network that maximizes aggregate welfare for the same entry cost. Here, there are 5 tiers, the cost coefficient in all tiers is $c = 0.1$, the probability of successful production is $q = 0.8$, $p_c = 50$, and $\alpha = 200$, $\beta = 2$. It can be readily verified that for these parameters Expression (10), and hence Assumption 1, hold.**

externality on the firms in the tier it enters, it may impose a positive externality on the rest of the supply chain that may outweigh any negative effects.

Table 1 reports the efficiency loss at equilibrium compared to the networks that maximize aggregate welfare for different values of the entry cost. As can be seen from the table, the inefficiency induced by endogenous entry can be substantial. This finding highlights the potential for welfare-increasing interventions, e.g., in the form of subsidies, favorable contract terms, or low-interest loans, that may incentivize entry in given tiers of the chain.

## 6. Concluding Remarks

This paper develops a model for the study of disruption risk in multi-tier supply chains. Production consists of $K + 1$ sequential tiers and in each tier multiple firms may compete with one another for the supply of an intermediate good. Our objective is to derive insights into the way network structure affects the output, prices, and expected profits of firms in the supply chain.

Using our equilibrium characterization, we establish that firms' profits may vary nonmonotonically in the reliability of production in different tiers. Moreover, we establish that the raw materials suppliers' and the retailers' incentives regarding the structure of the supply chain are typically misaligned. In particular, the retailers' profits are maximized for networks that are less diversified in downstream tiers than networks that maximize the suppliers' profits. Finally, we consider the process of endogenous supply chain formation, provide a characterization of equilibrium chains, and illustrate that equilibrium behavior may lead fewer firms to engage in production than what would be optimal for aggregate welfare.

We believe that our model and analysis could serve as a starting point for future work aimed at exploring risk management in complex supply networks. First, one of the main features in the model is that prices for the realized output of a tier are set so that the corresponding market clears. Although this might serve as a useful benchmark, it is a simplification of how many real-world supply chains operate. Incorporating the possibility of firms that enter into contractual agreements before supply uncertainty is realized, and renegotiate them ex-post depending on supply availability, is a promising direction to consider. Second, firms typically follow a number of operational strategies to mitigate the adverse effects of disruptive events. Evaluating the effectiveness of such strategies, e.g., holding excess inventory, and how they relate to a firm's position in the supply network could yield interesting prescriptive insights. Relatedly, our model and results may be useful for assessing the benefits for a firm of purchasing business interruption insurance, especially given that a firm's operations rely critically on the disruption and sourcing profiles of its direct and indirect suppliers.[18] Finally, we make a number of assumptions to make the analysis tractable and isolate the effects we aim to study. In particular, we assume that firms are symmetric and the outputs of firms in the same tier are perfectly substitutable. Moreover, we consider "all-or-nothing" disruptive events. Extending the model to incorporate asymmetries among firms and allowing for more general production functions and disruptive events are also quite promising (but potentially challenging) directions for future research.

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[18] Conversely, the model could be used to price such insurance instruments. Dong and Tomlin (2012) discuss the interplay between operational measures and insurance for managing a firm's exposure to disruption risk in a model that involves a single focal firm.

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# Online Appendix

## Supply Disruptions and Optimal Network Structures

### Appendix A: Proofs

### Proof of Theorem 1

The proof proceeds in three parts that (1) establish the essential uniqueness of the equilibrium under Assumption 1(i); (2) provide a characterization of equilibrium prices and procurement quantities when both Assumptions 1(i) and (ii) hold; and (3) show that if Assumption 1(ii) is violated, then there exist realizations of the disruptions for which the realized price for the output of at least one of the $K+1$ tiers is equal to zero, respectively.

The proof relies on formulating a convex optimization problem (given in (11)) and showing that its (unique) solution along with the solution to the corresponding dual problem can be used to construct a supply equilibrium. In particular, in establishing all three parts that follow, we use the problem's KKT conditions (that are necessary and sufficient for optimality).

We start by introducing some notation, stating the aforementioned optimization problem, and establishing two auxiliary lemmas. Then, we proceed to the proof of claims (1)-(3) above.

To simplify the exposition, we let $\eta_k(\omega) \triangleq \mathbb{P}(\hat{\omega}_k = \omega)$ denote the probability that state $\omega$ is realized after tiers $k+1, \cdots, K+1$ complete production. Note that $\eta(\omega_k) > 0$ for all $\omega_k \in \Omega_k$, since by definition $\Omega_k$ consists of states that are realized with nonzero probability. Also, given a state $\omega_k$, with some abuse of notation, we let $\omega_{k,j}$ denote whether firm $j \in Tier(k+1)$ experiences a disruptive event in state $\omega_k$ (i.e., $\omega_{k,j}$ is equal to the binary variable $z_{k+1,j}$ associated with $\omega_k = \{\omega_{k+1}, \{z_{k+1,i}\}_i\}$). In addition, we use notation $\omega_{k+1} \to \omega_k$ when state $\omega_k$ can be realized with positive probability assuming that the state describing the realizations of disruptions in tiers $k+2, \cdots, K+1$ is $\omega_{k+1}$. Finally, we let $\mathcal{S}$ denote the set of valid states.

We consider the following optimization problem:

$$
\begin{aligned}
\max \sum_{\omega_0 \in \Omega_0} \eta_0(\omega_0)\Big(\alpha x_0(\omega_0) - \frac{\beta}{2} x_0(\omega_0)^2\Big) - p_c \sum_{i \in Tier(K+1)} x_{K+1,i}(\omega_{K+1}) - \sum_{k=1}^{K+1} \sum_{i \in Tier(k)} \sum_{\omega_k \in \Omega_k} \eta_k(\omega_k) c(k) x_{k,i}(\omega_k)^2 \\
s.t. \sum_{i \in Tier(k)} x_{k,i}(\omega_k) \leq \sum_{j \in Tier(k+1) | \omega_{k+1} \to \omega_k, \omega_{k,j}=1} x_{k+1,j}(\omega_{k+1}), \qquad \text{for } 1 \leq k \leq K \text{ and } \forall \omega_k \in \Omega_k, \\
x_0(\omega_0) \leq \sum_{j \in Tier(1) | \omega_1 \to \omega_0, \omega_{0,j}=1} x_{1,j}(\omega_1), \qquad \forall \omega_0 \in \Omega_0, \\
x_0(\omega_0) \geq 0, \qquad x_{k,i}(\omega_k) \geq 0, \qquad \text{for } 1 \leq k \leq K+1, \text{ and } \forall i \in Tier(k), \forall \omega_0 \in \Omega_0, \forall \omega_k \in \Omega_k.
\end{aligned}
\tag{11}
$$

The following lemma summarizes a number of properties of this optimization problem that we use in our subsequent analysis.

Lemma 4. *Suppose Assumption 1(i) holds.*

*(a) Optimization problem* (11) *admits a unique optimal solution, which we denote by* $\mathbf{x}^\star$*. Moreover, this solution satisfies* $x^\star_{k,i}(\omega_k) = x^\star_{k,j}(\omega_k)$ *for all* $k \in \{1,\ldots K+1\}$ *and* $i,j \in Tier(k)$.

*(b) The nonnegativity constraints are not binding in the optimal solution for the decision variables that correspond to valid states, i.e.,* $x^\star_{k,i}(\omega_k) > 0$ *and* $x^\star_0(\omega_0) > 0$ *for* $\omega_k, \omega_0 \in \mathcal{S}$ *at the optimal solution.*

*Proof:*

*(a)* Note that the objective function is additively separable over the decision variables, i.e., $\{x_{k,i}(\omega_k)\}$ for $1 \le k \le K+1$, $i \in Tier(k)$, and $\omega_k \in \Omega_k$, and $x_0(\omega_0)$ for $\omega_0 \in \Omega_0$. Moreover, it is strictly concave in all the decision variables. Since the feasible set is convex, these observations imply that there is a unique optimal solution.[19]

We proceed by establishing that in the optimal solution

$$x^\star_{k,i}(\omega_k) = x^\star_{k,j}(\omega_k), \tag{12}$$

for all $i,j \in Tier(k)$, $\omega_k \in \Omega_k$, and $1 \le k \le K+1$. Suppose that this is not the case and $x^\star_{k,i}(\omega_k) > x^\star_{k,j}(\omega_k)$ for some $i,j \in Tier(k)$ and $\omega_k \in \Omega_k$. Then, another feasible solution $\mathbf{x}'$ with the same value for the objective function can be constructed by setting $x'_{k,i}(\omega_k) = x^\star_{k,j}(\omega_k)$ and $x'_{k,j}(\omega_k) = x^\star_{k,i}(\omega_k)$, and $\mathbf{x}'$ equal to the original solution $\mathbf{x}^\star$ for the remaining decision variables. This solution is also optimal; thus, we obtain a contradiction to its uniqueness.

*(b)* Next, we show that the nonnegativity constraints are not binding in the optimal solution for the decision variables that correspond to valid states. By way of contradiction, assume that this is not the case and consider the most upstream tier $k \in \{1,\cdots,K+1\}$ such that in an optimal solution $x^\star_{k,i}(\omega_k) = 0$ for some $i \in Tier(k)$, and valid state $\omega_k \in \Omega_k$. Then, by Expression (12), one of the following two cases should hold:

- $k < K+1$, i.e., $x^\star_{k,i}(\omega_k) = 0$ for all $i \in Tier(k)$ and $x^\star_{k+1,j}(\omega_{k+1}) > 0$ for any $j \in Tier(k+1)$ where $\omega_{k+1} \to \omega_k$; or
- $k = K+1$, i.e., $x^\star_{K+1,i}(\omega_{K+1}) = 0$ for all $i \in Tier(K+1)$.

Consider any valid states $\omega'_{k-1} \in \Omega_{k-1}, \ldots, \omega'_0 \in \Omega_0$ that can be reached (with nonzero probability) after $\omega_k$ is realized, i.e., $\omega_k \to \omega'_{k-1} \to \cdots \to \omega'_0$. Note that by feasibility in optimization problem (11), we have $x^\star_{\ell,i'}(\omega'_\ell) = 0 = x^\star_0(\omega'_0)$ for all $i' \in Tier(\ell)$ and $1 \le \ell < k$. Next, for each tier $\ell \le k$, fix a firm $j(\ell) \in Tier(\ell)$. Note that another feasible solution $\mathbf{x}'$ can be constructed by setting:

(i) $x'_{k,j(k)}(\omega_k) = \epsilon$,

[19] It can be readily seen that when the decision variables take large enough values, the objective becomes negative. Thus, it is sufficient to restrict attention to a bounded interval for the vector of decision variables $\mathbf{x}$. Thus, the existence of an optimal solution follows from the Weierstrass theorem.

(ii) $x'_{\ell,j(\ell)}(\omega'_\ell)=\epsilon$ for $1\le \ell<k$ and every state $\omega'_\ell$ such that firms $\{j(m)\}_{m=\ell+1}^k$ do not experience a disruption,

(iii) $x'_0(\omega'_0)=\epsilon$ for every state $\omega'_0$ such that firms $\{j(m)\}_{m=1}^k$ do not experience a disruption,

while leaving the remaining decision variables unchanged.

By Assumption 1(i), we have $\alpha\prod_{\ell=1}^{K+1}q(\ell)=\alpha(K+2)>p_c$. Hence, it can be seen that this feasible solution yields a strictly higher value for the objective function for sufficiently small $\epsilon$, thereby leading to a contradiction to $x^\star_{k,i}(\omega_k)=0$ for some $1<k\le K+1$, $i\in Tier(k)$, and valid state $\omega_k\in\Omega_k$ at the optimal solution of (11). Similarly, if all $x^\star_{k,i}(\omega_k)>0$ for valid $\omega_k\in\Omega_k$, and $x^\star_0(\omega_0)=0$ for some valid $\omega_0$, it can be readily seen that one can construct another solution $\mathbf{x}'$ with a strictly higher value for the objective function by setting $x'_0(\omega_0)=\epsilon$ for a sufficiently small $\epsilon>0$. Thus, we conclude that in the optimal solution of optimization problem (11), for valid states the nonnegativity constraints are not binding. □

Introducing Lagrange multipliers $\{\lambda_k(\omega_k)\}$ and $\{\lambda_0(\omega_0)\}$ for the first two sets of constraints, respectively, and $\{\mu_0(\omega_0)\}$ and $\{\mu_{k,i}(\omega_k)\}$ for the nonnegativity constraints, the necessary and sufficient (given that the problem's constraints are linear) KKT optimality conditions corresponding to optimization problem (11) can be written as follows:

$$-\eta_0(\omega_0)(\alpha-\beta x^\star_0(\omega_0))+\lambda_0(\omega_0)-\mu_0(\omega_0)=0, \tag{13a}$$

$$p_c+2c(K+1)x^\star_{K+1,i}(\omega_{K+1})-\sum_{\omega_K|\omega_{K+1}\to\omega_K,\omega_{K,i}=1}\lambda_K(\omega_K)-\mu_{K+1,i}(\omega_{K+1})=0, \tag{13b}$$

$$2c(k)\eta_k(\omega_k)x^\star_{k,i}(\omega_k)+\lambda_k(\omega_k)-\sum_{\omega_{k-1}|\omega_k\to\omega_{k-1},\omega_{k-1,i}=1}\lambda_{k-1}(\omega_{k-1})-\mu_{k,i}(\omega_k)=0, \tag{13c}$$

$$\lambda_k(\omega_k)\ge 0 \quad\perp\quad \sum_{i\in Tier(k)}x^\star_{k,i}(\omega_k)\le\sum_{j\in Tier(k+1)|\omega_{k+1}\to\omega_k,\omega_{k,j}=1}x^\star_{k+1,j}(\omega_{k+1}), \tag{13d}$$

$$\lambda_0(\omega_0)\ge 0 \quad\perp\quad x^\star_0(\omega_0)\le\sum_{j\in Tier(1)|\omega_1\to\omega_0,\omega_{0,j}=1}x^\star_{1,j}(\omega_1), \tag{13e}$$

$$\mu_{k,i}(\omega_k)\ge 0 \quad\perp\quad x^\star_{k,i}(\omega_k)\ge 0,\ \ \mu_0(\omega_0)\ge 0 \quad\perp\quad x^\star_0(\omega_0)\ge 0. \tag{13f}$$

Our next lemma provides a characterization of the optimal dual multipliers and relates the solution of the above system to the supply equilibrium.

Lemma 5. *Suppose Assumption 1(i) holds.*

*(a) The dual multipliers $\{\lambda_k(\omega_k)\}$ and $\{\lambda_0(\omega_0)\}$ corresponding to valid states and satisfying conditions (13a)–(13f) are unique. Moreover, $\mu_0(\omega_0)=\mu_{k,i}(\omega_k)=0$ for valid states $\omega_0\in\Omega_0\cap\mathcal{S},\omega_k\in\Omega_k\cap\mathcal{S}$, where $k\in\{1,\dots,K+1\}$ and $i\in Tier(k)$.*

*(b) Let $\{x^\star_{k,i}(\omega_k)\}\cup\{x^\star_0(\omega_0)\}$ and $\{\lambda_k(\omega_k)\}\cup\{\lambda_0(\omega_0)\}\cup\{\mu_k(\omega_k)\}\cup\{\mu_0(\omega_0)\}$ denote a solution of (13a)–(13f), and define $p_k(\omega_{k-1})=\lambda_{k-1}(\omega_{k-1})/\eta_{k-1}(\omega_{k-1})$. Then $\{p_k(\omega_{k-1}),x^\star_{k,i}(\omega_k)\}$ is a supply equilibrium.*

*(c) Conversely, suppose that* $\{p_k(\omega_{k-1}), x^\star_{k,i}(\omega_k)\}$ *is a supply equilibrium. Define*

$$\lambda_k(\omega_k) = p_{k+1}(\omega_k)\eta_k(\omega_k), \tag{14}$$

*and* $x^\star_0(\omega_0) = D(p_1(\omega_0))$*. Also, let* $\mu_{k,i}(\omega_k) = 0$ *if* $x^\star_{k,i}(\omega_k) > 0$ *for* $1 \le k \le K+1$*, and,* $\mu_0(\omega_0) = 0$ *if* $x^\star_0(\omega_0) > 0$*. Otherwise, if* $x^\star_{k,i}(\omega_k) = 0$*, let*

$$\mu_{k,i}(\omega_k) = \lambda_k(\omega_k) - \sum_{\omega_{k-1}|\omega_k \to \omega_{k-1}, \omega_{k-1,i}=1} \lambda_{k-1}(\omega_{k-1}), \tag{15}$$

*when* $1 \le k \le K$ *and* $\mu_{k,i}(\omega_k) = p_c - \sum_{\omega_{k-1}|\omega_k \to \omega_{k-1}, \omega_{k-1,i}=1} \lambda_{k-1}(\omega_{k-1})$ *when* $k = K+1$*. Similarly, let* $\mu_0(\omega_0) = \lambda_0(\omega_0) - \eta_0(\omega_0)\alpha$*, if* $x^\star_0(\omega_0) = 0$*.*

*Then,* $\{x^\star_{k,i}(\omega_k)\} \cup \{x^\star_0(\omega_0)\}$ *and* $\{\lambda_k(\omega_k)\} \cup \{\lambda_0(\omega_0)\} \cup \{\mu_k(\omega_k)\} \cup \{\mu_0(\omega_0)\}$ *is a solution of (13a)–(13f).*

*Proof:*

*(a)* By Lemma 4, optimization problem (11) admits a unique optimal solution $\{x^\star_{k,i}(\omega_k)\} \cup \{x^\star_0(\omega_0)\}$. Lemma 4 also implies that $x^\star_0(\omega_0) > 0$ for $\omega_0 \in \Omega_0 \cap \mathcal{S}$ and $x^\star_{k,i}(\omega_k) > 0$ for $i \in Tier(k)$, $\omega_k \in \Omega_k \cap \mathcal{S}$. Thus, by (13f), we have $\mu_0(\omega_0) = \mu_k(\omega_k) = 0$. Thus, given $x^\star_0(\omega_0)$, condition (13a) implies that $\lambda_0(\omega_0)$ is uniquely defined for $\omega_0 \in \Omega_0 \cap \mathcal{S}$. Suppose that $\{\lambda_\ell(\omega_\ell)\}_{\ell<k, \omega_\ell \in \Omega_\ell \cap \mathcal{S}}$ are uniquely defined for some $k$. Since $\mu_k(\omega_k) = 0$ for $\omega_k \in \Omega_k \cap \mathcal{S}$, given $x^\star_{k,i}(\omega_k)$ and $\{\lambda_\ell(\omega_\ell)\}_{\ell<k, \omega_\ell \in \Omega_\ell \cap \mathcal{S}}$, by conditions (13b) and (13c), dual multiplier $\lambda_k(\omega_k)$ is also uniquely defined for $\omega_k \in \Omega_k \cap \mathcal{S}$. Proceeding inductively, we conclude that dual variables $\{\lambda_k(\omega_k)\}$ and $\{\lambda_0(\omega_0)\}$ corresponding to valid states and satisfying KKT conditions (13a)–(13f) are uniquely defined.

*(b)* First consider $x^\star_0(\omega_0)$. Note that if $x^\star_0(\omega_0) > 0$, then by (13f) and (13a) we have $\mu_0(\omega_0) = 0$ and, in turn, $p_1(\omega_0) = (\alpha - \beta x^\star_0(\omega_0))$. Hence,

$$D(p_1(\omega_0)) = x^\star_0(\omega_0) \text{ for } \omega_0 \in \mathcal{S} \text{ and } x^\star_0(\omega_0) > 0. \tag{16}$$

On the other hand, if $x^\star_0(\omega_0) = 0$, by (13a), we obtain directly that $p_1(\omega_0) \ge \alpha$. Hence,

$$D(p_1(\omega_0)) = 0 = x^\star_0(\omega_0). \tag{17}$$

Expressions (16) and (17) together with KKT condition (13e) imply condition (iii) from the definition of a supply equilibrium (Definition 1). In addition, KKT condition (13d) readily implies condition (ii) from the definition of a supply equilibrium.

Finally, we establish that the combination of KKT conditions (13f), (13b), and (13c) imply condition (i) from the definition of a supply equilibrium. To see this, recall that the expected profit of firm $i \in Tier(k)$ when the realized state is $\omega_k$ after firms in tiers $k+1, \cdots, K+1$ complete production and firm $i$ procures $y_{k,i}$ can be written as follows

$$\begin{aligned}\bar{\pi}(i, \omega_k, y_{k,i}) &= \sum_{\omega_{k-1}|\omega_k \to \omega_{k-1}, \omega_{k-1,i}=1} \eta_{k-1}(\omega_{k-1}|\omega_k) p_k(\omega_{k-1}) y_{k,i} - p_{k+1}(\omega_k) y_{k,i} - c(k) y_{k,i}^2 \\ &= \sum_{\omega_{k-1}|\omega_k \to \omega_{k-1}, \omega_{k-1,i}=1} \frac{\eta_{k-1}(\omega_{k-1})}{\eta_k(\omega_k)} p_k(\omega_{k-1}) y_{k,i} - p_{k+1}(\omega_k) y_{k,i} - c(k) y_{k,i}^2,\end{aligned}$$

where $\eta_{k-1}(\omega_{k-1}|\omega_k) = \frac{\eta_{k-1}(\omega_{k-1})}{\eta_k(\omega_k)}$ is the probability state $\omega_{k-1}$ is realized, conditional on state $\omega_k$ being realized.

For $k < K+1$, by taking the derivative with respect to $y_{k,i}$ and substituting $p_{k+1}(\omega_k) = \lambda_k(\omega_k)/\eta_k(\omega_k)$ and $p_k(\omega_{k-1}) = \lambda_{k-1}(\omega_{k-1})/\eta_{k-1}(\omega_{k-1})$, we obtain

$$\frac{\partial \bar{\pi}(i, \omega_k, y_{k,i})}{\partial y_{k,i}}\bigg|_{y_{k,i}=x^\star_{k,i}(\omega_k)} = \sum_{\omega_{k-1}|\omega_k \to \omega_{k-1}, \omega_{k-1,i}=1} \frac{\lambda_{k-1}(\omega_{k-1})}{\eta_k(\omega_k)} - \frac{\lambda_k(\omega_k)}{\eta_k(\omega_k)} - 2c(k) x^\star_{k,i}(\omega_k) = -\frac{\mu_{k,i}(\omega_k)}{\eta_k(\omega_k)}, \tag{18}$$

where the last equality follows from (13c). Note that this equality implies that

$$x^\star_{k,i}(\omega_k) \in \arg\max_{y_{k,i} \geq 0} \bar{\pi}(i, \omega_k, y_{k,i}),$$

for all $k \in \{1, \cdots, K+1\}$. This is because if $x^\star_{k,i}(\omega_k) > 0$, then by (13f) we have $\mu_{k,i}(\omega_k) = 0$, and (18) implies that $x^\star_{k,i}(\omega_k)$ satisfies the first order optimality conditions. On the other hand, if $x^\star_{k,i}(\omega_k) = 0$, then from (18) we obtain that $\frac{\partial \bar{\pi}(i,\omega_k,y_{k,i})}{\partial y_{k,i}}\big|_{y_{k,i}=x^\star_{k,i}(\omega_k)} \leq 0$, which readily implies that $x^\star_{k,i}(\omega_k) \in \arg\max_{y_{k,i}\geq 0} \bar{\pi}(i, \omega_k, y_{k,i})$.

Following a similar approach but using $p_{K+2}(\omega_{K+1}) = p_c$ (and the fact that $\eta(\omega_{K+1}) = 1$), the derivative of the expected profit of a firm $i \in Tier(K+1)$ is given by:

$$\begin{aligned}\frac{\partial \bar{\pi}(i, \omega_{K+1}, y_{K+1,i})}{\partial y_{K+1,i}}\bigg|_{y_{K+1,i}=x^\star_{K+1,i}(\omega_{K+1})} &= \sum_{\omega_K|\omega_{K+1} \to \omega_K, \omega_{K,i}=1} \lambda_K(\omega_K) - p_c - 2c(K+1) x^\star_{K+1,i}(\omega_{K+1}) \\ &= -\mu_{K+1,i}(\omega_{K+1}),\end{aligned} \tag{19}$$

where the last equality follows from (13b). Once again this implies that $x^\star_{K+1,i}(\omega_{K+1}) \in \arg\max_{y_{K+1,i}\geq 0} \bar{\pi}(i, \omega_{K+1}, y_{K+1,i})$. Hence, the tuple $\{p_k(\omega_{k-1}), x^\star_{k,i}(\omega_k)\}$ also satisfies condition (i) of Definition 1; thus, it constitutes a supply equilibrium.

*(c)* First, note that $\mu_{k,i}(\omega_k)$ and $\mu_0(\omega_0)$ are nonnegative. This is immediate when $x^\star_{k,i}(\omega_k) > 0$ and $x^\star_0(\omega_0) > 0$. If, on the other hand, $x^\star_{k,i}(\omega_k) = 0$, condition (i) of a supply equilibrium in Definition 1 implies that

$$\mathbb{E}\left[p_k(\hat{\omega}_{k-1}) Z_{k,i} - p_{k+1}(\hat{\omega}_k) \,\middle|\, \hat{\omega}_k = \omega_k\right] \leq 0.$$

Using $\lambda_k(\omega_k) = p_{k+1}(\omega_k)\eta_k(\omega_k)$ from (14), we have

$$\begin{aligned}\mathbb{E}\left[p_k(\hat{\omega}_{k-1})Z_{k,i} - p_{k+1}(\hat{\omega}_k)\middle|\hat{\omega}_k = \omega_k\right] &= \sum_{\omega_{k-1}|\omega_k \to \omega_{k-1}, \omega_{k-1,i}=1} \frac{\lambda_{k-1}(\omega_{k-1})}{\eta_{k-1}(\omega_{k-1})}\frac{\eta_{k-1}(\omega_{k-1})}{\eta_k(\omega_k)} - \frac{\lambda_k(\omega_k)}{\eta_k(\omega_k)} \\ &= \frac{1}{\eta_k(\omega_k)}\left(\sum_{\omega_{k-1}|\omega_k \to \omega_{k-1}, \omega_{k-1,i}=1} \lambda_{k-1}(\omega_{k-1}) - \lambda_k(\omega_k)\right) \leq 0. \end{aligned} \tag{20}$$

Combining (20) with (15) implies that $\mu_{k,i}(\omega_k) \geq 0$ for all $\omega_k$ and $1 \leq k \leq K$. A similar argument readily applies to establish that $\mu_{K+1,i}(\omega_{K+1}) \geq 0$. Furthermore, if $x_0^\star(\omega_0) = D(p_1(\omega_0)) = 0$, then $p_1(\omega_0) \geq \alpha$. In turn, this implies

$$\mu_0(\omega_0) = \lambda_0(\omega_0) - \eta_0(\omega_0)\alpha = \eta_0(\omega_0)(p_1(\omega_0) - \alpha) \geq 0.$$

In addition to being nonnegative as we established above, by construction, variable $\mu_{k,i}(\omega_k)$ (similarly, $\mu_0(\omega_0)$) is equal to zero when $x_{k,i}^\star(\omega_k) > 0$ (similarly, $x_0^\star(\omega_0) > 0$). Thus, condition (13f) holds.

Moreover, the primal and dual variables as constructed above, satisfy conditions (13d) and (13e) given conditions (ii) and (iii) from the definition of a supply equilibrium, the nonnegativity of prices, and the fact that $x_0^\star(\omega_0) = D(p_1(\omega_0))$. In a similar fashion, note that for the given construction of variables $\mu_{k,i}(\omega_k)$ and $\mu_0(\omega_0)$, KKT conditions (13a)–(13c) are trivially satisfied when $x_{k,i}^\star(\omega_k) = 0$ (or $x_0^\star(\omega_0) = 0$). On the other hand, if $x_{k,i}^\star(\omega_k) > 0$ (or $x_0^\star(\omega_0) > 0$), condition (i) in Definition 1 implies that

$$\mathbb{E}\left[Z_{k,i}p_k(\hat{\omega}_{k-1}) - p_{k+1}(\hat{\omega}_k)\middle|\hat{\omega}_k = \omega_k\right] = 2c(k)x_{k,i}^\star(\omega_k),$$

for $1 \leq k \leq K$. Thus, (20) implies condition (13c) (a similar argument readily applies to establish that the constructed vector of dual variables satisfies also condition (13b)). Finally, when $x_0^\star(\omega_0) > 0$, condition (13a) readily follows by noting that $x_0^\star(\omega_0) = D(p_1(\omega_0))$.

Thus we conclude that $\{x_{k,i}^\star(\omega_k)\} \cup \{x_0^\star(\omega_0)\}$ together with the constructed $\{\lambda_k(\omega_k)\} \cup \{\lambda_0(\omega_0)\} \cup \{\mu_k(\omega_k)\} \cup \{\mu_0(\omega_0)\}$ is a solution of (13a)–(13f).

□

Using these auxiliary lemmas, we next complete the proof of the theorem.

**Part (1)**: *Suppose that Assumption 1(i) holds. Then, the supply equilibrium is essentially unique.*

Let $\{p_k(\omega_{k-1}), x_{k,i}(\omega_k)\}$ be a supply equilibrium, and consider the construction in Lemma 5(c). It follows by the lemma that $\{x_{k,i}(\omega_k)\} \cup \{x_0(\omega_0)\}$ and $\{\lambda_k(\omega_k)\} \cup \{\lambda_0(\omega_0)\} \cup \{\mu_k(\omega_k)\} \cup \{\mu_0(\omega_0)\}$ constitute a solution of the KKT conditions (13a)–(13f) associated with problem (11). Lemma 4(a) and Lemma 5(a) establish that the tuple of $\{x_{k,i}(\omega_k), x_0(\omega_0)\}$ and $\{\lambda_k(\omega_k), \lambda_0(\omega_0)\}$ for valid

$\omega_0, \omega_k$ that satisfy (13a)–(13f) is unique. In turn, this implies that all supply equilibria share the same $\{x_{k,i}(\omega_k)\}$. Moreover, since $\lambda_k(\omega_k) = p_{k+1}(\omega_k)\eta_k(\omega_k)$ by construction (and $\eta_k(\omega_k) > 0$), it also follows that all supply equilibria share the same $\{p_{k+1}(\omega_k)\}$ for all valid states $\{\omega_k\}$. Thus, we conclude that the supply equilibrium is essentially unique.

**Part (2)**: *Suppose that Assumptions 1(i) and (ii) hold. Then, the essentially unique equilibrium is characterized by Expressions (5), (6), and (7).*

By the expression for the procurement quantity of firm $i \in Tier(k)$ given in the statement of the theorem, i.e., (7), we have $\sum_{i\in Tier(k)} x_{k,i}(\omega_k) = s\prod_{\ell=k+1}^{K+1} \frac{n(\ell,\omega_k)}{n(\ell)}$ and

$$\sum_{i\in Tier(k+1)} z_{k+1,i}x_{k+1,i}(\omega_{k+1}) = s\frac{n(k+1,\omega_k)}{n(k+1)}\left(\prod_{\ell=k+2}^{K+1}\frac{n(\ell,\omega_{k+1})}{n(\ell)}\right) = s\prod_{\ell=k+1}^{K+1}\frac{n(\ell,\omega_k)}{n(\ell)},$$

where we make use of the fact that $\sum_{i\in Tier(k+1)} z_{k+1,i} = n(k+1,\omega_k)$, and $\omega_k = \{\omega_{k+1}, \{z_{k+1,i}\}_i\}$ and $\omega_{k+1}$ capture the same set of disruptions in tiers $\ell > k+1$, and hence $n(\ell,\omega_{k+1}) = n(\ell,\omega_k)$. These observations readily imply

$$\sum_{i\in Tier(k)} x_{k,i}(\omega_k) = \sum_{i\in Tier(k+1)} z_{k+1,i}x_{k+1,i}(\omega_{k+1}), \tag{21}$$

for $\omega_k = \{\omega_{k+1}, \{z_{k+1,i}\}\} \in \Omega_k$, and

$$D(p_1(\omega_0)) = \frac{\alpha}{\beta} - \frac{p_1(\omega_0)}{\beta} = s\prod_{\ell=1}^{K+1}\frac{n(\ell,\omega_0)}{n(\ell)} = \sum_{i\in Tier(1)} z_{1,i}x_{1,i}(\omega_1), \tag{22}$$

for $\omega_0 = \{\omega_1, \{z_{1,i}\}_i\} \in \Omega_0$. Thus, the constructed tuple satisfies conditions (ii) and (iii) defining a supply equilibrium (Definition 1). To complete the proof, it suffices to establish that the prices are nonnegative and

$$x_{k,i}(\omega_k) \in \arg\max_{y_{k,i}\geq 0} \bar{\pi}(i,\omega_k,y_{k,i}), \tag{23}$$

for all $k \in \{1,\dots,K+1\}$, $i \in Tier(k)$, and $\omega_k \in \Omega_k$.

We first establish that (23) holds. To this end, note that $\bar{\pi}(i,\omega_k,y_{k,i})$ is concave in $y_{k,i}$, as can be seen from (2). Thus, to verify (23), it suffices to check that the first order optimality conditions are satisfied by $y_{k,i} = x_{k,i}(\omega_k)$. In other words, it suffices to show that

$$0 \geq \left.\frac{\partial\bar{\pi}(i,\omega_k,y_{k,i})}{y_{k,i}}\right|_{y_{k,i}=x_{k,i}(\omega_k)} = \mathbb{E}\left[p_k(\hat{\omega}_{k-1})Z_{k,i} - p_{k+1}(\hat{\omega}_k) - 2c(k)x_{k,i}(\hat{\omega}_k)\middle|\hat{\omega}_k = \omega_k\right], \tag{24}$$

where the inequality holds with equality for $x_{k,i}(\omega_k) > 0$. Using the expression for the prices as given in (6), we have

$$\begin{aligned}\mathbb{E}\left[p_k(\hat{\omega}_{k-1})Z_{k,i}\middle|\hat{\omega}_k=\omega_k\right] &= \mathbb{E}\left[Z_{k,i}\left(\alpha(k)-\beta(k)s\prod_{\ell=k}^{K+1}\frac{n(\ell,\hat{\omega}_{k-1})}{n(\ell)}\right)\middle|\hat{\omega}_k=\omega_k\right]\\ &= \mathbb{E}\left[Z_{k,i}\left(\alpha(k)-\beta(k)s\frac{n(k,\hat{\omega}_{k-1})}{n(k)}\prod_{\ell=k+1}^{K+1}\frac{n(\ell,\hat{\omega}_k)}{n(\ell)}\right)\middle|\hat{\omega}_k=\omega_k\right] \\ &= \alpha(k)q(k)-\beta(k)sq(k)\frac{1+q(k)(n(k)-1)}{n(k)}\prod_{\ell=k+1}^{K+1}\frac{n(\ell,\omega_k)}{n(\ell)}.\end{aligned} \tag{25}$$

Here, the last equality follows since $n(k,\hat{\omega}_{k-1}) = \sum_{j\in Tier(k)} Z_{k,j}$ and

$$\mathbb{E}[Z_{k,i}n(k,\hat{\omega}_{k-1})\,|\,\hat{\omega}_k=\omega_k] = \mathbb{E}\left[Z_{k_i}+\sum_{j\in Tier(k)|j\neq i}Z_{k,i}Z_{k,j}\right] = q(k)+q(k)^2(n(k)-1). \tag{26}$$

Similarly, we have

$$\begin{aligned}\mathbb{E}\left[p_{k+1}(\hat{\omega}_k)+2c(k)x_{k,i}(\hat{\omega}_k)\middle|\hat{\omega}_k=\omega_k\right] &= p_{k+1}(\omega_k)+2c(k)x_{k,i}(\omega_k)\\ &= \alpha(k+1)-\beta(k+1)s\prod_{\ell=k+1}^{K+1}\frac{n(\ell,\omega_k)}{n(\ell)}+2c(k)\frac{s}{n(k)}\prod_{\ell=k+1}^{K+1}\frac{n(\ell,\omega_k)}{n(\ell)}.\end{aligned} \tag{27}$$

Using these two equalities and the fact that $\alpha(k)q(k) = \alpha(k+1)$, we can rewrite (24) as follows

$$0 \geq \beta(k+1)s\prod_{\ell=k+1}^{K+1}\frac{n(\ell,\omega_k)}{n(\ell)} - \beta(k)s\frac{q(k)^2(n(k)-1+\frac{1}{q(k)})}{n(k)}\prod_{\ell=k+1}^{K+1}\frac{n(\ell,\omega_k)}{n(\ell)} - 2c(k)\frac{s}{n(k)}\prod_{\ell=k+1}^{K+1}\frac{n(\ell,\omega_k)}{n(\ell)}. \tag{28}$$

Canceling out common terms and observing that $s > 0$ under Assumption 1(i) yields

$$0 \geq \beta(k+1)-\beta(k)\frac{q(k)^2(n(k)-1+1/q(k))}{n(k)}-\frac{2c(k)}{n(k)}. \tag{29}$$

Note that this inequality always holds with equality as can be seen from the definition of $\beta(k+1)$ in (4). Thus, we conclude that (24) and, hence, (23) hold.

It remains to show that the prices are nonnegative. Note that for any $\omega_k$, we have

$$p_{k+1}(\omega_k) \geq \alpha(k+1)-\beta(k+1)s = \alpha(k+1)-\beta(k+1)\frac{\alpha(K+2)-p_c}{\beta(K+2)} > 0, \tag{30}$$

where the inequality follows from Assumption 1(ii). Thus, the prices given in (6) are, in fact, strictly positive. Therefore, the essentially unique equilibrium is characterized by Expressions (5), (6), and (7) when Assumptions 1(i) and (ii) hold.

**Part (3)**: *If Assumption 1(i) holds but (ii) does not, then at any supply equilibrium, there exists at least one tier $k' \in \{1,\dots,K+1\}$, such that $p_{k'}(\bar{\omega}_{k'-1}) = 0$, where $\bar{\omega}_{k'-1} \in \Omega_{k'-1}$ is the state where no firm experiences a disruption in tiers $\{k',\dots,K+1\}$.*

First, assume that $\frac{\alpha(k)}{\beta(k)} \geq \frac{\alpha(K+2)-p_c}{\beta(K+2)}$, for $k \in \{1,\dots,K+1\}$, with equality for some $k = k'$. Consider the characterization provided by Expressions (5), (6), and (7). The proof of Part (2) implies (after changing the inequality in (30) to a weak inequality) that Expressions (5), (6), and (7) once again describe the essentially unique equilibrium. However, since $\alpha(k')/\beta(k') = (\alpha(K+2)-p_c)/\beta(K+2)$, it can be readily seen that in state $\bar{\omega}_{k'-1}$ where no firm in tiers $k \geq k'$ experiences a disruption, we have $p_{k'}(\omega_{k'}) = 0$; hence the claim follows.

Suppose instead that there exists some $k'$ such that $\frac{\alpha(k')}{\beta(k')} < \frac{\alpha(K+2)-p_c}{\beta(K+2)}$ and let $\{x^\star_{k,i}(\omega_k)\}$ for $1 \leq k \leq K+1$ and $\{x^\star_0(\omega_0)\}$ denote the unique solution of optimization problem (11). We claim that there exists some valid state $\omega_\ell \in \Omega_\ell$ for $0 \leq \ell \leq K$, for which one of the first two constraints of (11) is not binding, i.e.,

$$\sum_{i \in Tier(\ell)} x^\star_{\ell,i}(\omega_\ell) < \sum_{j \in Tier(\ell+1) | \omega_{\ell+1} \to \omega_\ell, \omega_{\ell,j}=1} x^\star_{\ell+1,j}(\omega_{\ell+1}), \quad \text{or}$$
$$x^\star_0(\omega_0) < \sum_{j \in Tier(1) | \omega_1 \to \omega_0, \omega_{0,j}=1} x^\star_{1,j}(\omega_1), \quad \text{if } \ell = 0. \tag{31}$$

By way of contradiction, suppose that the first two constraints in (11) are binding for any valid state. Recall that, as established in Lemma 4(b), the nonnegativity constraints are not binding at the optimal solution for valid states; thus, $s = \sum_{i \in Tier(K+1)} x^\star_{K+1,i}(\omega_{K+1}) > 0$. Since the first two constraints are binding in (11) for any valid state (and the optimal solution is such that $x^\star_{k,i}(\omega_k) = x^\star_{k,j}(\omega_k)$ for all $i,j \in Tier(k)$, by Lemma 4(a)), we obtain that

$$x^\star_{k,i}(\omega_k) = \frac{s}{n(k)} \prod_{\ell=k+1}^{K+1} \frac{n(\ell,\omega_k)}{n(\ell)} \text{ for } 1 \leq k \leq K+1 \quad \text{and} \quad x^\star_0(\omega_0) = s \prod_{\ell=1}^{K+1} \frac{n(\ell,\omega_0)}{n(\ell)}. \tag{32}$$

Let $\{\lambda_k(\omega_k), \lambda_0(\omega_0), \mu_{k,i}(\omega_k), \mu_0(\omega_0)\}$ denote the set of dual variables that satisfy (13a)–(13f). Observe that by (32) we have $x^\star_{k,i}(\omega_k), x^\star_0(\omega_0) > 0$ for valid states $\omega_0, \omega_k$. Hence, by (13f) we obtain $\mu_0(\omega_0) = \mu_k(\omega_k) = 0$ for valid states.

We claim that

$$\lambda_k(\omega_k) = \eta_k(\omega_k)\alpha(k+1) - \eta_k(\omega_k)\beta(k+1)s \prod_{\ell=k+1}^{K+1} \frac{n(\ell,\omega_k)}{n(\ell)}, \tag{33}$$

for any valid state $\omega_k$ and $0 \leq k \leq K$. We prove that (33) holds by induction. First, consider the case when $k = 0$ and let $\omega_0 \in \Omega_0$ be some valid state. Then, using the expression for $x^\star_0(\omega_0)$ in (32) (and recalling $\mu_0(\omega_0) = 0$) we obtain from (13a) that

$$\lambda_0(\omega_0) = \eta_0(\omega_0)\alpha - \eta_0(\omega_0)\beta x^\star_0(\omega_0) = \eta_0(\omega_0)\alpha(1) - \eta_0(\omega_0)\beta(1)s \prod_{\ell=1}^{K+1} \frac{n(\ell,\omega_0)}{n(\ell)}.$$

Suppose that (33) holds for some $k \in \{0, 1, \dots, K-1\}$ (induction hypothesis). We will establish that it holds for $k+1$, as well. Note that for a valid state $\omega_{k+1}$, we have

$$\begin{aligned}
\lambda_{k+1}(\omega_{k+1}) &= \sum_{\omega_k | \omega_{k+1} \to \omega_k, \omega_{k,i}=1} \lambda_k(\omega_k) - 2c(k+1)\eta_{k+1}(\omega_{k+1}) x^{\star}_{k+1,i}(\omega_{k+1}) \\
&= \sum_{\omega_k | \omega_{k+1} \to \omega_k, \omega_{k,i}=1} \left( \eta_k(\omega_k)\alpha(k+1) - \eta_k(\omega_k)\beta(k+1) s \prod_{\ell=k+1}^{K+1} \frac{n(\ell, \omega_k)}{n(\ell)} \right) - 2c(k+1)\eta_{k+1}(\omega_{k+1}) x^{\star}_{k+1,i}(\omega_{k+1}) \\
&= \alpha(k+1) q(k+1) \eta_{k+1}(\omega_{k+1}) - s\beta(k+1) \left( \prod_{\ell=k+2}^{K+1} \frac{n(\ell, \omega_{k+1})}{n(\ell)} \right) \left( \sum_{\omega_k | \omega_{k+1} \to \omega_k, \omega_{k,i}=1} \eta_k(\omega_k) \frac{n(k+1, \omega_k)}{n(k+1)} \right) \\
&\qquad - 2c(k+1)\eta_{k+1}(\omega_{k+1}) \frac{s}{n(k+1)} \prod_{\ell=k+2}^{K+1} \frac{n(\ell, \omega_{k+1})}{n(\ell)} \\
&= \alpha(k+1) q(k+1) \eta_{k+1}(\omega_{k+1}) - s\beta(k+1) \left( \prod_{\ell=k+2}^{K+1} \frac{n(\ell, \omega_{k+1})}{n(\ell)} \right) \eta_{k+1}(\omega_{k+1}) \left( \frac{q(k+1)(1+q(k+1)(n(k+1)-1))}{n(k+1)} \right) \\
&\qquad - 2c(k+1)\eta_{k+1}(\omega_{k+1}) \frac{s}{n(k+1)} \prod_{\ell=k+2}^{K+1} \frac{n(\ell, \omega_{k+1})}{n(\ell)} \\
&= \alpha(k+2)\eta_{k+1}(\omega_{k+1}) - s\eta_{k+1}(\omega_{k+1})\beta(k+2) \prod_{\ell=k+2}^{K+1} \frac{n(\ell, \omega_{k+1})}{n(\ell)}.
\end{aligned} \tag{34}$$

Here, the first equality follows by (13c) and the fact that $\mu_{k+1,i}(\omega_{k+1}) = 0$ for any valid state $\omega_{k+1}$. The second equality follows from the induction hypothesis. The third equality follows by using (32) and the following two observations:

(i) $\sum_{\omega_k | \omega_{k+1} \to \omega_k, \omega_{k,i}=1} \eta_k(\omega_k)$ is equal to the probability that state $\omega_{k+1}$ is realized and firm $i \in Tier(k+1)$ does not experience a disruptive event, which can alternatively be expressed by $\eta_{k+1}(\omega_{k+1}) q(k+1)$;

(ii) Also, $n(\ell, \omega_{k+1}) = n(\ell, \omega_k)$ for any $\ell \geq k+2$ and $\omega_{k+1} \to \omega_k$.

The fourth equality follows by observing that

$$\begin{aligned}
\sum_{\omega_k | \omega_{k+1} \to \omega_k, \omega_{k,i}=1} \eta_k(\omega_k) n(k+1, \omega_k) &= \mathbb{P}(Z_{k+1,i}=1)\eta_{k+1}(\omega_{k+1}) \sum_{\omega_k | \omega_{k+1} \to \omega_k, \omega_{k,i}=1} \left( \eta_k(\omega_k | \omega_{k+1}) \frac{n(k+1, \omega_k)}{\mathbb{P}(Z_{k+1,i}=1)} \right) \\
&= \mathbb{P}(Z_{k+1,i}=1)\eta_{k+1}(\omega_{k+1}) \mathbb{E}[n(k+1, \hat{\omega}_k) | Z_{k+1,i}=1, \hat{\omega}_{k+1} = \omega_{k+1}] \\
&= \mathbb{P}(Z_{k+1,i}=1)\eta_{k+1}(\omega_{k+1}) \mathbb{E}[\sum_{j \in Tier(k+1)} Z_{k+1,j} | Z_{k+1,i}=1] \\
&= \eta_{k+1}(\omega_{k+1}) q(k+1)(1 + q(k+1)(n(k+1)-1)).
\end{aligned} \tag{35}$$

The fifth equality follows from (3) and (4). Thus, the induction hypothesis holds for $k+1$ as well; i.e., Expression (33) holds for any $k \in \{1, \cdots, K\}$. Finally, by Expression (13b) we have

$$p_c = \sum_{\omega_K | \omega_{K+1} \to \omega_K, \omega_{K,i}=1} \lambda_K(\omega_K) - 2c(K+1) x^{\star}_{K+1,i}(\omega_{K+1}),$$

given that $\mu_{K+1,i}(\omega_{K+1})=0$. Using (33) to express $\lambda_K(\omega_K)$ and following a similar reasoning as in (34), yields

$$p_c=\alpha(K+2)-\beta(K+2)s, \text{ or alternatively } s=\frac{\alpha(K+2)-p_c}{\beta(K+2)}.$$

Let $\bar{\omega}_{k'-1}$ denote the state where no firms experience a disruption in tiers $k',\dots,K+1$. Given that we have assumed that $\frac{\alpha(k')}{\beta(k')}<\frac{\alpha(K+2)-p_c}{\beta(K+2)}$, using (33) we obtain

$$\frac{\lambda_{k'-1}(\bar{\omega}_{k'-1})}{\eta_{k'-1}(\bar{\omega}_{k'-1})}=\alpha(k')-\beta(k')s=\alpha(k')-\beta(k')\frac{\alpha(K+2)-p_c}{\beta(K+2)}<0.$$

However, given that $\lambda_{k'-1}(\omega_{k'-1})$ is nonnegative, we arrive at a contradiction to our original hypothesis that at the unique solution of (11) the first two constraints of the optimization problem are binding for any valid state. In other words, we conclude that (31) holds.

To finalize the claim, let $\omega_k$ denote a valid state where the first constraint (second if $k=0$) of (11) is not binding at an optimal solution. If there are multiple such states, let $\omega_k$ denote the one that corresponds to the largest $k$. Also, let $\bar{\omega}_k$ denote the state where no firm in any tier $l>k$ experiences a disruption. We claim the aforementioned constraint is also not binding for $\bar{\omega}_k$.

First, note that if $\bar{\omega}_k=\omega_k$, the claim trivially follows. Next, suppose $\bar{\omega}_k\neq\omega_k$, and the aforementioned constraint is binding for $\bar{\omega}_k$. Observe that by our choice of $k$ and $\bar{\omega}_k$, we have

$$s=\sum_{i\in Tier(k)}x^\star_{k,i}(\bar{\omega}_k)>\sum_{i\in Tier(k)}x^\star_{k,i}(\omega_k). \tag{36}$$

Let $\mathcal{T}$ denote the set of states reachable from $\omega_k$, i.e., $\mathcal{T}=\{\omega_l|\omega_k\to\omega_{k-1}\to\cdots\to\omega_l, l<k\}\cup\{\omega_k\}$. In addition, for any $\omega_l\in\mathcal{T}$, let $m(\omega_l)$ denote the state that is identical to $\omega_l$ in terms of disruption realizations in tiers $l+1,\dots,k$, but involves no disruptions in tiers $k+1,\dots,K+1$. Note that $m(\omega_k)=\bar{\omega}_k$. Let $\mathcal{T}'=\{m(\omega_l)|\omega_l\in\mathcal{T}\}$. Clearly $m(\cdot)$ is a bijection between $\mathcal{T}$ and $\mathcal{T}'$, and $m(\omega_l)$ is reachable from $\bar{\omega}_k$ if and only if $\omega_l$ is reachable from $\omega_k$.

Observe that in (11), by our choice of $\omega_k$, the upper bound constraint on $\sum_{i\in Tier(k)}x_{k,i}(\omega_k)$ is not binding. Since (11) is a convex optimization problem, $\mathbf{x}^\star$ remains the unique optimal solution even when the constraint that sets an upper bound to $\sum_{i\in Tier(k)}x_{k,i}(\omega_k)$ is removed. We refer to the optimization problem that is obtained from (11) after removing the aforementioned constraint as (11'). We construct another solution $\mathbf{x}'$ for (11'), by setting

- $x'_{l,j}(\omega_l)=x^\star_{l,j}(\omega_l)$ for any $\omega_l\notin\mathcal{T}\cup\mathcal{T}'$,
- $x'_{l,j}(\omega_l)=\dfrac{\eta_k(\omega_k)x^\star_{l,j}(\omega_l)+\eta_k(\bar{\omega}_k)x^\star_{l,j}(m(\omega_l))}{\eta_k(\omega_k)+\eta_k(\bar{\omega}_k)}$ for $\omega_l\in\mathcal{T}$,
- $x'_{l,j}(\omega_l)=\dfrac{\eta_k(\bar{\omega}_k)x^\star_{l,j}(\omega_l)+\eta_k(\omega_k)x^\star_{l,j}(m^{-1}(\omega_l))}{\eta_k(\omega_k)+\eta_k(\bar{\omega}_k)}$ for $\omega_l\in\mathcal{T}'$,

for any tier $l$, $j \in Tier(l)$, and defining $x'_0(\omega_0)$ similarly. Intuitively, $\mathbf{x}'$ is obtained from $\mathbf{x}^\star$ by taking convex combinations of the decision variables between states in $\mathcal{T}$ and $\mathcal{T}'$ (and weighing decision variables corresponding to $\mathcal{T}$ by $\eta_k(\omega_k)$ and those in $\mathcal{T}'$ by $\eta_k(\bar{\omega}_k)$), while leaving the remaining decision variables intact.

The feasibility of $\mathbf{x}'$ in (11') readily follows from (36) and the fact that in (11') no constraint imposes an upper bound on $\sum_{i\in Tier(k)} x'_{k,i}(\omega_k)$. Furthermore, it can be seen that in (11) (and similarly in (11')) the objective function can be expressed as follows:

$$f + \eta_k(\omega_k) g(\{x_0\}_{\omega_0\in\mathcal{T}}, \{x_{l,i}(\omega_l)\}_{i\in Tier(l),\omega_l\in\mathcal{T},l>0}) + \eta_k(\bar{\omega}_k) g(\{x_0\}_{\omega_0\in\mathcal{T}'}, \{x_{l,i}(\omega_l)\}_{i\in Tier(l),\omega_l\in\mathcal{T}',l>0}), \tag{37}$$

where $f$ is a function of only $(\{x_0\}_{\omega_0\notin\mathcal{T}\cup\mathcal{T}'}, \{x_{l,i}(\omega_l)\}_{i\in Tier(l),\omega_l\notin\mathcal{T}\cup\mathcal{T}',l>0})$ and $f$ and $g$ are concave functions. On the other hand, because of concavity and the construction of $\mathbf{x}'$, it follows that $\mathbf{x}'$ has a weakly higher objective value in (11') than $\mathbf{x}^\star$. This contradicts that fact that $\mathbf{x}^\star$ is the unique optimal solution in (11'). Therefore, we obtain a contradiction, and conclude that the first constraint (second if $k=0$) of (11) is not binding at $\bar{\omega}_k$.

Note that this implies by (13d)-(13e) that $\lambda_k(\bar{\omega}_k)=0$ (or $\lambda_0(\bar{\omega}_0)=0$). Since $\bar{\omega}_k$ is a valid state its equilibrium price is uniquely defined by $p_{k+1}(\bar{\omega}_k)=\lambda_k(\bar{\omega}_k)/\eta(\bar{\omega}_k)$ (by Lemma 5(b)). Hence, we obtain $p_{k+1}(\bar{\omega}_k)=0$. Thus, we conclude that for a state where no disruptions take place in tiers $\{k+1,\dots,K+1\}$, the equilibrium price is zero, and the claim follows. Q.E.D.

**Proof of Corollary 1**

Before proving the corollary, we introduce additional notation and state and prove an auxiliary lemma. Let $\mu(k) \triangleq \mathbb{E}[Z_{k,i}x_{k,i}(\hat{\omega}_k)]$ denote the expected equilibrium production output of firm $i$ in tier $k$. Using the fact that $\hat{\omega}_k$ and $Z_{k,i}$ are independent (and the fact that $\mathbb{E}[Z_{k,i}]=q(k)$), $\mu(k)$ can alternatively be expressed as $\mu(k)=q(k)\mathbb{E}[x_{k,i}(\hat{\omega}_k)]$. Furthermore, let

$$\hat{\theta}(k) \triangleq \mathbb{E}\left[\left(Z_{k,i}x_{k,i}(\hat{\omega}_k)\right)^2\right] = q(k)\mathbb{E}\left[x^2_{k,i}(\hat{\omega}_k)\right],$$

for any $i\in Tier(k)$. Finally, for any tier $k$ such that $n(k)>1$, let

$$\theta(k) \triangleq \mathbb{E}[Z_{k,i}Z_{k,j}x_{k,i}(\hat{\omega}_k)x_{k,j}(\hat{\omega}_k)] = q(k)^2\mathbb{E}[x_{k,i}(\hat{\omega}_k)x_{k,j}(\hat{\omega}_k)],$$

where $i,j\in Tier(k)$ and $i\neq j$. We next state an auxiliary lemma, which we then use in the proof of the corollary.

Lemma 6. *Suppose Assumption 1 holds. In the equilibrium, for any tier $1\le k\le K+1$ we have*
*(i)* $\mu(k) = \dfrac{s}{n(k)}\prod_{\ell=k}^{K+1} q(\ell)$;

*(ii)* $\theta(k) = \left(\frac{sq(k)}{n(k)}\right)^2 \prod_{\ell=k+1}^{K+1} \left(\frac{q(\ell)^2}{n(\ell)}\right)\left(n(\ell) - 1 + \frac{1}{q(\ell)}\right)$, *for* $n(k) > 1$;

*(iii)* $\hat{\theta}(k) = \frac{s^2 q(k)}{n(k)^2} \prod_{\ell=k+1}^{K+1} \left(\frac{q(\ell)^2}{n(\ell)}\right)\left(n(\ell) - 1 + \frac{1}{q(\ell)}\right)$, *and thus for* $n(k) > 1$, *we have* $\hat{\theta}(k) = \frac{\theta(k)}{q(k)}$.

*Proof:*

*(i)* By Theorem 1, we have that $x_{k,i}(\omega_k) = \frac{s}{n(k)} \prod_{\ell=k+1}^{K+1} \frac{n(\ell,\omega_k)}{n(\ell)}$, for every tier $k$ when the realized state is $\omega_k$. This immediately implies that

$$\mu(k) = q(k)\mathbb{E}[x_{k,i}(\hat{\omega}_k)] = q(k)\mathbb{E}\left[\frac{s}{n(k)} \prod_{\ell=k+1}^{K+1} \frac{n(\ell,\hat{\omega}_k)}{n(\ell)}\right] = q(k)\frac{s}{n(k)} \prod_{\ell=k+1}^{K+1} q(\ell) = \frac{s}{n(k)} \prod_{\ell=k}^{K+1} q(\ell),$$

given that disruptive events are independent and $\mathbb{E}[n(\ell,\hat{\omega}_k)] = \mathbb{E}\big[\sum_{i\in Tier(\ell)} Z_{\ell,i}\big] = q(\ell)n(\ell)$, for $\ell \geq k+1$.

*(ii), (iii)* First, note that for tier $K+1$ we have $\theta(K+1) = \left(\frac{sq(K+1)}{n(K+1)}\right)^2$ if $n(K+1) > 1$ and $\hat{\theta}(K+1) = \frac{q(K+1)}{n(K+1)^2} s^2$, given that $s_i = \frac{s}{n(K+1)}$ for any firm $i \in Tier(K+1)$ and $\hat{\omega}_{K+1} = \omega_{K+1} = \emptyset$. For any other tier $k$, we can write the following recursive equation for $\theta(k)$

$$\begin{aligned}
\theta(k) &= q(k)^2\mathbb{E}[x_{k,i}(\hat{\omega}_k)x_{k,j}(\hat{\omega}_k)] = q(k)^2\mathbb{E}\left[\left(\frac{s}{n(k)} \prod_{\ell=k+1}^{K+1} \frac{n(\ell,\hat{\omega}_k)}{n(\ell)}\right)\left(\frac{s}{n(k)} \prod_{\ell=k+1}^{K+1} \frac{n(\ell,\hat{\omega}_k)}{n(\ell)}\right)\right] \\
&= q(k)^2\mathbb{E}\left[\left(\frac{s}{n(k)}\frac{n(k+1,\hat{\omega}_k)}{n(k+1)} \prod_{\ell=k+2}^{K+1} \frac{n(\ell,\hat{\omega}_k)}{n(\ell)}\right)\left(\frac{s}{n(k)}\frac{n(k+1,\hat{\omega}_k)}{n(k+1)} \prod_{\ell=k+2}^{K+1} \frac{n(\ell,\hat{\omega}_k)}{n(\ell)}\right)\right] \\
&= \left(\frac{q(k)}{n(k)}\right)^2\mathbb{E}\left[\left(n(k+1,\hat{\omega}_k)\frac{s}{n(k+1)} \prod_{\ell=k+2}^{K+1} \frac{n(\ell,\hat{\omega}_k)}{n(\ell)}\right)\left(n(k+1,\hat{\omega}_k)\frac{s}{n(k+1)} \prod_{\ell=k+2}^{K+1} \frac{n(\ell,\hat{\omega}_k)}{n(\ell)}\right)\right] \\
&= \left(\frac{q(k)}{n(k)}\right)^2\mathbb{E}\left[\left(\sum_{t\in Tier(k+1)} Z_{k+1,t}x_{k+1,t}(\hat{\omega}_{k+1})\right)\left(\sum_{t\in Tier(k+1)} Z_{k+1,t}x_{k+1,t}(\hat{\omega}_{k+1})\right)\right] \\
&= \left(\frac{q(k)}{n(k)}\right)^2\mathbb{E}\left[\sum_{t,u\in Tier(k+1),t\neq u} Z_{k+1,t}Z_{k+1,u}x_{k+1,t}(\hat{\omega}_{k+1})x_{k+1,u}(\hat{\omega}_{k+1}) + \sum_{t\in Tier(k+1)} \big(Z_{k+1,t}x_{k+1,t}(\hat{\omega}_{k+1})\big)^2\right] \\
&= \left(\frac{q(k)}{n(k)}\right)^2\Big(n(k+1)\big(n(k+1)-1\big)\theta(k+1) + n(k+1)\hat{\theta}(k+1)\Big),
\end{aligned} \tag{38}$$

where the expression in the fourth line follows from the fact that $n(k+1,\hat{\omega}_k) = \sum_{t\in Tier(k+1)} Z_{k+1,t}$ and

$$\frac{s}{n(k+1)} \prod_{\ell=k+2}^{K+1} \frac{n(\ell,\hat{\omega}_k)}{n(\ell)} = \frac{s}{n(k+1)} \prod_{\ell=k+2}^{K+1} \frac{n(\ell,\hat{\omega}_{k+1})}{n(\ell)} = x_{k+1,t}(\hat{\omega}_{k+1}), \tag{39}$$

for every firm $t \in Tier(k+1)$ (since $\hat{\omega}_k$ and $\hat{\omega}_{k+1}$ both encode the same disruptions in tiers above $k+1$). Following the same approach for $\hat{\theta}(k)$ we obtain that

$$\hat{\theta}(k) = \left(\frac{q(k)}{n(k)^2}\right)\Big(n(k+1)\big(n(k+1)-1\big)\theta(k+1) + n(k+1)\hat{\theta}(k+1)\Big). \tag{40}$$

Expressions (38) and (40) imply that $\theta(k)=q(k)\hat{\theta}(k)$. Exploiting this observation, (40) can be written as

$$\hat{\theta}(k)=\Big(\frac{q(k)}{n(k)^2}\Big)\Big(n(k+1)\big(n(k+1)-1\big)q(k+1)+n(k+1)\Big)\hat{\theta}(k+1). \tag{41}$$

Using this equality recursively with the boundary condition $\hat{\theta}(K+1)=\frac{q(K+1)}{n(K+1)^2}s^2$, part (iii) of the lemma follows. Together with the fact that $\theta(k)=q(k)\hat{\theta}(k)$, this implies part (ii); thus, completing the proof of the lemma. □

Next, using Lemma 6 we provide an expression for the (unconditional) expected profits of firms in different tiers. Consider firm $i$ in tier $k\in\{1,\dots,K\}$. Recall that its expected profit is given as follows:

$$\pi(k)=\mathbb{E}\Bigg[p_k(\hat{\omega}_{k-1})Z_{k,i}x_{k,i}(\hat{\omega}_k)-p_{k+1}(\hat{\omega}_k)x_{k,i}(\hat{\omega}_k)-c(k)x_{k,i}^2(\hat{\omega}_k)\Bigg].$$

As before we have $n(k,\hat{\omega}_{k-1})=\sum_{t\in Tier(k)}Z_{k,t}$ and

$$\frac{s}{n(k)}\prod_{\ell=k+1}^{K+1}\frac{n(\ell,\hat{\omega}_{k-1})}{n(\ell)}=\frac{s}{n(k)}\prod_{\ell=k+1}^{K+1}\frac{n(\ell,\hat{\omega}_k)}{n(\ell)}=x_{k,t}(\hat{\omega}_k),$$

for every firm $t\in Tier(k)$ (see Expression (39)). Using these observations, we obtain

$$s\prod_{\ell=k}^{K+1}\frac{n(\ell,\hat{\omega}_{k-1})}{n(\ell)}=n(k,\hat{\omega}_{k-1})\frac{s}{n(k)}\prod_{\ell=k+1}^{K+1}\frac{n(\ell,\hat{\omega}_{k-1})}{n(\ell)}=\sum_{t\in Tier(k)}Z_{k,t}x_{k,t}(\hat{\omega}_k)=\sum_{t\in Tier(k-1)}x_{k-1,t}(\hat{\omega}_{k-1}), \tag{42}$$

where the last equality follows from Definition 1 after noting that under Assumption 1 $p_{k+1}(\omega_k)>0$ for any state $\omega_k\in\Omega_k$. Together with Theorem 1 this equality in turn implies that

$$p_k(\hat{\omega}_{k-1})=\alpha(k)-\beta(k)\sum_{t\in Tier(k)}Z_{k,t}x_{k,t}(\hat{\omega}_k)=\alpha(k)-\beta(k)\sum_{t\in Tier(k-1)}x_{k-1,t}(\hat{\omega}_{k-1}).$$

Using these expressions (and the corresponding expressions for $p_{k+1}(\hat{\omega}_k)$), the profits can be expressed as follows:

$$\begin{aligned}\pi(k)&=\mathbb{E}\Bigg[p_k(\hat{\omega}_{k-1})Z_{k,i}x_{k,i}(\hat{\omega}_k)-p_{k+1}(\hat{\omega}_k)x_{k,i}(\hat{\omega}_k)-c(k)x_{k,i}^2(\hat{\omega}_k)\Bigg]\\
&=\mathbb{E}\left[\alpha(k)Z_{k,i}x_{k,i}(\hat{\omega}_k)\right]-\mathbb{E}\Bigg[\beta(k)Z_{k,i}x_{k,i}(\hat{\omega}_k)\sum_{t\in Tier(k)}Z_{k,t}x_{k,t}(\hat{\omega}_k)\Bigg]-\mathbb{E}\left[\alpha(k+1)x_{k,i}(\hat{\omega}_k)\right]\\
&\qquad+\mathbb{E}\Bigg[\beta(k+1)x_{k,i}(\hat{\omega}_k)\sum_{t\in Tier(k)}x_{k,t}(\hat{\omega}_k)\Bigg]-\mathbb{E}\left[c(k)x_{k,i}^2(\hat{\omega}_k)\right]\\
&=\mathbb{E}\left[\alpha(k)Z_{k,i}x_{k,i}(\hat{\omega}_k)\right]-\mathbb{E}\Bigg[\beta(k)\Bigg(\sum_{t|t\neq i}Z_{k,i}Z_{k,t}x_{k,i}(\hat{\omega}_k)x_{k,t}(\hat{\omega}_k)+\big(Z_{k,i}x_{k,i}(\hat{\omega}_k)\big)^2\Bigg)\Bigg]\\
&\qquad-\mathbb{E}\left[\alpha(k+1)x_{k,i}(\hat{\omega}_k)\right]+\mathbb{E}\Bigg[\beta(k+1)x_{k,i}(\hat{\omega}_k)\sum_{t\in Tier(k)}x_{k,t}(\hat{\omega}_k)\Bigg]-\mathbb{E}\left[c(k)x_{k,i}^2(\hat{\omega}_k)\right].\end{aligned} \tag{43}$$

Recalling the definitions of $\mu(k), \theta(k)$, and $\hat{\theta}(k)$, we can rewrite (43) as follows:

$$\begin{aligned}\pi(k) = \alpha(k)\mu(k) - \beta(k)\left((n(k)-1)\,\theta(k) + \hat{\theta}(k)\right) - \frac{\alpha(k+1)}{q(k)}\mu(k)\\ + \frac{\beta(k+1)}{q(k)^2}\left[(n(k)-1)\theta(k) + q(k)\hat{\theta}(k)\right] - \frac{c(k)}{q(k)}\hat{\theta}(k).\end{aligned} \tag{44}$$

Canceling common terms using (3) and noting that $\theta(k) = q(k)\hat{\theta}(k)$, the expression for $\pi(k)$ simplifies to

$$\begin{aligned}\pi(k) &= \frac{\beta(k+1)}{q(k)}n(k)\hat{\theta}(k) - \beta(k)\left((n(k)-1)\,\hat{\theta}(k)q(k) + \hat{\theta}(k)\right) - \frac{c(k)}{q(k)}\hat{\theta}(k)\\ &= \hat{\theta}(k)\left(\frac{\beta(k+1)}{q(k)}n(k) - \beta(k)\Big((n(k)-1)q(k)+1\Big) - \frac{c(k)}{q(k)}\right).\end{aligned} \tag{45}$$

Note that the expression defining $\beta(k)$, i.e., Expression (4), yields

$$\beta(k+1)n(k) - \beta(k)(n(k)-1)q(k)^2 - \beta(k)q(k) = 2c(k).$$

Using this observation, (45) can be written as $\pi(k) = \hat{\theta}(k)\frac{c(k)}{q(k)}$. Combining this with the expression for $\hat{\theta}(k)$, i.e., $\hat{\theta}(k) = \frac{s^2 q(k)}{n(k)^2}\prod_{\ell=k+1}^{K+1}\left(\frac{q(\ell)^2}{n(\ell)}\right)\left(n(\ell)-1+\frac{1}{q(\ell)}\right)$, yields

$$\pi(k) = s^2\frac{c(k)}{n(k)^2}\prod_{\ell=k+1}^{K+1} q(\ell)^2\left(\frac{n(\ell)-1+\frac{1}{q(\ell)}}{n(\ell)}\right), \tag{46}$$

which completes the proof of the claim for firms in any tier $k \in \{1,\dots,K\}$.

Finally, consider tier $k = K+1$, and recall that by Theorem 1, we have

$$p_{K+2}(\hat{\omega}_{K+1}) = p_c = \alpha(K+2) - \beta(K+2)s = \alpha(K+2) - \beta(K+2)\sum_{t\in Tier(K+1)} x_{K+1,t}(\hat{\omega}_{K+1}).$$

Using this observation the profits can once again be expressed as in (43). Hence, following the same steps as before yields the profit expression in (46) for $k = K+1$. Thus, the claim follows for $k = K+1$ as well. Q.E.D.

**Proof of Proposition 1**

Recall that according to Theorem 1, equilibrium supply $s$ is given by $s = \frac{\alpha(K+2)-p_c}{\beta(K+2)}$. Using (4) recursively to state $\beta(K+2)$ more explicitly, supply $s$ can be expressed as follows:

$$s = \left(\alpha\prod_{\ell=1}^{K+1} q(\ell) - p_c\right)\left(\sum_{r=1}^{K+1}\frac{2c(r)}{n(r)}\prod_{\ell=r+1}^{K+1}\frac{q(\ell)^2}{n(\ell)}\left(n(\ell)-1+\frac{1}{q(\ell)}\right) + \beta\prod_{\ell=1}^{K+1}\frac{q(\ell)^2}{n(\ell)}\left(n(\ell)-1+\frac{1}{q(\ell)}\right)\right)^{-1}.$$

The first part of the proposition follows immediately by observing that the cost coefficients appear only in the denominator; thus, increasing the cost coefficient corresponding to any tier decreases equilibrium supply $s$ (provided that both before and after the update of the cost coefficient Assumption 1 is satisfied). Similarly, increasing the number of firms $n(\ell)$ in tier $\ell$ leads to a decrease in the term $\frac{n(\ell)-1+1/q(\ell)}{n(\ell)}$; thus, equilibrium supply is increasing in $n(\ell)$ for any tier $\ell$.

To establish the last part of the proposition, we consider tier $k$ and define the following terms to simplify exposition:

$$A_k \triangleq \sum_{r=k}^{K+1} \frac{2c(r)}{n(r)} \prod_{\ell=r+1}^{K+1} \frac{q(\ell)^2}{n(\ell)} \left(n(\ell) - 1 + \frac{1}{q(\ell)}\right) \tag{47}$$

$$B_k \triangleq \beta \prod_{\ell=1,\ell\neq k}^{K+1} \frac{q(\ell)^2}{n(\ell)} \left(n(\ell) - 1 + \frac{1}{q(\ell)}\right) + \sum_{r=1}^{k-1} \frac{2c(r)}{n(r)} \prod_{\ell=r+1,\ell\neq k}^{K+1} \frac{q(\ell)^2}{n(\ell)} \left(n(\ell) - 1 + \frac{1}{q(\ell)}\right). \tag{48}$$

Note that $A_k$ and $B_k$ are strictly positive and independent of $q(k)$. We can rewrite $s$ in terms of $A_k$ and $B_k$ as follows:

$$s = \frac{\alpha \prod_{\ell=1}^{K+1} q(\ell) - p_c}{\left(A_k + B_k \frac{q(k)^2}{n(k)} \left(n(k) - 1 + \frac{1}{q(k)}\right)\right)} = \frac{n(k)\left(\alpha \prod_{\ell=1}^{K+1} q(\ell) - p_c\right)}{\left(A_k n(k) + B_k q(k)\Big(1 + (n(k)-1)q(k)\Big)\right)}. \tag{49}$$

The partial derivative of $s$ with respect to $q(k)$ can be expressed as follows:

$$\frac{\partial s}{\partial q(k)} = \frac{n(k)\left(A_k n(k) \alpha \prod_{\ell=1,\ell\neq k}^{K+1} q(\ell) + B_k\Big(\big(1 + 2(n(k)-1)q(k)\big)p_c - \big(n(k)-1\big)\alpha q(k)^2 \prod_{\ell=1,\ell\neq k}^{K+1} q(\ell)\Big)\right)}{\Big(A_k n(k) + B_k q(k)\big(1 + (n(k)-1)q(k)\big)\Big)^2}. \tag{50}$$

First, note that the denominator of (50) is always positive. Second, the numerator is concave (quadratic) in $q(k)$ and positive at $q(k) = 0$. Thus, it follows that there exists some $\tilde{q} > 0$ such that for $q(k) \leq \tilde{q}$, Expression (50) is positive, whereas for $q(k) > \tilde{q}$, (50) is negative. For some modeling primitives, $\tilde{q} < 1$ and, thus, aggregate supply $s$ at equilibrium first increases ((50) is positive) and then decreases in $q(k)$ ((50) is negative). We provide an illustration of the (potential) nonmonotonicity of $s$ in $q(k)$ in Figure 2. Q.E.D.

**Proof of Proposition 2**

According to Corollary 1, the expression for the profits of firms in tier $\ell$ is given as follows:

$$\pi(\ell) = s^2 \frac{c(\ell)}{n(\ell)^2} \prod_{m=\ell+1}^{K+1} \frac{q(m)^2}{n(m)} \left(n(m) - 1 + \frac{1}{q(m)}\right). \tag{51}$$

We analyze the effect of increasing $q(k)$ on $\pi(\ell)$ by considering the following cases separately: (a) $\ell < k$ and (b) $\ell \geq k$.

*Case (a) $\ell < k$:* Recall that $s$ can be stated in terms of $A_k$ and $B_k$ defined in Expressions (47) and (48), respectively, as follows (see (49)):

$$s = \frac{\alpha \prod_{\ell=1}^{K+1} q(\ell) - p_c}{\left(A_k + B_k q(k)^2 \left(\frac{n(k)-1+1/q(k)}{n(k)}\right)\right)}.$$

Moreover, we can rewrite $\pi(\ell)$ as

$$\pi(\ell)=s^2\frac{c(\ell)}{n(\ell)^2}\frac{q(k)^2}{n(k)}\left(n(k)-1+\frac{1}{q(k)}\right)\prod_{m=\ell+1,m\neq k}^{K+1}\frac{q(m)^2}{n(m)}\left(n(m)-1+\frac{1}{q(m)}\right),$$

or, equivalently, as

$$\pi(\ell)=\Gamma\left(q(k)^2\left(n(k)-1+\frac{1}{q(k)}\right)\right)\left[\frac{\alpha\prod_{\ell=1}^{K+1}q(\ell)-p_c}{\left(A_k+B_kq(k)^2\left(\frac{n(k)-1+1/q(k)}{n(k)}\right)\right)}\right]^2,$$

where $\Gamma$ is a constant independent of $q(k)$. The partial derivative of $\pi(\ell)$ with respect to $q(k)$ can be written as follows:

$$\begin{aligned}\frac{\partial\pi(\ell)}{\partial q(k)}&=\Gamma\Big(A_kn(k)+B_kq(k)\big(q(k)(n(k)-1)+1\big)\Big)^{-3}\\&\quad n(k)^2(\alpha\prod_{\ell=1}^{K+1}q(\ell)-p_c)\Bigg[B_kq(k)(q(k)(n(k)-1)+1)\left(\alpha\prod_{\ell=1}^{K+1}q(\ell)+p_c+2(n(k)-1)q(k)p_c\right)+\\&\quad A_kn(k)\left(\alpha\prod_{\ell=1}^{K+1}q(\ell)\big(3+4q(k)(n(k)-1)\big)-p_c\big(1+2q(k)(n(k)-1)\big)\right)\Bigg]>0,\end{aligned}$$

where the fact that $\frac{\partial\pi(\ell)}{\partial q(k)}>0$ follows since both the numerator and denominator in the expression above are positive.

*Case (b) $\ell\geq k$:* In this case, Expression (51) directly implies that firms' profits in tier $\ell$ depend on $q(k)$ only through term $s$. Therefore, from Proposition 1, we conclude that $\pi(\ell)$ is, in general, nonmonotonic in $q(k)$. For completeness, we provide an example that illustrates this nonmonotonicity in Figure 2. Q.E.D.

**Proof of Proposition 3**

According to Corollary 1, the expression for the profits of firms in tier $\ell$ is given as follows:

$$\pi(\ell)=s^2\frac{c(\ell)}{n(\ell)^2}\prod_{m=\ell+1}^{K+1}\frac{q(m)^2}{n(m)}\left(n(m)-1+\frac{1}{q(m)}\right).\tag{52}$$

We analyze the effect of increasing $n(k)$ on $\pi(\ell)$ by considering the following three cases separately: (a) $k>\ell$, (b) $k=\ell$, and (c) $k<\ell$.

*Case (a) $k>\ell$ :* Collecting the terms that depend on $n(k)$ in (52), it can be seen that for $k>\ell$, $\pi(\ell)$ depends on $n(k)$ only through term $s^2\frac{n(k)-1+1/q(k)}{n(k)}$. Thus, the sign of $\frac{\partial\pi(\ell)}{\partial n(k)}$ is the same as the sign of $\frac{\partial}{\partial n(k)}\left(s^2\frac{n(k)-1+1/q(k)}{n(k)}\right)$ and this holds for all tiers $\ell$ downstream of $k$. Note that we can write

$$s^2\frac{n(k)-1+1/q(k)}{n(k)}=\frac{\left(\alpha\prod_{\ell=1}^{K+1}q(\ell)-p_c\right)^2\left(n(k)^2-n(k)+n(k)/q(k)\right)}{\left(A_{k+1}n(k)+D_k+B_kq(k)^2\left(n(k)-1+\frac{1}{q(k)}\right)\right)^2},$$

where $A_k$ and $B_k$ are defined as in (47) and (48); and

$$D_k \triangleq 2c(k) \prod_{m=k+1}^{K+1} \frac{q(m)^2}{n(m)} \left( n(m) - 1 + \frac{1}{q(m)} \right). \tag{53}$$

Thus,

$$\left(\alpha \prod_{\ell=1}^{K+1} q(\ell) - p_c\right)^{-2} \frac{\partial}{\partial n(k)} \left( s^2 \frac{n(k) - 1 + 1/q(k)}{n(k)} \right) =$$
$$= \frac{(q(k)-1)\Big[A_{k+1}n(k) - B_k q(k)(q(k)n(k) + 1 - q(k))\Big] + D_k\Big(2n(k)q(k) - q(k) + 1\Big)}{q(k)\Big(A_{k+1}n(k) + D_k + B_k q(k)^2\Big(n(k) - 1 + \frac{1}{q(k)}\Big)\Big)^3},$$

which is positive for large enough $q(k) < 1$ (since it is a continuous function and takes strictly positive values for $q(k) = 1$). Conversely, one can identify conditions under which the numerator is negative. For example, it can be readily seen that this is the case if $q(k) \ll 1$ and $c(k)$ is sufficiently small.[20] In this case, the terms in the numerator involving $B_k$ and $D_k$ are small, and the first (negative) term dominates, thereby yielding nonmonotonicity of profits in $n(k)$.

*Case (b) $k = \ell$ :* Using $A_k, B_k, D_k$ (as defined in (47), (48), and (53), respectively), we can rewrite $s$ as follows:

$$s = \frac{\alpha \prod_{\ell=1}^{K+1} q(\ell) - p_c}{\left(A_{k+1} + \frac{D_k}{n(k)} + B_k q(k)^2 \left(\frac{n(k)-1+\frac{1}{q(k)}}{n(k)}\right)\right)}$$
$$= \frac{n(k)\Big(\alpha \prod_{\ell=1}^{K+1} q(\ell) - p_c\Big)}{\Big(A_{k+1}n(k) + D_k + B_k q(k)^2 \Big(n(k) - 1 + \frac{1}{q(k)}\Big)\Big)}.$$

Thus, the expression for the firms' profits in tier $k$ can be written as

$$\pi(k) = \frac{\Big(\alpha \prod_{\ell=1}^{K+1} q(\ell) - p_c\Big)^2}{\Big(A_{k+1}n(k) + D_k + B_k q(k)^2 \Big(n(k) - 1 + \frac{1}{q(k)}\Big)\Big)^2} \frac{D_k}{2}.$$

Finally, by noting that $A_{k+1}$, $B_k$, and $D_k$ are independent of $n(k)$, we obtain that $\frac{\partial \pi(k)}{\partial n(k)} < 0$ (since $n(k)$ appears only in the denominator of the expression).

*Case (c) $k < \ell$:* Ignoring the terms that do not depend on $n(k)$ in (52), it can be seen that for $k < \ell$, $\pi(\ell)$ depends on $n(k)$ only through term $s$. Thus, the claim follows directly by noting that the aggregate supply of raw materials $s$ is increasing in $n(k)$ as shown in Proposition 1. Q.E.D.

[20] Note that it is straightforward to construct problem instances where having small $q(k)$ and $c(k)$ does not violate Assumption 1. In particular, Lemma 3 provides sufficient conditions under which the assumption holds for any network structure and choice of cost parameters. Furthermore, this lemma shows that even for $q(k) \ll 1$ for sufficiently small $p_c$ the assumption continues to hold.

**Proof of Lemmas 1 and 2**

First, note that $\mathbb{E}[U] = \mu(1)$, and according to Lemma 6 we have $\mathbb{E}[U] = s\prod_{\ell=1}^{K+1} q(\ell)$. In addition, note that according to the definition of the coefficient of variation, we have

$$CV(U) = \sqrt{\frac{\mathrm{Var}(U)}{\mathbb{E}[U]^2}} = \sqrt{\frac{\mathbb{E}[U^2]-\mathbb{E}[U]^2}{\mathbb{E}[U]^2}} = \sqrt{\frac{\hat{\theta}(1)\Big(n(1)+n(1)(n(1)-1)q(1)\Big)}{\Big(s\prod_{\ell=1}^{K+1} q(\ell)\Big)^2} - 1}. \tag{54}$$

Observe that Lemma 6 implies that $\hat{\theta}(1) = \frac{s^2 q(1)}{n(1)^2}\prod_{\ell=2}^{K+1}\left(\frac{q(\ell)^2}{n(\ell)}\right)\left(n(\ell)-1+\frac{1}{q(\ell)}\right)$. Substituting this expression for $\hat{\theta}(1)$ in (54), it immediately follows that

$$CV(U) = \left(\prod_{\ell=1}^{K+1}\frac{1}{n(\ell)}\left(n(\ell)-1+\frac{1}{q(\ell)}\right)-1\right)^{\frac{1}{2}}. \tag{55}$$

Finally, Corollary 1 together with Expression (55) yield for the profits of a retailer:

$$\pi(1) = s^2\frac{c(1)}{n(1)}\frac{1}{n(1)-1+1/q(1)}\left(CV(U)^2+1\right)\prod_{\ell=2}^{K+1} q(\ell)^2.$$

Q.E.D.

**Proof of Lemma 3**

First, note that $p_c < \alpha\prod_{\ell=1}^{K+1} q(\ell)$ in (10) implies the first part of Assumption 1. To show that (10) guarantees that the second part of Assumption 1 also holds, note that

$$\frac{\beta(k)}{\beta(K+2)} < \frac{1}{\prod_{\ell=k}^{K+1}\frac{q(\ell)^2}{n(\ell)}\Bigg(n(\ell)-1+1/q(\ell)\Bigg)} \le \frac{1}{\prod_{\ell=k}^{K+1} q(\ell)^2}, \tag{56}$$

where the first inequality follows since $c(\ell) > 0$ for $1 \le \ell \le K+1$ and the second follows since $\frac{1}{n(\ell)}\Bigg(n(\ell)-1+1/q(\ell)\Bigg) \ge 1$ for any $q(\ell)$ such that $0 < q(\ell) \le 1$.

On the other hand, when $p_c > \alpha\Bigg(\prod_{\ell=1}^{K+1} q(\ell) - \prod_{\ell=1}^{K+1} q(\ell)^2\Bigg)$, we have

$$\frac{\alpha(k)}{\alpha(K+2)-p_c} > \frac{\alpha\prod_{\ell=1}^{k-1} q(\ell)}{\alpha\prod_{\ell=1}^{K+1} q(\ell)^2} = \frac{1}{\prod_{\ell=1}^{k-1} q(\ell)}\frac{1}{\prod_{\ell=k}^{K+1} q(\ell)^2} > \frac{\beta(k)}{\beta(K+2)},$$

where the last inequality follows directly from (56) given that $0 < q(\ell) \le 1$ for $1 \le \ell \le K+1$. Q.E.D.

**Proof of Proposition 4**

According to Corollary 1, a raw materials supplier's profits are given as

$$\pi(K+1) = s^2\frac{c(K+1)}{n(K+1)^2}.$$

Thus, since $n(K+1)$ is assumed to be fixed and given, it follows that the networks that maximize $s$ also maximize $\pi(K+1)$. Q.E.D.

**Proof of Proposition 5**

The fact that $\mathcal{V}_S = \mathcal{V}_{supply}$ follows directly from Proposition 4. Note that in the absence of any disruption risk the expression for a retailer's expected profit simplifies to $\pi(1) = s^2 \frac{c(1)}{n(1)^2}$ (see Corollary 1). Thus, the network that maximizes the retailers' profits is the same as the one that maximizes the supply procured in the upstream tier, which, in turn, is the same as the one that maximizes the suppliers' profits.

Finally, when $q = 1$, the expression for the supply level $s$ simplifies as follows (see Theorem 1):

$$s = \frac{\alpha - p_c}{\beta(K+2)}.$$

Thus, when the cost coefficients for all tiers are equal, the supply level $s$ is maximized when

$$\beta(K+2) = \sum_{r=1}^{K+1} \frac{2c}{n(r)} + \beta, \tag{57}$$

is minimized. In turn, this directly implies that, given $n(1), n(K+1)$, the optimal network is such that the number of firms in any two intermediate tiers differs by at most 1, i.e., Expression (57) is minimized when the number of firms in any two tiers $k, \ell \in \{2, \ldots, K\}$ satisfies $|n(k) - n(\ell)| \leq 1$. Q.E.D.

**Proof of Theorem 2**

Proposition 4 readily implies that $s_{\mathcal{N}_R} \leq s_{\mathcal{N}_S}$. By way of contradiction, assume that $CV(U_{\mathcal{N}_S}) > CV(U_{\mathcal{N}_R})$. Then, since the supply $s$ is weakly larger and $CV$ is strictly larger in $\mathcal{N}_S$, Lemma 2 implies that retailers can obtain a strictly larger expected profit in $\mathcal{N}_S$. This contradicts our assumption about the optimality of $\mathcal{N}_R$ for retailers' profits. Thus, we must have $CV(U_{\mathcal{N}_S}) \leq CV(U_{\mathcal{N}_R})$, which concludes the proof of part (i) of the theorem.

To establish part (ii), consider $\mathcal{N}_R$ and assume by way of contradiction that there exist two consecutive tiers $k-1$ and $k$ (with $2 < k < K+1$) such that $n(k) < n(k-1)$. In what follows, we show that swapping the number of firms in tiers $k-1$ and $k$ leads to a network that generates higher (expected) profits for the retailers. When the disruption risk is the same for all tiers, Lemma 1 implies that such a swap does not change $CV(U)$. Hence, it follows from Lemma 2 that retailers' profits are only affected through the $s$ term in the expression for $\pi(1)$. Theorem 1 implies that when $q(\ell) = q$ and $c(\ell) = c$ for every tier $\ell$, $s = (pq^{K+1} - p_c)/\beta(K+2)$, where by using Theorem 1 to express $\beta(K+2)$ more explicitly we obtain:

$$\beta(K+2) = \left( \sum_{r=1}^{K+1} \frac{2cq^{2(K-r+1)}}{n(r)} \prod_{\ell=r+1}^{K+1} \frac{n(\ell) - 1 + 1/q}{n(\ell)} + \beta q^{2(K+1)} \prod_{\ell=1}^{K+1} \frac{n(\ell) - 1 + 1/q}{n(\ell)} \right).$$

It is straightforward to see that when $q < 1$ the aforementioned swap strictly decreases term $\sum_{r=1}^{K+1} \frac{2cq^{2(K-r+1)}}{n(r)} \prod_{\ell=r+1}^{K+1} \frac{n(\ell)-1+1/q}{n(\ell)}$. Given that the last term in the expression for $s$ remains the same after the swap, this implies that both supply $s$ and (through Lemma 2) the retailers' profits strictly increase. Thus, we obtain a contradiction to the optimality of $\mathcal{N}_R$ for retailers' profits. Hence, it cannot be the case that $n(k) < n(k-1)$ for two consecutive tiers.

A similar argument can be used to establish that $\mathcal{N}_S$ also takes the form of an inverted pyramid. Suppose that this is not the case. The same approach as before implies that a swap between two consecutive tiers $k-1$ and $k$ with $n(k) < n(k-1)$ (and $2 < k < K+1$) strictly increases $s$. However, by Proposition 4, $\mathcal{N}_S$ maximizes $s$, thereby yielding a contradiction. Q.E.D.

**Proof of Proposition 6**

Let the probability of a disruptive event be equal to $q$ and the production cost coefficients be equal to $c$ for all tiers. To simplify exposition, we use the shorthand notation

$$\phi \triangleq \beta(K+2) = \left( \sum_{r=1}^{K+1} \frac{2c}{n(r)} q^{2(K+1-r)} \prod_{\ell=r+1}^{K+1} \frac{n(\ell)-1+1/q}{n(\ell)} + \beta q^{2(K+1)} \prod_{\ell=1}^{K+1} \frac{n(\ell)-1+1/q}{n(\ell)} \right), \quad (58)$$

where the expression for $\beta(K+2)$ is obtained by recursively using (4). Note that by Theorem 1 we have $s = \frac{\left(\alpha \prod_{\ell=1}^{K+1} q(\ell) - p_c\right)}{\phi}$. Thus, the structure that maximizes $s$ also minimizes $\phi$. The following lemma is useful in the analysis that follows.

Lemma 7. *Consider two network structures $\mathcal{N}$ and $\mathcal{N}'$ with the same total number of firms. Suppose that the numbers of firms in different tiers satisfy $n(k-1) \le n(k)$, $n'(k-1) \le n'(k)$ for $2 < k < K+1$ and $n(k) = n'(k)$ for $k \in \{1, K+1\}$. Let $\phi$ and $\phi'$ be the corresponding values of Expression (58) for $\mathcal{N}$ and $\mathcal{N}'$ respectively. Assume that there exist two tiers $k_1, k_2$ such that $K+1 > k_2 > k_1 > 1$ and*

$$\begin{aligned} &n(k_1) < n'(k_1) \text{ and } n(k_2) > n'(k_2), \\ &n(\ell) = n'(\ell) \quad \forall \ell \in \{k_1+1, \dots, k_2-1\}, \\ \text{and } &n(\ell) \le n'(\ell) \;\; \forall \ell < k_1. \end{aligned}$$

*If in $\mathcal{N}$ removing one firm from $k_2$ and adding it to $k_1$ weakly increases $\phi$, then in $\mathcal{N}'$ removing one firm from $k_1$ and adding it to $k_2$ weakly decreases $\phi'$; thus, it increases supply $s$.*

*Proof:* To simplify exposition, we define the following set of terms:

$$\begin{aligned} \gamma_1 &\triangleq \left( \sum_{r=1}^{k_1-1} \frac{2c}{n(r)} q^{2(K+1-r)} \prod_{\ell=r+1}^{k_1-1} \frac{n(\ell)-1+1/q}{n(\ell)} + \beta q^{2(K+1)} \prod_{\ell=1}^{k_1-1} \frac{n(\ell)-1+1/q}{n(\ell)} \right), \\ \gamma_2 &\triangleq \sum_{r=k_1+1}^{k_2-1} \frac{2c}{n(r)} q^{2(K+1-r)} \prod_{\ell=r+1}^{k_2-1} \frac{n(\ell)-1+1/q}{n(\ell)}, \\ \gamma_3 &\triangleq \sum_{r=k_2+1}^{K+1} \frac{2c}{n(r)} q^{2(K+1-r)} \prod_{\ell=r+1}^{K+1} \frac{n(\ell)-1+1/q}{n(\ell)}, \end{aligned}$$

$$\zeta_1 \triangleq \prod_{\ell=k_1+1}^{k_2-1} \frac{n(\ell)-1+1/q}{n(\ell)},$$
$$\zeta_2 \triangleq \prod_{\ell=k_2+1}^{K+1} \frac{n(\ell)-1+1/q}{n(\ell)}.$$

Similarly, let $\gamma_1'$, $\gamma_2'$, $\gamma_3'$, $\zeta_1'$, and $\zeta_2'$ denote the corresponding quantities for network structure $\mathcal{N}'$. Finally, let $\hat{\phi}$ denote the value of Expression (58) corresponding to the network that results from moving one firm from tier $k_2$ to tier $k_1$ in network $\mathcal{N}$. Then, according to the assumption in the lemma, we have $\hat{\phi} \geq \phi$. Next, we write $\phi$ and $\hat{\phi}$ in terms of $\gamma_1$, $\gamma_2$, $\gamma_3$, $\zeta_1$ and $\zeta_2$. We have

$$\phi = \gamma_1 \left( \frac{\Big(n(k_1)-1+1/q\Big)\Big(n(k_2)-1+1/q\Big)}{n(k_1)n(k_2)} \right) \zeta_1\zeta_2 + 2c\left(\frac{q^{2(K+1-k_1)}}{n(k_1)}\right)\zeta_1\left(\frac{n(k_2)-1+1/q}{n(k_2)}\right)\zeta_2$$
$$+\gamma_2\left(\frac{n(k_2)-1+1/q}{n(k_2)}\right)\zeta_2 + 2c\left(\frac{q^{2(K+1-k_2)}}{n(k_2)}\right)\zeta_2+\gamma_3.$$

Similarly, $\hat{\phi}$ is given by

$$\hat{\phi} = \gamma_1 \left( \frac{\Big(n(k_1)+1/q\Big)\Big(n(k_2)-2+1/q\Big)}{\Big(n(k_1)+1\Big)\Big(n(k_2)-1\Big)} \right) \zeta_1\zeta_2 + 2c\left(\frac{q^{2(K+1-k_1)}}{\big(n(k_1)+1\big)}\right)\zeta_1\left(\frac{n(k_2)-2+1/q}{\big(n(k_2)-1\big)}\right)\zeta_2$$
$$+\gamma_2\left(\frac{n(k_2)-2+1/q}{\big(n(k_2)-1\big)}\right)\zeta_2 + 2c\left(\frac{q^{2(K+1-k_2)}}{\big(n(k_2)-1\big)}\right)\zeta_2+\gamma_3.$$

Note that $\phi-\hat{\phi}\leq 0$ implies $(\phi-\hat{\phi})/\zeta_2 \leq 0$ which using the above expressions can be expressed as follows:

$$\gamma_1\zeta_1\left(\frac{\Big(n(k_1)-1+1/q\Big)\Big(n(k_2)-1+1/q\Big)}{n(k_1)n(k_2)} - \frac{\Big(n(k_1)+1/q\Big)\Big(n(k_2)-2+1/q\Big)}{\Big(n(k_1)+1\Big)\Big(n(k_2)-1\Big)}\right)$$
$$+\left(\frac{2cq^{2(K+1-k_1)}\zeta_1\big(n(k_2)-1+1/q\big)+n(k_1)\Big(\gamma_2\big(n(k_2)-1+1/q\big)+2cq^{2(K+1-k_2)}\Big)}{n(k_1)n(k_2)}\right)$$
$$-\left(\frac{2cq^{2(K+1-k_1)}\zeta_1\big(n(k_2)-2+1/q\big)+\big(n(k_1)+1\big)\Big(\gamma_2\big(n(k_2)-2+1/q\big)+2cq^{2(K+1-k_2)}\Big)}{\big(n(k_1)+1\big)\big(n(k_2)-1\big)}\right)\leq 0.$$

Multiplying this expression by $q^2\Big(n(k_1)n(k_2)\big(n(k_1)+1\big)\big(n(k_2)-1\big)\Big)$ and rearranging terms, we conclude that the above inequality holds if and only if the following holds:

$$\gamma_1\zeta_1(1-q)\Big(n(k_2)-n(k_1)-1\Big)\Big(q\big(n(k_2)+n(k_1)-1\big)+1\Big)-\gamma_2(1-q)qn(k_1)\big(n(k_1)+1\big)$$
$$+\zeta_1 2cqq^{2(K+1-k_1)}\Big(\Big((n(k_2)-1)^2q+n(k_2)-1\Big)-n(k_1)(1-q)\Big) \tag{59}$$
$$-2cq^2q^{2(K+1-k_2)}n(k_1)\big(n(k_1)+1\big)\leq 0.$$

Consider network structure $\mathcal{N}'$. Let $\hat{\phi}'$ denote the value of Expression (58) corresponding to the network that results from moving one firm from tier $k_1$ to tier $k_2$ in network $\mathcal{N}'$. By way of contradiction, assume that by moving one firm from tier $k_1$ to $k_2$, we have $\phi' < \hat{\phi}'$ or equivalently

$$0 < \hat{\phi}' - \phi'.$$

Once again writing the difference in terms of $\gamma_1'$, $\gamma_2'$, $\gamma_3'$, $\zeta_1'$, $\zeta_2'$, and rearranging terms we obtain that this inequality holds if and only if

$$\begin{aligned}&\gamma_1'\zeta_1'(1-q)\Big(n'(k_2)-n'(k_1)+1\Big)\Big(q\big(n'(k_2)+n'(k_1)-1\big)+1\Big)-\gamma_2'(1-q)q\big(n'(k_1)-1\big)n'(k_1)\\&+\zeta_1'2cqq^{2(K+1-k_1)}\bigg(\Big(\big(n'(k_2)\big)^2q+n'(k_2)\Big)-(n'(k_1)-1)(1-q)\bigg)\\&-2cq^2q^{2(K+1-k_2)}\big(n'(k_1)-1\big)n'(k_1)>0.\end{aligned} \tag{60}$$

However, the inequality above cannot hold in light of inequality (59). In particular, note that since $n(\ell)=n'(\ell)$ for every $\ell \in \{k_1+1,\dots,k_2-1\}$, we have $\gamma_2=\gamma_2'$, and $\zeta_1=\zeta_1'$. Also since $n(\ell)\le n'(\ell)$, for every tier $\ell<k_1$, we have $\gamma_1\ge\gamma_1'$. Finally, since $n'(k_2)+1\le n(k_2)$ and $n'(k_1)-1\ge n(k_1)$, the left-hand side of the inequality (60) is less than or equal to the left-hand side of inequality (59), which is a contradiction. So, it must be the case that $\phi'\ge\hat{\phi}'$ and, thus, the proof of the lemma is complete. □

Using the lemma above we complete the proof of the proposition. Let $k$ be the first tier such that $n_S(k)\ne n_R(k)$ (thus, for every $\ell<k$, we have $n_S(\ell)=n_R(\ell)$). By way of contradiction, assume that $n_S(k)<n_R(k)$. Since the total number of firms in the two networks is the same, there should be a tier $k_2>k$ such that $n_S(k_2)>n_R(k_2)$. Consider the most downstream such tier $k_2$. Also consider the largest $k_1<k_2$ such that $n_S(k_1)<n_R(k_1)$. Note that for every tier between $k_1$ and $k_2$ the number of firms in the two networks is equal, and also for every $k'<k_1$ we have $n_S(k')\le n_R(k')$. Moreover, by Proposition 4, $\mathcal{N}_S$ is a network that maximizes the supply level $s$, and moving one firm from tier $k_2$ and adding it to tier $k_1$ should not strictly decrease $\phi$ (and strictly increase the supply $s$). Note that by Theorem 2, for any $\ell>\ell'$ such that $\ell,\ell'\in\{2,\dots,K\}$, we have $n_S(\ell)\ge n_S(\ell')$ and $n_R(\ell)\ge n_R(\ell')$. Thus, we can use Lemma 7 and conclude that in network $\mathcal{N}_R$ moving one firm from tier $k_1$ to tier $k_2$ does not decrease the supply level $s$.

Note that since $k_1<k_2$, we have $n_R(k_1)\le n_R(k_2)$. We claim that in $\mathcal{N}_R$ if we move a firm from tier $k_1$ to tier $k_2$, the following term strictly increases:

$$\prod_{\ell=1}^{K+1}\left(\frac{n_R(\ell)-1+1/q}{n_R(\ell)}\right).$$

Note that by Lemma 1 this implies that the aforementioned change in number of firms in different tiers strictly increases $CV$ of the output of the chain. In order to show that the term above increases, it is enough to show that

$$\left(\frac{n_R(k_1)-1+1/q}{n_R(k_1)}\right)\left(\frac{n_R(k_2)-1+1/q}{n_R(k_2)}\right)<\left(\frac{n_R(k_1)-2+1/q}{n_R(k_1)-1}\right)\left(\frac{n_R(k_2)+1/q}{n_R(k_2)+1}\right).$$

The inequality above holds since for $q<1$ we have

$$\left(\frac{n_R(k_1)-2+1/q}{n_R(k_1)-1}\right)\left(\frac{n_R(k_2)+1/q}{n_R(k_2)+1}\right)-\left(\frac{n_R(k_1)-1+1/q}{n_R(k_1)}\right)\left(\frac{n_R(k_2)-1+1/q}{n_R(k_2)}\right)$$
$$=\frac{\big(n_R(k_2)-n_R(k_1)+1\big)\big(1-q\big)\Big(\big(n_R(k_2)+n_R(k_1)-1\big)q+1\Big)}{\big(n_R(k_1)-1\big)n_R(k_1)n_R(k_2)\big(n_R(k_2)+1\big)q^2}>0.$$

Summarizing, in network $\mathcal{N}_R$ moving one firm from tier $k_1$ to tier $k_2$ weakly increases $s$ and strictly increases the $CV$ of the output. By Lemma 2, we conclude that this increases the retailers' profits. This contradicts our original assumption that $\mathcal{N}_R$ is a network that maximizes the profits of the retailer. Thus, we obtain a contradiction, and conclude that $n_S(k)\geq n_R(k)$. Hence, the claim follows. Q.E.D.

**Proof of Corollary 2**

Observe that the claim trivially holds if $\mathcal{N}_S=\mathcal{N}_R$. Suppose this is not the case. Then, the fact that $\mathcal{N}_S$ and $\mathcal{N}_R$ have the same number of firms, together with Proposition 6 implies the claim. Q.E.D.

**Proof of Theorem 3**

Note that Expression (1) implies that in the end consumer market at most $\alpha/\beta$ units of final goods are sold. Thus, the total monetary transfer from the end consumers to the rest of the supply chain is bounded by $\alpha^2/\beta$. In turn, this implies that at most $\bar{N}=\alpha^2/\beta\kappa$ firms find it optimal to enter the supply chain at equilibrium.

In addition, recall from Proposition 3 that for a given network there exists $q'<1$ such that if $q\geq q'$, then adding a firm to a tier of the supply chain increases the profits of firms in all other tiers. Since at most $\bar{N}$ firms participate in the production process, there are finitely many networks to consider. Hence, there exists some $\hat{q}<1$ such that if $q\geq\hat{q}$, then adding a firm to a tier of the supply chain increases the profits of firms in all other tiers for networks with at most $\bar{N}$ firms. Note that by construction all networks in $\mathcal{W}_\kappa$ have at most $\bar{N}$ firms, and satisfy the aforementioned property. In the remainder of the proof we focus on $q\geq\hat{q}$, and exploit this observation.

In particular, start with a network in $\mathcal{W}_\kappa$ (recall that $\mathcal{W}_\kappa$ is nonempty), and assume that a firm can enter some tier $i$ and earn more than $\kappa$ (if there is no such firm, then the network we start with is an equilibrium network). Assume that this firm joins tier $i$; then Proposition 3 implies that no firm in any other tier finds it optimal to leave the chain. Furthermore, the profits in tier $i$ are decreasing yet they have to be at least as high as the cost of entry $\kappa$ (as otherwise the new firm would not enter). Thus, we obtain another network structure in $\mathcal{W}_\kappa$. Proceeding sequentially (and keeping in mind that for any network in $\mathcal{W}_\kappa$, Proposition 3 applies by our choice of $\hat{q}$), after a finite number of firms enter the chain, we reach an outcome where additional entry would not be profitable (as the maximum total profits for the firms are bounded, and the entry cost $\kappa > 0$).[21] Note that this is an equilibrium outcome (hence, an equilibrium exists) since at each step we obtain a new network in $\mathcal{W}_\kappa$, and reach an outcome where no further entry is profitable.

Finally, we show that there exists an equilibrium network that is maximal. The claim trivially follows if the equilibrium is unique. Suppose that there are two arbitrary equilibrium structures $\mathcal{N}$ and $\mathcal{N}'$, and define structure $\mathcal{N}_{\max}$ as follows: for every tier $1 \le \ell \le K+1$, let $n_{\max}(\ell) = \max\{n(\ell), n'(\ell)\}$. We claim that in the structure $\mathcal{N}_{\max}$ we have $\pi_{\max}(\ell) \ge \kappa$ for all $1 \le \ell \le K+1$. To see this, consider an arbitrary tier $1 \le \ell \le K+1$, and without loss of generality assume that $n(\ell) \ge n'(\ell)$. Then since $n_{\max}(\ell) = n(\ell)$ and for every other tier $k \ne \ell$ we have $n_{\max}(k) \ge n(k)$, it follows from Proposition 3 that $\pi_{\max}(\ell) \ge \pi(\ell) \ge \kappa$. Since $\ell$ is arbitrary it follows that $\mathcal{N}_{\max}$ belongs to $\mathcal{W}_\kappa$. Note that if no other firm finds it profitable to enter $\mathcal{N}_{\max}$, then this network is an equilibrium. Otherwise, after a finite number of (sequential) entries to $\mathcal{N}_{\max}$ as explained before, we reach an equilibrium structure where each tier has weakly more number of firms than the corresponding tier in $\mathcal{N}$ and $\mathcal{N}'$. The above argument implies that given any equilibrium structures $\mathcal{N}$ and $\mathcal{N}'$, we can obtain another equilibrium structure that has (weakly) more firms than both networks in all tiers. Since firms pay an entry cost $\kappa > 0$, it follows that the number of firms in an equilibrium network is bounded, and there are finitely many equilibria. Thus, we conclude that the maximal equilibrium structure exists. Q.E.D.

**Proof of Proposition 7**

Consider an equilibrium chain $\mathcal{N}_\mathcal{E}$. Let $\hat{\pi}_\mathcal{E}(k)$ and $\hat{s}_\mathcal{E}$ denote the profits of firms in tier $k$ and the level of supply procured by tier $K+1$, respectively, in the network that results after an additional firm enters tier $k$ of $\mathcal{N}_\mathcal{E}$. Using Corollary 1, we obtain

$$\begin{aligned}\kappa > \hat{\pi}_\mathcal{E}(k) &= \hat{s}_\mathcal{E}^2 \frac{c(k)}{(n_\mathcal{E}(k)+1)^2} \prod_{\ell=k+1}^{K+1} q^2\Big(\frac{n_\mathcal{E}(\ell) - 1 + 1/q}{n_\mathcal{E}(\ell)}\Big) \\ &> s_\mathcal{E}^2 \frac{c(k)}{(n_\mathcal{E}(k)+1)^2} \prod_{\ell=k+1}^{K+1} q^2\Big(\frac{n_\mathcal{E}(\ell) - 1 + 1/q}{n_\mathcal{E}(\ell)}\Big),\end{aligned} \tag{61}$$

where the second inequality follows from the fact that $\hat{s}_\mathcal{E} > s_\mathcal{E}$ (the level of supply procured by the firms in tier $K+1$ increases when a firm enters the chain as can be directly seen from Proposition 1).

[21] The number of firms that enter each tier cannot grow to infinity. To see this note that according to Corollary 1, we have for the firms' profits in tier $1 \le k \le K+1$,

$$\pi(k) = s^2 \frac{c(k)}{n(k)^2} \prod_{\ell=k+1}^{K+1} \frac{q(\ell)^2}{n(\ell)} \left(n(\ell) - 1 + \frac{1}{q(\ell)}\right).$$

In addition, the supply $s$ is bounded from above for any allocation of firms into the different tiers. Thus, as the number of firms in tier $k$ increases, $\pi(k)$ approaches zero and thus at some point firms' expected profits become smaller than the entry cost $\kappa$.

On the other hand, since $\mathcal{N}_\mathcal{E}$ is an equilibrium network structure, we have that the profits $\pi_\mathcal{E}(k+1)$ of firms in tier $k+1$ (at the corresponding supply equilibrium) are such that $\pi_\mathcal{E}(k+1) \geq \kappa$. This further implies that

$$\pi_\mathcal{E}(k+1) = s_\mathcal{E}^2 \frac{c(k+1)}{n_\mathcal{E}(k+1)^2} \prod_{\ell=k+2}^{K+1} q^2 \Big(\frac{n_\mathcal{E}(\ell) - 1 + 1/q}{n_\mathcal{E}(\ell)}\Big) \geq \kappa. \tag{62}$$

Expressions (61) and (62) yield the following inequality:

$$\frac{c(k+1)}{n_\mathcal{E}(k+1)^2} \prod_{\ell=k+2}^{K+1} q^2 \Big(\frac{n_\mathcal{E}(\ell) - 1 + 1/q}{n_\mathcal{E}(\ell)}\Big) > \frac{c(k)}{(n_\mathcal{E}(k)+1)^2} \prod_{\ell=k+1}^{K+1} q^2 \Big(\frac{n_\mathcal{E}(\ell) - 1 + 1/q}{n_\mathcal{E}(\ell)}\Big),$$

which, in turn, yields

$$\frac{(n_\mathcal{E}(k)+1)^2}{n_\mathcal{E}(k+1)^2} > \frac{c(k)}{c(k+1)} q^2 \Big(\frac{n_\mathcal{E}(k+1) - 1 + 1/q}{n_\mathcal{E}(k+1)}\Big). \tag{63}$$

Finally, inequality (63) implies that

$$n_\mathcal{E}(k) > q n_\mathcal{E}(k+1) \sqrt{\frac{c(k)}{c(k+1)}} \sqrt{\frac{n_\mathcal{E}(k+1) - 1 + 1/q}{n_\mathcal{E}(k+1)}} - 1. \tag{64}$$

Following a similar approach and noting that since $\mathcal{N}_\mathcal{E}$ is an equilibrium chain, no firm has an incentive to enter tier $k+1$ (i.e., using an inequality similar to (61) for tier $k+1$, and an inequality similar to (62) for tier $k$), we can establish that

$$n_\mathcal{E}(k) < q(n_\mathcal{E}(k+1)+1) \sqrt{\frac{c(k)}{c(k+1)}} \sqrt{\frac{n_\mathcal{E}(k+1) - 1 + 1/q}{n_\mathcal{E}(k+1)}}, \tag{65}$$

which completes the first part of the proposition.

For the second part, we assume that $c(k) = c$ for all $1 \leq k \leq K+1$. Then, note that (64) directly implies that

$$\Big\lfloor q \cdot n_\mathcal{E}(k+1) \Big\rfloor \leq n_\mathcal{E}(k),$$

since $\frac{n_\mathcal{E}(k+1)-1+1/q}{n_\mathcal{E}(k+1)} \geq 1$. Also note that if

$$q\Big(n_\mathcal{E}(k+1)+1\Big) \sqrt{\frac{n_\mathcal{E}(k+1) - 1 + \frac{1}{q}}{n_\mathcal{E}(k+1)}} - q n_\mathcal{E}(k+1) \leq 1, \tag{66}$$

then by noting that $n_\mathcal{E}(k)$ is an integer, (65) immediately implies

$$\Big\lceil q \cdot n_\mathcal{E}(k+1) \Big\rceil \geq n_\mathcal{E}(k).$$

Straightforward algebra implies that inequality (66) holds for any $q \leq 1$. Summarizing, we have established that

$$\Big\lfloor q \cdot n_\mathcal{E}(k+1) \Big\rfloor \leq n_\mathcal{E}(k) \leq \Big\lceil q \cdot n_\mathcal{E}(k+1) \Big\rceil. \tag{67}$$

Q.E.D.

## Appendix B: Discussion and Additional Results

Appendix B discusses three results that were omitted from the main body of the paper. In particular, Appendix B.1 establishes a connection between balanced networks and the coefficient of variation of the output that reaches the downstream consumer market. Appendix B.2 considers the extension of our benchmark model to the case where disruptive events in a tier are correlated. Finally, in Appendix B.3 we illustrate potential prescriptive implications of our results.

### B.1. Balanced Networks and Coefficient of Variation

In this section, we establish that more balanced structures induce smaller $CV$. Throughout the section we consider networks with a fixed number $N$ of firms. Before we state our results formally, we define a preorder on network structures with the same number of firms that aims to capture how "balanced" they are.

Definition 2. Suppose networks $\mathcal{N}_1$ and $\mathcal{N}_2$ have the same number of firms. Network $\mathcal{N}_1$ is more balanced than network $\mathcal{N}_2$ if for any two tiers $k_1$ and $k_2$ we have

$$|n_1(k_2)-n_1(k_1)|\le|n_2(k_2)-n_2(k_1)|,$$

where $n_1(k)$ and $n_2(k)$ denote the number of firms in tier $k$ in networks $\mathcal{N}_1$ and $\mathcal{N}_2$, respectively. The result that follows relates this preorder to the coefficient of variation of the corresponding outputs.

Lemma 8. *Let $q(k)=q<1$ for all tiers $1\le k\le K+1$ and assume that network $\mathcal{N}_1$ is more balanced than network $\mathcal{N}_2$. Then, the coefficient of variation of the supply chain's output corresponding to $\mathcal{N}_1$ is (weakly) smaller than that corresponding to $\mathcal{N}_2$, i.e., $CV(U_{\mathcal{N}_1})\le CV(U_{\mathcal{N}_2})$.*

*Proof:* The claim trivially holds if $\mathcal{N}_1=\mathcal{N}_2$. Suppose this is not the case. Consider the following sets $S_1$ and $S_2$:

$$\begin{aligned} S_1 &\triangleq \{1\le k\le K+1 | n_2(k)<n_1(k)\},\\ S_2 &\triangleq \{1\le k\le K+1 | n_2(k)>n_1(k)\}. \end{aligned} \tag{68}$$

Since $\mathcal{N}_1\neq\mathcal{N}_2$ and these networks have the same number $N$ of firms, it follows that $S_1,S_2\neq\emptyset$.

First, note that for any two tiers $k_1\in S_1$ and $k_2\in S_2$ we should have $n_2(k_1)<n_2(k_2)$, since otherwise we would have $|n_2(k_1)-n_2(k_2)|<|n_1(k_1)-n_1(k_2)|$, which contradicts the assumption that network $\mathcal{N}_1$ is more balanced than $\mathcal{N}_2$. To see this, assume, by way of contradiction, that there exist tiers $k_1\in S_1$ and $k_2\in S_2$ such that $n_2(k_1)\ge n_2(k_2)$. Since we have $n_1(k_1)>n_2(k_1)$ and $n_1(k_2)<n_2(k_2)$ we obtain

$$n_1(k_2)<n_2(k_2)\le n_2(k_1)<n_1(k_1),$$

which in turn implies that $|n_2(k_1)-n_2(k_2)|<|n_1(k_1)-n_1(k_2)|$, contradicting the assumption that $\mathcal{N}_1$ is more balanced.

Fix some $k_1 \in S_1$ and $k_2 \in S_2$. We claim:

$$\left(\frac{n_2(k_1)-1+\frac{1}{q}}{n_2(k_1)}\right)\left(\frac{n_2(k_2)-1+\frac{1}{q}}{n_2(k_2)}\right) \geq \left(\frac{n_2(k_1)+\frac{1}{q}}{n_2(k_1)+1}\right)\left(\frac{n_2(k_2)-2+\frac{1}{q}}{n_2(k_2)-1}\right). \tag{69}$$

To see this, note that we have

$$\begin{aligned}&\left(\frac{n_2(k_1)-1+\frac{1}{q}}{n_2(k_1)}\right)\left(\frac{n_2(k_2)-1+\frac{1}{q}}{n_2(k_2)}\right) - \left(\frac{n_2(k_1)+\frac{1}{q}}{n_2(k_1)+1}\right)\left(\frac{n_2(k_2)-2+\frac{1}{q}}{n_2(k_2)-1}\right)\\ &\qquad = \frac{\big(n_2(k_2)-n_2(k_1)-1\big)\big(1-q\big)\Big(\big(n_2(k_1)+n_2(k_2)-1\big)q+1\Big)}{n_2(k_1)\big(n_2(k_1)+1\big)\big(n_2(k_2)-1\big)n_2(k_2)q^2} \geq 0,\end{aligned} \tag{70}$$

where the inequality holds since $n_2(k_2) > n_2(k_1)$.

Finally, Expression (55) together with the inequality above imply that in $\mathcal{N}_2$ after moving a firm from $k_2$ to $k_1$, the $CV$ of the resulting network is at most as high as the $CV$ of the original network. Finally, noting that network $\mathcal{N}_1$ can be obtained from $\mathcal{N}_2$ after moving a certain number of firms belonging to tiers in $S_2$ to tiers in $S_1$ one after the other, we conclude that $CV(U_{\mathcal{N}_2}) \geq CV(U_{\mathcal{N}_1})$ (note that after each such move, the following still holds $|n_1(k_1)-n_1(k_2)| \leq |n_2(k_1)-n_2(k_2)|$ for $k_1 \in S_1$ and $k_2 \in S_2$.) □

This result establishes that when a network becomes less balanced, e.g., due to having too few firms in one tier, while having many more in another, then the coefficient of variation of the output increases. The increase in the output variability is intuitive, since if the network has a tier with a few firms, in case of disruption in this tier, the output is significantly reduced.

Our next result complements the previous one by showing that the $CV$ minimizing networks are the most balanced ones, in the sense that the number of firms in different tiers is (almost) the same.[22]

Lemma 9. *Let $q(k) = q < 1$ for all tiers $1 \leq k \leq K+1$, and denote by $\mathcal{N}$ the network with $N$ firms that minimizes $CV$. The network $\mathcal{N}$ is such that for any two tiers $1 \leq k_1, k_2 \leq K+1$, we have $|n(k_1)-n(k_2)| \leq 1$.*

*Proof:* By way of contradiction, assume that for two tiers $k_1$ and $k_2$, $|n(k_1)-n(k_2)| \geq 2$, and without loss of generality assume $n(k_2) > n(k_1)$. Then moving one firm from tier $k_2$ to tier $k_1$ decreases $CV$. Note that to establish this claim it is enough to show that

$$\left(\frac{n_2(k_1)-1+\frac{1}{q}}{n_2(k_1)}\right)\left(\frac{n_2(k_2)-1+\frac{1}{q}}{n_2(k_2)}\right) > \left(\frac{n_2(k_1)+\frac{1}{q}}{n_2(k_1)+1}\right)\left(\frac{n_2(k_2)-2+\frac{1}{q}}{n_2(k_2)-1}\right), \tag{71}$$

which follows from equality (70) by noting that $n(k_2) > n(k_1)+1$. □

[22] Note that for a given number $N$ of firms, it may not be possible to have the same number of firms in all tiers. To account for this possibility, our result allows for the number of firms in different tiers to differ by one.

## B.2. Correlated Disruptions

Assume that for any two firms $i, j$ in $Tier(k)$ we have $Cov(Z_{k,i}, Z_{k,j}) = \delta(k)$, where we recall that $Z_{k,i}$ is a Bernoulli random variable that captures whether a disruptive event has occurred in firm $i$. Suppose that independence of disruptions across tiers is still preserved (i.e., $Cov(Z_{k,i}, Z_{\ell,j}) = 0$ for $k \neq \ell$). Theorem 4 below generalizes Theorem 1 to this case where disruptive events in a tier may be correlated. Before stating the theorem, we introduce some notation (analogous to (3) and (4)) and state an assumption analogous to Assumption 1.

In particular, we let

$$\hat{\alpha}(k) = \alpha \prod_{\ell=1}^{k-1} q(\ell),$$

$$\hat{\beta}(k) = \begin{cases} \beta & \text{if } k = 1, \\ \hat{\beta}(k-1)\dfrac{\delta(k-1)+q(k-1)^2}{n(k-1)}\left(n(k-1)-1+\dfrac{q(k-1)}{\delta(k-1)+q(k-1)^2}\right)+\dfrac{2c(k-1)}{n(k-1)} & \text{if } 1 < k \leq K+2. \end{cases}$$

In addition, we state the following assumption.

Assumption 2. *The supply chain network is such that:*

*(i)* $\hat{\alpha}(K+2) > p_c$,

*(ii)* $\dfrac{\hat{\alpha}(k)}{\hat{\beta}(k)} > \dfrac{\hat{\alpha}(K+2)-p_c}{\hat{\beta}(K+2)}$, *for* $k \in \{1,\dots,K+1\}$.

Then, we obtain the following theorem, which is analogous to Theorem 1.

Theorem 4. *Suppose that Assumption 2(i) holds. Then, the supply equilibrium is essentially unique. In addition, if Assumption 2(ii) holds, the (essentially unique) equilibrium can be characterized as follows:*

*(i) The aggregate supply $s$ of raw materials is given by*

$$s = \frac{\hat{\alpha}(K+2)-p_c}{\hat{\beta}(K+2)}. \tag{72}$$

*(ii) The price for the output of tier $k$ when the state is $\omega_{k-1}$ is given by*

$$p_k(\omega_{k-1}) = \hat{\alpha}(k) - \hat{\beta}(k) s \prod_{\ell=k}^{K+1} \frac{n(\ell,\omega_{k-1})}{n(\ell)} > 0, \tag{73}$$

*for all $k \in \{1,\dots,K+1\}$ and $\omega_{k-1} \in \Omega_{k-1}$. Here, we let $n(\ell,\omega_{k-1})$ denote the number of firms in tier $\ell \geq k$ that did not experience a disruption at state $\omega_{k-1}$.*

*(iii) The procurement quantity of firm $i$ in tier $k$ when the state is $\omega_k$ is given by*

$$x_{k,i}(\omega_k) = \frac{s}{n(k)} \prod_{\ell=k+1}^{K+1} \frac{n(\ell,\omega_k)}{n(\ell)}, \tag{74}$$

*for all $k \in \{1,\dots,K+1\}$ and $\omega_k \in \Omega_k$.*

*Finally, if Assumption 1(ii) does not hold, then at any supply equilibrium, there exists at least one tier $k' \in \{1, \ldots, K+1\}$, such that $p_{k'}(\bar{\omega}_{k'-1}) = 0$, where $\bar{\omega}_{k'-1} \in \Omega_{k'-1}$ is the state where no firm experiences a disruption in tiers $\{k', \ldots, K+1\}$.*

It can be readily seen that the proofs of the essential uniqueness of the equilibrium (under Assumption 2(i)) and its characterization when Assumption 2(ii) does not hold follow using essentially the same arguments as in the proofs of the corresponding claims in Theorem 1 (with the only difference that we need to take into account the correlation between disruptions in the same tier when writing Expression (35)). Similarly, in order to establish that the prices and procurement quantities given in Expressions (72), (73), and (74) constitute an equilibrium, as in the proof of Theorem 1, we use the first order optimality conditions for the firms' expected profit maximization problems. In the setting of Theorem 4, when evaluating firms' expected profits, we need to take into account the correlation in disruptions affecting firms in the same tier. In particular, this implies that instead of Expression (26), we have

$$\mathbb{E}[Z_{k,i} n(k, \hat{\omega}_{k-1}) \mid \hat{\omega}_k = \omega_k] = \mathbb{E}\left[Z_{k_i} + \sum_{j \in Tier(k) | j \neq i} Z_{k,i} Z_{k,j}\right] = q(k) + \Bigg(q(k)^2 + \delta(k)\Bigg)(n(k) - 1).$$

Since the proof generally follows the same steps as the proof of Theorem 1, it is omitted for brevity.

### B.3. Prescriptive Implications

The closed-form characterization of profits given in Corollary 1 is useful for understanding how implementing a strategic initiative may impact firms' profits, and consequently inform managerial decision making. In this section, we illustrate this observation by investigating how investing in improving the monitoring/production reliability of a given tier (or other interventions that effectively lead to a higher $q(k)$) may affect the profits of a firm, and how a manager should prioritize investing in such interventions among different tiers of the chain.

In particular, we consider a retailer investing in decreasing the disruption risk associated with one of the intermediate stages of production. Assuming that such efforts incur the same cost irrespective of the tier they are targeted to, we provide some understanding of the potential return investing in a tier would yield as a function of the primitives of the environment.

Formally, we say that it is optimal for a downstream retailer to invest in decreasing the disruption risk of tier $k$ if $\frac{d \log \pi(1)}{dq(k)}$ is maximized for $k$. Intuitively, this is the tier where a marginal improvement in disruption probability has the largest impact on the retailers' profits. The proposition below provides a characterization of such an optimal tier for the retailers as a function of the modeling primitives. To simplify the exposition and obtain a crisp characterization, the proposition focuses on the setting where the variable costs associated with production are positive but negligible for all intermediate stages of productions, i.e., $c(k) \to 0$ for $k > 1$, whereas $c(1) > 0$ (note that this implies that roughly the entire surplus generated by the supply network goes to the downstream retailers).[23]

PROPOSITION 8. *Consider a setting where $c(k) \to 0$ for all $1 < k \le K+1$. Then, it is optimal for the downstream retailer to invest in decreasing the disruption risk of tier $k$ that maximizes the following expression:*

$$\frac{1}{q(k)}\left(\frac{2\alpha\prod_{\ell=1}^{K+1} q(\ell)}{\alpha\prod_{\ell=1}^{K+1} q(\ell) - p_c} - 2 + \frac{1/q(k)}{n(k) - 1 + \frac{1}{q(k)}}\right). \tag{75}$$

*Proof:* The profits for a downstream retailer are given by the following expression from Corollary 1:

$$\pi(1) = s^2 \frac{c(1)}{n(1)^2} \prod_{\ell=2}^{K+1} \frac{q(\ell)^2}{n(\ell)}\Big(n(\ell) - 1 + 1/q(\ell)\Big). \tag{76}$$

Also, the aggregate supply $s$ at equilibrium is given by

$$s = \frac{\alpha\prod_{\ell=1}^{K+1} q(\ell) - p_c}{\beta(K+2)}.$$

We are interested in finding the tier $k$ for which the derivative $\frac{d\pi(1)}{dq(k)}$ is maximized. Note that this is equivalent to finding $k$ for which $\frac{d\log\pi(1)}{dq(k)}$ is maximized. We can write the logarithm of Expression (76) as

$$\begin{aligned}\log\pi(1) =& 2\log\left(\alpha\prod_{\ell=1}^{K+1} q(\ell) - p_c\right) - 2\log\left(\beta\prod_{\ell=1}^{K+1} q(\ell)^2\frac{n(\ell) - 1 + \frac{1}{q(\ell)}}{n(\ell)} + \sum_{r=1}^{K+1}\frac{2c(r)}{n(r)}\prod_{\ell=r+1}^{K+1} q(\ell)^2\frac{n(\ell) - 1 + \frac{1}{q(\ell)}}{n(\ell)}\right) \\ &+ \log\left(\frac{c(1)}{n(1)^2}\prod_{\ell=2,\ell\neq k}^{K+1} q(\ell)^2\frac{n(\ell) - 1 + \frac{1}{q(\ell)}}{n(\ell)}\right) + \log\left(q(k)^2\frac{n(k) - 1 + \frac{1}{q(k)}}{n(k)}\right).\end{aligned}$$

Thus, we can write $\frac{d\log\pi(1)}{dq(k)}$ as follows:

$$\begin{aligned}\frac{\partial\log\pi(1)}{\partial q(k)} =& \frac{2\alpha\prod_{\ell=1,\ell\neq k}^{K+1} q(\ell)}{\alpha\prod_{\ell=1}^{K+1} q(\ell) - p_c} + \frac{2q(k)(n(k)-1)+1}{q(k)^2\big(n(k) - 1 + \frac{1}{q(k)}\big)} \\ &- 2\frac{\beta\frac{2q(k)\left(n(k)-1\right)+1}{n(k)}\prod_{\ell=1,\ell\neq k}^{K+1} q(\ell)^2\frac{n(\ell)-1+\frac{1}{q(\ell)}}{n(\ell)} + \sum_{r=1}^{k-1}\frac{2c(r)}{n(r)}\frac{2q(k)\left(n(k)-1\right)+1}{n(k)}\prod_{\ell=r+1,\ell\neq k}^{K+1} q(\ell)^2\frac{n(\ell)-1+\frac{1}{q(\ell)}}{n(\ell)}}{\beta\prod_{\ell=1}^{K+1} q(\ell)^2\frac{n(\ell)-1+\frac{1}{q(\ell)}}{n(\ell)} + \sum_{r=1}^{K+1}\frac{2c(r)}{n(r)}\prod_{\ell=r+1}^{K+1} q(\ell)^2\frac{n(\ell)-1+\frac{1}{q(\ell)}}{n(\ell)}}.\end{aligned}$$

Then, by using $c(k) \to 0$ for $k > 1$, we can rewrite the above as follows:

$$\frac{\partial\log\pi(1)}{\partial q(k)} = \frac{2\alpha\prod_{\ell=1,\ell\neq k}^{K+1} q(\ell)}{\alpha\prod_{\ell=1}^{K+1} q(\ell) - p_c} - \frac{2q(k)(n(k)-1)+1}{q(k)^2\big(n(k) - 1 + \frac{1}{q(k)}\big)},$$

[23] Similar insights hold in the general setting as well but obtaining analytical expressions is much more challenging.

which, in turn, implies that it is optimal for a downstream retailer to invest in decreasing the disruption risk of tier $k$ for which the following expression is maximized

$$\frac{1}{q(k)}\left(\frac{2\alpha\prod_{\ell=1}^{K+1}q(\ell)}{\alpha\prod_{\ell=1}^{K+1}q(\ell)-p_c}-\frac{2(n(k)-1)+1/q(k)}{n(k)-1+\frac{1}{q(k)}}\right).$$

Finally, we can rewrite the above as

$$\frac{1}{q(k)}\left(\frac{2\alpha\prod_{\ell=1}^{K+1}q(\ell)}{\alpha\prod_{\ell=1}^{K+1}q(\ell)-p_c}-2+\frac{1/q(k)}{n(k)-1+\frac{1}{q(k)}}\right),$$

which completes the proof of the claim. □

As one would intuitively expect, if the number of firms in each tier is the same, Expression (75) implies that it is optimal to invest in the tier for which the disruption risk is the highest. On the other hand, this conclusion no longer holds if the number of firms in different tiers is different. In particular, it may no longer be optimal to invest in the tier that is most prone to disruptions, i.e., the tier with the lowest $q(k)$, or the retailers' direct suppliers, i.e., firms in tier 2. In general, retailers may find it optimal to concentrate their efforts on a different tier depending on the environment they operate in. We illustrate this with an example where the chain consists of $K+1=3$ tiers with $n(3)=5$ firms, $n(2)\in\{3,5,7\}$ firms, and $n(1)=1$ retailer. For each value of $n(2)$ we compare (75) for tiers $k=2$ and $k=3$, and obtain $(q(2),q(3))$ pairs for which it is optimal to invest in tier $k=2$ or $k=3$. The results are illustrated in Figure 4.

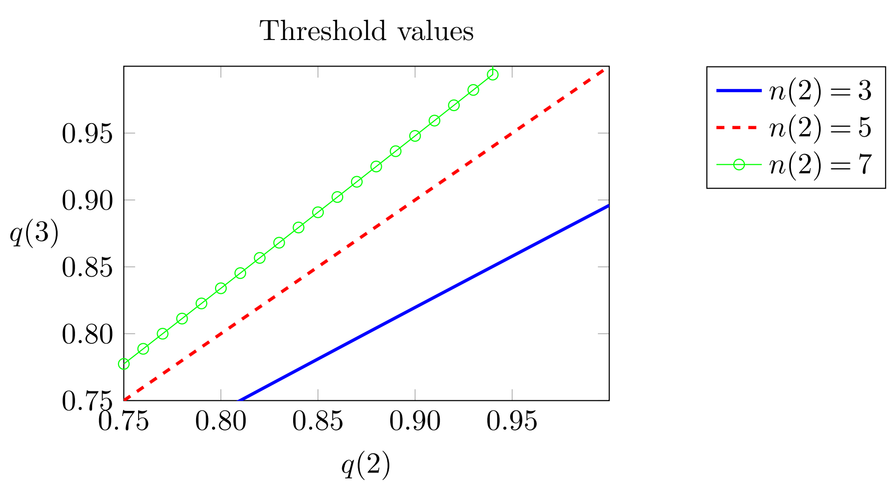


**Figure 4** **Each of the curves in the figure corresponds to a different value for $n(2)$, i.e., $n(2)\in\{3,5,7\}$, when $n(3)=5$. For $(q(2),q(3))$ pairs that lie above the corresponding curve, it is optimal to invest in tier 2, whereas otherwise it is optimal to invest in tier 3. Here, the probability of successful production in tier 1 is $q(1)=1$, $p_c=0.5$, and $\alpha=2$.**

Despite its simplicity, this example clearly illustrates the importance of results such as Theorem 1 and Corollary 1 in prescribing managerial decision-making in multi-tier supply chains: although intuitively it may seem best for a firm to focus on its direct suppliers (as their operations have an immediate impact on the firm's profits), this often leads to suboptimal returns. Thus, developing an understanding of the firm's supply chain structure and taking into account how relationships between its direct and indirect suppliers affect its profits may be necessary for its strategic planning.