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# Optimal Pricing in Networks with Externalities

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We study the optimal pricing strategies of a monopolist selling a divisible good (service) to consumers who are embedded in a social network. A key feature of our model is that consumers experience a (positive) *local network effect*. In particular, each consumer's usage level depends directly on the usage of her *neighbors* in the social network structure. Thus, the monopolist's optimal pricing strategy may involve offering discounts to certain agents who have a *central* position in the underlying network. Our results can be summarized as follows. First, we consider a setting where the monopolist can offer individualized prices and derive a characterization of the optimal price for each consumer as a function of her network position. In particular, we show that it is optimal for the monopolist to charge each agent a price that consists of three components: (i) a nominal term that is independent of the network structure, (ii) a discount term proportional to the influence that this agent exerts over the rest of the social network (quantified by the agent's *Bonacich centrality*), and (iii) a markup term proportional to the influence that the network exerts on the agent. In the second part of the paper, we discuss the optimal strategy of a monopolist who can only choose a single uniform price for the good and derive an algorithm polynomial in the number of agents to compute such a price. Third, we assume that the monopolist can offer the good in two prices, full and discounted, and we study the problem of determining which set of consumers should be given the discount. We show that the problem is NP-hard; however, we provide an explicit characterization of the set of agents who should be offered the discounted price. Next, we describe an approximation algorithm for finding the optimal set of agents. We show that if the profit is nonnegative under any feasible price allocation, the algorithm guarantees at least 88% of the optimal profit. Finally, we highlight the value of network information by comparing the profits of a monopolist who does not take into account the network effects when choosing her pricing policy to those of a monopolist who uses this information optimally.

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## 1. Introduction

Inarguably, social networks that describe the pattern and level of interaction of a set of agents[1] are instrumental in the propagation of information and act as conduits of influence among its members. Their importance is best exemplified by the overwhelming success of online social networking communities such as Facebook and Twitter. The ubiquity of these Internet-based services that are built around social networks has made possible the collection of vast amounts of data on the structure and intensity of social interactions. The question that arises naturally is whether firms can intelligently use the available data to improve their business strategies.

In this paper, we focus on the question of using the potentially available data on network interactions to improve the pricing strategies of a seller who offers a divisible good (service). A main feature of the products we consider is that they exhibit a local (positive) *network effect*: increasing the usage level of a consumer has a positive impact on the usage levels of her peers. As concrete examples of such goods, consider online games (e.g., World of Warcraft, Second Life) and social networking tools and communities (e.g., online dating services, employment websites, etc.). More generally, the local network effect can capture *word-of-mouth* communication among agents: agents typically form their opinions about the quality of a product based on the information they obtain from their peers.

How can a monopolist exploit the above network effects and maximize her revenues? In particular, in such a setting it is plausible that an optimal pricing strategy may involve favoring certain agents by offering the good at a discounted price and subsequently exploiting the positive effect of their usage on the rest of the consumers. At its extreme, such a scheme would offer the product for free to a subset of consumers, hoping that this would have a large positive impact on the purchasing decisions of the rest. Although such strategies have been used extensively in practice, mainly in the form of ad hoc or heuristic mechanisms, the available data enable companies to effectively target the agents to maximize that impact.

The goal of the present paper is to characterize optimal pricing strategies as a function of the underlying social interactions in a stylized model, which features consumers that are embedded in a given social network and influencing each other's decisions. In particular, a monopolist first chooses a pricing strategy and then consumers choose their usage levels so as to maximize their own utility. We capture the local positive network effect by assuming that a consumer's utility is increasing in the usage level of her peers. Specifically, we assume that a consumer's utility function is quadratic in her own usage level and linear in the usage level of her peers. This linear-quadratic functional form is important for tractability and enables us to obtain structural insights on the optimal pricing strategies.

We study three variations of the baseline model by imposing different assumptions on the set of available pricing strategies that the monopolist can implement. First, we allow the monopolist to set an individual price for each of the consumers. We show that the optimal price for each agent can be decomposed into three components: a fixed cost that does not depend on the network structure, a markup, and a discount. Both the markup and the discount are proportional to the *Bonacich centrality* of the agent's neighbors in the social network structure, which is a sociological measure of network influence. The Bonacich centrality measure, introduced by Bonacich (1987), can be understood in terms of a random walk on the underlying network structure. The agents with the highest centrality are the ones that are visited by the random walk most frequently. Intuitively, an agent is *central* in the Bonacich sense if she is connected to other central agents. In social networks, two agents with the same degree, i.e., same number of direct connections, may not exert the same level of influence when one of these agents is connected to more central agents. The Bonacich centrality measure captures this phenomenon and identifies agents whose peers are also central as the central agents. Informally, we show that agents get a discount proportional to the amount they influence their peers to purchase the product, and they receive a markup if they are strongly influenced by other agents in the network. Our results also provide an economic foundation for this sociological measure of influence.

Perfect price differentiation is typically hard to implement. Therefore, in the second part of the paper we study a setting where the monopolist offers a single uniform price for the good. Intuitively, this price might make the product unattractive for a subset of consumers who end up not purchasing, but the monopolist recovers the revenue losses from the rest of the consumers. We develop an algorithm that finds the optimal single price in time polynomial in the number of agents. The algorithm considers different subsets of the consumers and finds the optimal price provided that only the consumers in the given subset purchase a positive amount of the good. First, we show that given a subset $S$ we can find the optimal price $p_S$ under the above constraint in closed form. Then, we show that we only need to consider a small number of such subsets. In particular, we rank the agents with respect to a weighted centrality index and iteratively construct a sequence of sets by removing the consumer with the smallest such index at each step. The optimal solution is then obtained by comparing the profits the monopolist makes for each of those subsets.

Finally, we consider an intermediate setting, where the monopolist can choose one of a small number of prices for each agent. For expositional purposes, we restrict the discussion to two prices, *full* and *discounted*. Unlike the previous two settings, we first consider the case when the two prices are exogenously given, and we study the problem of determining the optimal subset of consumers to offer the discounted price. We show that the problem is NP-hard,[2] and we provide an approximation algorithm that recovers (in polynomial time) at least 88% of the optimal revenue for the case of two prices. Finally, we discuss how we can relax the assumption of exogenous prices and provide a simple procedure that can be used to search for the best full and discounted prices.

To further highlight the importance of network effects, we compare the profits of a monopolist who ignores them when choosing her pricing policy to those of a monopolist who exploits them optimally. We are able to provide a concise characterization of the discrepancy in profits as a function of the level of interaction between the agents. Informally, the value of information about the network structure increases with the level of asymmetry of interactions among the agents.

As mentioned above, a main feature of our model is the positive impact of a consumer's purchasing decision to the purchasing behavior of other consumers. This effect, known as *network externality*, is extensively studied in the literature (e.g., Farrell and Saloner 1985, Katz and Shapiro 1986, and, more recently, Johari and Kumar 2010). However, the network effects in those studies are of *global* nature, i.e., the utility of a consumer depends directly on the behavior of the whole set of consumers. In our model, consumers interact directly only with a subset of agents. Although interaction is local for each consumer, her utility may depend on the global structure of the network, because each consumer potentially interacts indirectly with a much larger set of agents than just her peers.

Given a set of prices, our model takes the form of a *network game* among agents that interact locally. A recent series of papers studies such games, e.g., Ballester et al. (2006), Bramoullé and Kranton (2007), Corbo et al. (2007), Galeotti and Goyal (2009). A key modeling assumption in Ballester et al. (2006), Bramoullé and Kranton (2007), Corbo et al. (2007) and Bramoullé et al. (2012), which we also adopt in our setting, is that the best reply functions are linear. Ballester et al. (2006) were the first to note the linkage between Bonacich centrality and Nash equilibrium outcomes in a single-stage game with local payoff complementarities. Our characterization of optimal prices when the monopolist can perfectly price differentiate is reminiscent of their results, because prices are inherently related to the Bonacich centrality of each consumer. However, both the motivation and the analysis are quite different, because ours is a two-stage game, where a monopolist chooses prices to maximize her revenue subject to equilibrium constraints. Also, Bramoullé and Kranton (2007), Corbo et al. (2007), and, most recently, Bramoullé et al. (2012) study a similar game to the one in Ballester et al. (2006) and interpret their results in terms of public good provision. A number of recent papers (Campbell 2012, Galeotti et al. 2010, and Sundararajan 2007) have a similar motivation to ours, but take a completely different approach: they make the assumption of limited knowledge of the social network structure, i.e., they assume that only the degree distribution is known, and thus derive optimal pricing strategies that depend on this first degree measure of influence of a consumer. In our model, we make the assumption that the monopolist has complete knowledge of the social network structure and, thus, obtain qualitatively different results: the degree is not the appropriate measure of influence, but rather prices are proportional to the Bonacich centrality of the agents. On the technical side, note that assuming more global knowledge of the network structure increases the complexity of the problem in the following way: if only the degree of an agent is known, then essentially there are as many different *types* of agents as there are different degrees. This is no longer true when more is known: then, two agents of the same degree may be of different type because of the difference in the characteristics of their neighbors, and therefore, optimal prices charged to agents may be different.

Furthermore, there is a recent stream of literature in computer science that studies a set of algorithmic questions related to marketing strategies over social networks. Kempe et al. (2003) discuss optimal *network-seeding* strategies over social networks, when consumers act myopically according to a prespecified rule of thumb. In particular, they distinguish between two basic models of diffusion: the *linear threshold model*, which assumes that an agent adopts a behavior as soon as adoption in her neighborhood of peers exceeds a given threshold; and *independent cascade model*, which assumes that an adopter *infects* each of her neighbors with a given probability. The main question they ask is finding the optimal set of initial adopters, when their number is given, so as to maximize the eventual adoption of the behavior, when consumers behave according to one of the diffusion models described above. They show that the problem of *influence maximization* is NP-hard and provide a greedy heuristic that achieves a solution that is provably within 63% of the optimal.

More recently, Hartline et al. (Hartline et al. 2008—see also follow-up work by Akhlaghpour et al. 2010 and Haghpanah et al. 2011) discuss the optimal marketing strategies of a monopolist. Specifically, they assume a general model of influence, where an agent's willingness to pay for the good is given by a function of the subset of agents that have already bought the product, i.e., $u_i: 2^V \to \mathbb{R}_+$, where $u_i$ is the willingness to pay for agent $i$ and $V$ is the set of consumers. They restrict the monopolist to the following set of *marketing strategies*: the seller visits the consumers in some sequence and makes a take-it-or-leave-it offer to each one of them. Both the sequence of visits as well as the prices are chosen by the monopolist. They provide a dynamic programming algorithm that outputs the optimal pricing strategy for a symmetric setting, i.e., when the agents are ex ante identical (the sequence of visits is irrelevant in this setting). Not surprisingly, the optimal strategy offers discounts to the consumers that are visited earlier in the sequence and then extracts revenue from the rest. The general problem, when agents are heterogeneous, is NP-hard; thus, they consider approximation algorithms. They show, in particular, that *influence-and-exploit* strategies that offer the product for free to a strategically chosen set $A$, and then offer the myopically optimal price to the remaining agents, provably achieve a constant factor approximation of the optimal revenues under some assumptions on the influence model. However, this paper does not provide a qualitative insight on the relation between optimal strategies and the structure of the social network. In contrast, we are mainly interested in characterizing the optimal strategies as a function of the underlying network.

The most closely related paper to ours is an independent work by Bloch and Quérou (2011). They study a pricing setting with linear utility functions and private valuations. They consider externalities resulting either from local network interactions or from prices (aspiration-based reference pricing), and they distinguish between a single, global monopoly and several local monopolies. Their results also point out the importance of network centrality for pricing decisions; however, their model and overall pricing game is considerably different than ours.

The rest of paper is organized as follows. Section 2 introduces the model. In §3, we begin our analysis by characterizing the usage level of the consumers at equilibrium given the vector of prices chosen by the monopolist. In §4 we turn attention to the pricing stage (first stage of the game) and characterize the optimal strategy for the monopolist under three different settings: when the monopolist can perfectly price discriminate (§4.1), when the monopolist chooses a single uniform price for all consumers (§4.2), and finally, when the monopolist can choose between two exogenously given prices, the full and the discounted (§4.3). In §5, we compare the profits of a monopolist that has no information about the network structure (and thus chooses her pricing strategy as if consumers did not interact with one another) with those of a monopolist that has full knowledge over the network structure and can perfectly price discriminate consumers. Finally, we conclude in §6. To ease exposition of our results, we decided to relegate the proofs to the appendix.

## 2. Model

The society consists of a set $\mathcal{I} = \{1, \dots, n\}$ of agents embedded in a social network represented by the adjacency matrix $G$. The $ij$th entry of $G$, denoted by $g_{ij}$, represents the *strength* of the influence of agent $j$ on $i$. We assume that $g_{ij} \geqslant 0$ for all $i, j$, and we normalize $g_{ii} = 0$ for all $i$. A monopolist introduces a *divisible* good in the market and chooses a vector $\mathbf{p}$ of prices from the set of allowable *pricing strategies* $\mathbf{P}$. In its full generality, $\mathbf{p} \in \mathbf{P}$ is simply a mapping from the set of agents to $\mathbb{R}^n$, i.e., $\mathbf{p}$: $\mathcal{I} \to \mathbb{R}^n$. In particular, $\mathbf{p}(i)$, or equivalently $p_i$, is the price that the monopolist offers to agent $i$ for one unit of the divisible good. Then, the agents choose the amount of the divisible good they will purchase at the announced price. Their utility is given by an expression of the following form:

$$u_i(x_i, \mathbf{x_{-i}}, p_i) = a_i x_i - b_i x_i^2 + x_i \cdot \sum_{j \in \{1, \dots, n\}} g_{ij} \cdot x_j - p_i x_i, \quad (1)$$

where $x_i \in [0, \infty)$ is the amount of the divisible good that agent $i$ chooses to purchase, and $\mathbf{x_{-i}}$ denotes the consumption levels of all agents but $i$. The first two terms represent the utility that agent $i$ derives from consuming $x_i$ units of the good irrespective of the consumption of her peers, and are quantified by the model parameters $a_i$ and $b_i$. The third term represents the (positive) network effect of her social group and, finally, the last term is the cost of usage. The quadratic form of the utility function allows for tractable analysis, but also serves as a good second-order approximation for the broader class of concave payoffs.

We next describe the two-stage *pricing-consumption* game that models the interaction between the agents and the monopolist:

*Stage* 1 (*Pricing*). The monopolist chooses the pricing strategy $\mathbf{p}$ so as to maximize profits, i.e., $\max_{\mathbf{p} \in \mathbf{P}} \sum_i p_i x_i - c x_i$, where $c$ denotes the marginal cost of producing a unit of the good and $x_i$ denotes the amount of the good that agent $i$ purchases in the second stage of the game.

*Stage* 2 (*Consumption*), Agent $i$ chooses to purchase $x_i$ units of the good, so as to maximize her utility given the prices chosen by the monopolist and $\mathbf{x}_{-i}$, i.e.,

$$x_i \in \arg\max_{y_i \in [0, \infty)} u_i(y_i, \mathbf{x_{-i}}, p_i).$$

We are interested in the *subgame-perfect* equilibria of the two-stage pricing-consumption game.

For a fixed vector of prices $\mathbf{p} = [p_i]_i$ chosen by the monopolist, the equilibria of the second-stage game, referred to as the consumption equilibria, are defined as follows:

Definition 1 (Consumption Equilibrium). For a given vector of prices $\mathbf{p}$, a vector $\mathbf{x}$ is a consumption equilibrium if, for all $i \in \mathcal{I}$,

$$x_i \in \arg\max_{y_i \in [0, \infty)} u_i(y_i, \mathbf{x_{-i}}, p_i).$$

We denote the set of consumption equilibria at a given price vector $\mathbf{p}$ by $C[\mathbf{p}]$.

For a given vector of prices $\mathbf{p}$, we denote by $\mathcal{G} = \{\mathcal{I}, \{u_i\}_{i \in \mathcal{I}}, [0, \infty)_{i \in \mathcal{I}}\}$ the second-stage game where the set of players is $\mathcal{I}$, each player $i \in \mathcal{I}$ chooses her strategy (consumption level) from the set $[0, \infty)$, and her utility function, $u_i$, has the form in (1). We make the following assumption that ensures that in this game the optimal consumption level of each agent is bounded.

Assumption 1. *For all* $i \in \mathcal{I}$, $2b_i > \sum_{j \in \mathcal{I}} g_{ij}$.

Note that this assumption does not imply that the network effects are small. In particular, an agent whose neighbors consume large amounts of the good might be deriving most of her utility through the networks effects. The assumption imposes that after a significant amount of consumption, there is negative marginal utility in increasing one's consumption. The importance of Assumption 1 is evident from the following example: assume that the adjacency matrix, which represents the level of influence among agents, takes the following simple form: $g_{ij} = 1$ for all $i, j$ such that $i \neq j$, i.e., $G$ represents a complete graph with unit weights. Also, assume that $0 < b_i = b < (n-1)/2$ and $0 < a_i = a$ for all $i \in \mathcal{I}$. It can be seen from (1) that in this setting, for any given vector of prices $\mathbf{p}$, starting from consumption levels $x_i = x_0$ for all $i \in \mathcal{I}$, all agents have incentive to increase their consumptions provided that $x_0$ is sufficiently large. Thus, if Assumption 1 does not hold, in the consumption game, consumers may choose to unboundedly increase their usage irrespective of the vector of prices.[3]

We begin our analysis by the second stage (the consumption subgame, studied in §3) and then discuss the optimal pricing policies for the monopolist given that agents purchase according to the consumption equilibrium of the subgame defined by the monopolist's choice of prices. As already mentioned in the introduction, we consider three variants of the monopolist's pricing problem. In the first, we allow the monopolist to fully price discriminate among the users, i.e., charge an individual price for each one of them (§4.1). Then, we study a setting where the monopolist offers a single uniform price for the good (§4.2) and, finally, we consider an intermediate setting where the monopolist offers two (exogenous) prices. The question then becomes who is offered the discount (§4.3). Because our objective is to obtain qualitative insights on the structure of the monopolist's optimal pricing policies, we mainly conduct our analysis for the cases of perfect price discrimination and two exogenous prices, under simplifying assumptions that allow us to state our results in closed form. In particular, we assume that it is always optimal for the monopolist to set prices that induce positive consumption by all agents, i.e., the optimal solution is interior (Assumptions 2 and 3 below). However, we also provide a discussion of how our analysis can be extended if this assumption is relaxed.

## 3. Consumption Equilibria

In this section, we study the second stage of the game defined in §2 under Assumption 1 and characterize the equilibria of the consumption game among the agents for a given vector of prices $\mathbf{p}$. In particular, we show that the equilibrium is unique and we provide a closed-form expression for it. To express the results in a compact form, we define the vectors $\mathbf{x}, \mathbf{a}, \mathbf{p} \in \mathbb{R}^n$ such that $\mathbf{x} = [x_i]_i$, $\mathbf{a} = [a_i]_i$, $\mathbf{p} = [p_i]_i$. We also define a matrix $\Lambda \in \mathbb{R}^{n\times n}$ as follows:

$$\Lambda_{i,j} = \begin{cases} 2b_i & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Let $\beta_i(\mathbf{x_{-i}})$ denote the best response of agent $i$ when the rest of the agents choose consumption levels represented by the vector $\mathbf{x_{-i}}$. From (1) it follows that:

$$\beta_i(\mathbf{x_{-i}}) = \max\left\{\frac{a_i - p_i}{2b_i} + \frac{1}{2b_i}\sum_{j\in\mathcal{I}} g_{ij}x_j, 0\right\}. \tag{2}$$

Our first result shows that the equilibrium of the consumption game is unique for any price vector.

Theorem 1. *Under Assumption* 1, *the game* $\mathcal{G} = \{\mathcal{I}, \{u_i\}_{i\in\mathcal{I}}, [0,\infty)_{i\in\mathcal{I}}\}$ *has a unique equilibrium.*

Intuitively, Theorem 1 follows from the fact that increasing one's consumption incurs a positive externality on her peers, which further implies that the game involves strategic complementarities, and therefore the equilibria are ordered. The proof exploits this monotonic ordering to show that the equilibrium is actually unique.[4]

We conclude this section by characterizing the unique equilibrium of $\mathcal{G}$. Suppose that $\mathbf{x}$ is this equilibrium, and $x_i > 0$ only for $i \in S$. Then, it follows that

$$\begin{aligned} x_i = \beta_i(\mathbf{x_{-i}}) &= \frac{a_i - p_i}{2b_i} + \frac{1}{2b_i}\sum_{j\in\mathcal{I}} g_{ij}x_j \\ &= \frac{a_i - p_i}{2b_i} + \frac{1}{2b_i}\sum_{j\in S} g_{ij}x_j \end{aligned} \tag{3}$$

for all $i \in S$. Denoting by $\mathbf{x}_S$ the vector of all $x_i$ such that $i \in S$, and defining the vectors $\mathbf{a}_S$, $\mathbf{b}_S$, $\mathbf{p}_S$ and the matrices $G_S$, $\Lambda_S$ similarly, Equation (3) can be rewritten as

$$\Lambda_S\mathbf{x}_S = \mathbf{a}_S - \mathbf{p}_S + G_S\mathbf{x}_S. \tag{4}$$

Note that Assumption 1 holds for the graph restricted to the agents in $S$; hence, $I - \Lambda_S^{-1}G_S$ is invertible (cf. Lemma 4 in the appendix). Therefore, (4) implies that

$$\mathbf{x}_S = (\Lambda_S - G_S)^{-1}(\mathbf{a}_S - \mathbf{p}_S). \tag{5}$$

Therefore, the unique equilibrium of the consumption game takes the following form:

$$\begin{aligned} &\mathbf{x}_S = (\Lambda_S - G_S)^{-1}(\mathbf{a}_S - \mathbf{p}_S), \\ &\mathbf{x}_{\mathcal{I}-\mathbf{S}} = \mathbf{0}, \end{aligned} \tag{6}$$

for some subset $S$ of the set of agents $\mathcal{I}$. This characterization suggests that consumptions of players (weakly) decrease with the prices. The following lemma, which is used in the subsequent analysis, formalizes this fact.

Lemma 1. *Let* $\mathbf{x}(\mathbf{p})$ *denote the unique consumption equilibrium in the game where each player* $i \in \mathcal{I}$ *is offered the price* $p_i$. *Then,* $x_i(\mathbf{p})$ *is weakly decreasing in* $\mathbf{p}$ *for all* $i \in \mathcal{I}$, *i.e., if* $\hat{\mathbf{p}}_j \geqslant \mathbf{p}_j$ *for all* $j \in \mathcal{I}$, *then* $x_i(\hat{\mathbf{p}}) \leqslant x_i(\mathbf{p})$.

## 4. Optimal Pricing

In this section, we turn attention to the first stage of the game, where a monopolist sets the vector of prices. We distinguish between three different scenarios. In the first subsection, we assume that the monopolist can *perfectly price discriminate* the agents, i.e., there is no restriction imposed on the prices. In the second subsection, we consider the problem of choosing a single uniform price, whereas in the third we allow the monopolist to choose between two exogenous prices, $p_L$ and $p_H$, for each consumer. In our terminology, in the first case $\mathbf{P} = \mathbb{R}^{|\mathcal{I}|}$, in the second $\mathbf{P} = \{(p,\ldots,p)\}$, for $p \in [0,\infty)$, and finally, in the third, $\mathbf{P} = \{p_L, p_H\}^{|\mathcal{I}|}$.

### 4.1. Perfect Price Discrimination

In this section, we first make the following assumption that ensures that in the absence of network effects, the monopolist would find it optimal to charge individual prices low enough so that all consumers purchase a positive amount of the good.

Assumption 2. *For all* $i \in \mathcal{I}$, $a_i > c$.

Assumption 2 also guarantees that when the monopolist sets prices optimally, all consumers purchase a positive amount of the good, i.e., the equilibrium features an *interior* consumption vector (see Lemma 7 in the appendix). This enables us to state our results in closed form and draw explicitly the connection between the optimal pricing strategy of the monopolist and a measure of network influence, the *Bonacich centrality*, defined below. Later in this section, we relax this assumption and show that a similar relation between the prices and the network influence can still be established.

We start by providing a preliminary characterization of optimal prices (Theorem 2) under Assumption 2. When stating our results, we use the shorthand notation $\mathbf{1}$ to denote the vector of all 1s.

THEOREM 2. *Under Assumptions* 1 *and* 2, *the optimal prices are given by*

$$\mathbf{p} = \mathbf{a} - (\Lambda - G)\left(\Lambda - \frac{G+G^T}{2}\right)^{-1} \frac{\mathbf{a} - c\mathbf{1}}{2}. \tag{7}$$

The following corollary is an immediate consequence of Theorem 2.

COROLLARY 1. *Let Assumptions* 1 *and* 2 *hold. Moreover, assume that the interaction matrix $G$ is symmetric. Then, the optimal prices satisfy*

$$\mathbf{p} = \frac{\mathbf{a} + c\mathbf{1}}{2},$$

*i.e., the optimal prices do not depend on the network structure.*

This result implies that when players affect each other in the same way, i.e., when the interaction matrix $G$ is symmetric, then the graph topology has no effect on the optimal prices. Intuitively, this happens because when deciding what price to offer to an agent, the monopolist considers the trade-off between the profit loss due to (potentially) subsidizing the agent and the increase in profits due to the influence this agent exerts over her peers. The profit loss is proportional to the consumption of the agent, and it increases with the influence of the network on this agent. The profit increase term, on the other hand, relates to the influence of the agent on the rest of the network. When matrix $G$ is symmetric, these opposing effects cancel each other and the optimal prices do not depend on the network structure.

It can also be seen from Corollary 1 that agents with large $a_i$ parameters are charged with higher prices, i.e., the monopolist charges higher prices to agents who value the good the most (absent of network effects, for low consumption levels). Finally, Assumption 2 implies that in the symmetric setting the optimal prices are always above the cost parameter $c$.

Next, we present two results that build on Theorem 2 and better illustrate the effect of the network structure on prices. Before doing so, we provide the definition of Bonacich Centrality (see also Bonacich 1987), which we use subsequently to obtain an alternative characterization of the optimal prices.

DEFINITION 2 (BONACICH CENTRALITY). For a network with (weighted) adjacency matrix $G$ and scalar $\alpha$, the Bonacich centrality vector of parameter $\alpha$ is given by $\mathcal{K}(G, \alpha) = (I - \alpha G)^{-1}\mathbf{1}$ provided that $(I - \alpha G)^{-1}$ is well defined and nonnegative.

To gain a better understanding of the concept of Bonacich centrality, consider a random walk, which has a uniform initial distribution and is defined over the network. Note that when the spectral radius of $\alpha G$ matrix is smaller than 1, we have $(I - \alpha G)^{-1} = \sum_{k=0}^{\infty} (\alpha G)^k$. The Bonacich centrality of a node is proportional to the expected number of visits (weighted by $\alpha^k$ at time $k$) of the random walk to this node.

To simplify exposition, we first present a characterization of the optimal prices when agents differ only in terms of their network position, i.e., $a_i = a_0$, $b_i = b_0$ for all $i \in \mathcal{I}$.

THEOREM 3. *Under Assumptions* 1, 2, *and when $a_i = a_0$, $b_i = b_0$ for all $i \in \mathcal{I}$, the vector of optimal prices is given by*

$$\mathbf{p} = \frac{a_0 + c}{2}\mathbf{1} + \frac{a_0 - c}{8b_0} G\mathcal{K}\left(\frac{G+G^T}{2}, \frac{1}{2b_0}\right) - \frac{a_0 - c}{8b_0} G^T\mathcal{K}\left(\frac{G+G^T}{2}, \frac{1}{2b_0}\right).$$

The network $(G + G^T)/2$ is the average interaction network, and it represents the average interaction between pairs of agents in network $G$. Intuitively, the centrality $\mathcal{K}((G + G^T)/2, 1/(2b_0))$ measures how "central" each agent is with respect to the average interaction network.

The optimal prices in Theorem 3 have three components. The first component can be thought of as a nominal price, which is charged to all agents irrespective of the network structure. The second term is a markup that the monopolist can impose on the price of a consumer due to the utility this agent derives from her peers. Finally, the third component can be seen as a discount term that is offered to a consumer, because increasing her consumption increases the consumption level of her peers. Note that Theorem 3 suggests that for a given agent $i$ the optimal markup term can be obtained by multiplying the Bonacich centrality vector $\mathcal{K}$ with the $i$th row of $G$, whereas the discount term involves the $i$th column of $G$. In other words, the markup term is proportional to the amount the agent is influenced by her central peers, whereas prices offered are discounted proportionally to the influence the agent exerts on central agents. Therefore, it follows that the agents that are offered the most favorable prices are the ones that *influence* highly central agents.

Optimal prices have some interesting properties. For instance, Example 1 demonstrates that the optimal solution may involve offering some individuals prices below marginal cost $c$. Hence, the monopolist may sell the good to some agents at a loss to ensure larger profits from the remaining agents.

EXAMPLE 1. Consider a network with three agents and assume that the problem parameters are such that $c = 1$, $a_1 = a_2 = a_3 = 2$, $b_1 = b_2 = b_3 = 6$, and the weight matrix is given by

$$G = \begin{bmatrix} 0 & 10 & 0 \\ 1 & 0 & 1 \\ 0 & 10 & 0 \end{bmatrix}.$$

The $G$ matrix describes a line network where agent 2 is neighbors with agents 1 and 3 and can exert larger influence over these agents compared to what they can exert over her. It can be readily seen that the problem parameters satisfy Assumptions 1 and 2; hence, Theorem 3 applies. The optimal price vector for this problem instance is given by:

$$[p_1, p_2, p_3] \approx [2.12, 0.56, 2.12].$$

The price offered by the monopolist to agent 2 is lower than the cost ($c = 1$), because it is in the interest of the monopolist to incentivize this agent, who has a large influence over her peers, to purchase larger amounts of the good.

Note that Theorem 3 can be modified in a simple way to relate the optimal prices to centrality measures in the underlying graph when agents are not symmetric with respect to their $\{a_i\}$ and $\{b_i\}$ parameters. In particular, when the parameters $\{a_i\}$ and $\{b_i\}$ are not identical, the discount and markup terms are proportional to a weighted version of the Bonacich centrality measure, defined below.

Definition 3 (Weighted Bonacich Centrality). For a network with (weighted) adjacency matrix $G$, diagonal matrix $D$, and weight vector $\mathbf{v}$, the weighted Bonacich centrality vector is given by $\tilde{\mathcal{K}}(G, D, \mathbf{v}) = (I - GD)^{-1}\mathbf{v}$ provided that $(I - GD)^{-1}$ is well defined and nonnegative.

We next characterize the optimal prices in terms of the weighted Bonacich centrality measure.

Theorem 4. *Under Assumptions* 1 *and* 2 *the vector of optimal prices is given by*

$$\mathbf{p} = \frac{\mathbf{a} + c\mathbf{1}}{2} + G\Lambda^{-1}\tilde{\mathcal{K}}(\tilde{G}, \Lambda^{-1}, \tilde{\mathbf{v}}) - G^T\Lambda^{-1}\tilde{\mathcal{K}}(\tilde{G}, \Lambda^{-1}, \tilde{\mathbf{v}}),$$

*where* $\tilde{G} = (G + G^T)/2$ *and* $\tilde{\mathbf{v}} = (\mathbf{a} - c\mathbf{1})/2$.

We next discuss the solution of the optimal pricing problem when Assumption 2 does not hold. Recall that this assumption is used to guarantee that all agents purchase positive amounts of the good at the optimal solution, and hence, when we relax it, the optimal solution may involve agents not purchasing the good at all.

Assume that the monopolist knows in advance the set of agents (say $S$) who purchase a positive amount of the good at an optimal solution. Then, she can offer prices that are arbitrarily large to the remaining agents, and the optimal prices and consumption levels for agents in $S$ satisfy the best response condition in (4). The proofs of Theorems 2, 3, and 4 all rely on (4), and Assumption 2 is only used to guarantee that $S$ is the set of all agents. Hence, these theorems still hold, when we restrict attention to the agents in set $S$. Thus, the optimal prices offered to agents in $S$ can still be expressed in terms of the centralities of these agents, where centralities are now defined with respect to the subgraph of agents in $S$.

We next focus on the question of finding the set of agents who purchase a positive amount of the good at the optimal solution. Assume that the consumption vector at an equilibrium is equal to $\mathbf{x}$. We can relate $\mathbf{x}$ with a price vector using Equations (2) and (4). Note that because some agents may not consume the good, the price vector corresponding to this equilibrium need not be unique. In particular, we can increase the prices offered to the agents who do not consume the good, and the corresponding consumption levels remain the same. However, the corresponding equilibrium profits are unique, because only the prices offered to agents who do not consume the good can take different values.

Exploiting this relation between the consumption vectors and the corresponding prices, we can rewrite the profit maximization problem of the firm by using consumption levels as variables. Assume, for instance, that the monopolist wants to guarantee a consumption vector of $\mathbf{x}$, and hence sets prices according to $\mathbf{p} = \mathbf{a} - (\Lambda - G)\mathbf{x}$ (using (4)). Substituting this, her profit can be written as

$$\mathbf{x}^T\mathbf{p} - c\mathbf{x}^T\mathbf{1} = \mathbf{x}^T(\mathbf{a} - c\mathbf{1}) + \mathbf{x}^T(G - \Lambda)\mathbf{x}.$$

Thus, to find the optimal consumption levels, the monopolist needs to solve the following optimization problem:

$$\max_{\mathbf{x}\geqslant 0} \mathbf{x}^T(\mathbf{a} - c\mathbf{1}) - \mathbf{x}^T(\Lambda - G)\mathbf{x}.$$

Furthermore, a price vector corresponding to the optimal consumption levels can be calculated (using (4)) as $\mathbf{p} = \mathbf{a} - (\Lambda - G)\mathbf{x}$. Note that this problem is a convex optimization problem as long as $(\Lambda - G) + (\Lambda - G)^T = 2\Lambda - (G + G^T)$ is positive semidefinite.

Thus, we conclude that the monopolist can solve for the optimal prices even when Assumption 2 does not hold, provided that $\{b_i\}$ and $\{g_{ij}\}$ are such that $2\Lambda - (G + G^T)$ is positive semidefinite. Additionally, the optimal prices and centralities of agents are still related as in Theorems 3 and 4, after restricting attention to the set of agents who purchase the good.

### 4.2. Choosing a Single Uniform Price

In this subsection we characterize the equilibria of the pricing-consumption game, when the monopolist can only set a single uniform price, i.e., $p_i = p_0$ for all $i$. In this case, for any fixed $p_0$, the payoff function of agent $i$ is given by

$$u_i(x_i, \mathbf{x}_{-\mathbf{i}}, p_i) = a_i x_i - b_i x_i^2 + x_i \cdot \sum_{j\in\{1,\dots,n\}} g_{ij} \cdot x_j - p_0 x_i,$$

and the payoff function for the monopolist is given by

$$\max_{p_0\in[0,\infty)} \quad (p_0 - c)\sum_i x_i \qquad \text{s.t.} \quad \mathbf{x} \in C[\mathbf{p}_0],$$

where $\mathbf{p}_0 = (p_0, \dots, p_0)$. Note that Theorem 1 implies that even when the monopolist offers a single price, the consumption game has a unique equilibrium point.

Following a similar approach to §4.1, we first assume that all agents purchase a positive amount of the good at the optimal solution. It follows from (5) that when all prices are set to $p_0$, provided that all agents consume the good, the corresponding consumption vector is given by $\mathbf{x} = (\Lambda - G)^{-1}(\mathbf{a} - p_0\mathbf{1})$. Thus, the profit function takes the form $(p_0 - c)\mathbf{1}^T(\Lambda - G)^{-1}(\mathbf{a} - p_0\mathbf{1})$. Using first-order conditions, in this case the optimal price $p_0$ is simply given as

$$p_0 = \frac{1}{2}\frac{\mathbf{1}^T(\Lambda - G)^{-1}(\mathbf{a} + c\mathbf{1})}{\mathbf{1}^T(\Lambda - G)^{-1}\mathbf{1}}. \tag{8}$$

Next, we analyze the case where some agents potentially do not purchase the good at the optimal solution. We show that applying first-order conditions iteratively, the set of agents that consume the good at the optimal solution, and the optimal price can be computed. Moreover, the optimal price turns out to have a similar form to the one provided in (8).

We begin our analysis by a lemma that states that the consumption of each agent decreases monotonically in the price.

Lemma 2. *Let $\mathbf{x}(\mathbf{p_0})$ denote the unique equilibrium in the game where $p_i = p_0$ for all $i$. Then, $x_i(\mathbf{p_0})$ is weakly decreasing in $p_0$ for all $i \in \mathcal{I}$ and strictly decreasing for all $i$ such that $x_i(\mathbf{p_0}) > 0$.*

Next, we introduce the notion of the centrality gain.[5]

Definition 4 (Centrality Gain). In a network with (weighted) adjacency matrix $G$, for any diagonal matrix $D$ and weight vector $\mathbf{v}$, the centrality gain of agent $i$ is defined as

$$H_i(G, D, \mathbf{v}) = \frac{\tilde{\mathcal{K}}_i(G, D, \mathbf{v})}{\tilde{\mathcal{K}}_i(G, D, \mathbf{1})}.$$

The following theorem provides a characterization of the consumption vector at equilibrium as a function of the single uniform price $p$.

Theorem 5. *Consider game $\bar{\mathcal{G}} = \{\mathcal{I}, \{u_i\}_{i\in\mathcal{I}}, [0, \infty)_{i\in\mathcal{I}}\}$, and define*

$$D_1 = \operatorname*{argmin}_{i\in\mathcal{I}} H_i(G, \Lambda^{-1}, \mathbf{a}) \quad \text{and} \quad p_1 = \min_{i\in\mathcal{I}} H_i(G, \Lambda^{-1}, \mathbf{a}).$$

*Moreover, let $I_k = \mathcal{I} - \bigcup_{i=1}^{k} D_i$ and define*

$$D_{k+1} = \operatorname*{arg\,min}_{i\in I_k} H_i(G_{I_k}, \Lambda_{I_k}^{-1}, \mathbf{a}_{I_k}) \quad \text{and}$$

$$p_{k+1} = \min_{i\in I_k} H_i(G_{I_k}, \Lambda_{I_k}^{-1}, \mathbf{a}_{I_k}),$$

*for $k \in \{1, 2, \ldots, n-1\}$. Then,*

(1) *$p_k$ strictly increases in $k$.*

(2) *Given a $p$ such that $p < p_1$, all agents purchase a positive amount of the good, i.e., $x_i(p) > 0$ for all $i \in \mathcal{I}$, where $\mathbf{x}(\mathbf{p})$ denotes the unique consumption equilibrium at price $p$. If $k \geqslant 1$, and $p$ is such that $p_k \leqslant p \leqslant p_{k+1}$, then $x_i(p) > 0$ if and only if $i \in I_k$. Moreover, the corresponding consumption levels are given as in (6), where $S = I_k$.*

Theorem 5 also suggests a polynomial-time algorithm for computing the optimal uniform price $p_{\text{opt}}$. Intuitively, the algorithm sequentially removes consumers with the lowest centrality gain and computes the optimal price for the remaining consumers under the assumption that the price is low enough so that only these agents purchase a positive amount of the good at the associated consumption equilibrium. In particular, using Theorem 5, it is possible to identify the set of agents who purchase a positive amount of the good for price ranges $[p_k, p_{k+1}]$, $k \in \{1, \ldots, n-1\}$. Observe that given a set of players, who purchase a positive amount of the good, the equilibrium consumption levels can be obtained in closed form as a linear function of the offered price, and, thus, the profit function of the monopolist takes a quadratic form in the price. It follows that for each price range, the maximum profit can be found by solving a quadratic optimization problem. Thus, Theorem 5 suggests Algorithm 1 for finding the optimal single uniform price $p_{\text{opt}}$.

**Algorithm 1** (Compute the optimal single uniform price $p_{\text{opt}}$)

*Step* 1. Preliminaries:
—Initialize the set of *active* agents: $S := \mathcal{I}$.
—Initialize $k = 1$ and $p_0 = 0$,
$p_1 = \min_{i\in\mathcal{I}} H_i(G_{\mathcal{I}}, \Lambda_{\mathcal{I}}^{-1}, \mathbf{a}_{\mathcal{I}})$
—Initialize the monopolist's revenues with $Re_{\text{opt}} = 0$ and $p_{\text{opt}} = 0$.

*Step* 2.
—Let $\hat{p} = \frac{1}{2}(\mathbf{1}^T(\Lambda_S - G_S)^{-1}(\mathbf{a}_S + c\mathbf{1}))/(\mathbf{1}^T(\Lambda_S - G_S)^{-1}\mathbf{1})$
—**IF** $\hat{p} \geqslant p_k$, let $p = p_k$.
**ELSE IF** $\hat{p} \leqslant p_{k-1}$, let $p = p_{k-1}$ **ELSE** $p = \hat{p}$.
—$Re = (p - c)\mathbf{1}^T \cdot (\Lambda_S - G_S)^{-1}(\mathbf{a}_S - p\mathbf{1})$.
—**IF** $Re > Re_{\text{opt}}$ **THEN** $Re_{\text{opt}} = Re$ and $p_{\text{opt}} = p$.
—$D = \arg\min_{i\in S} H_i(G_S, \Lambda_S^{-1}, \mathbf{a}_S)$ and $S := S - D$.
—Increase $k$ by 1 and let
$p_k = \min_{i\in S} H_i(G_S, \Lambda_S^{-1}, \mathbf{a}_S)$.
—Return to **STEP 2** if $S \neq \varnothing$ **ELSE Output** $p_{\text{opt}}$.

The algorithm solves a series of subproblems, where the monopolist is constrained to choose a price $p$ in a given interval $[p_k, p_{k+1}]$ with appropriately chosen endpoints. In particular, from Theorem 5 it follows that we can choose those endpoints, so as to ensure that only a particular set $S$ of agents purchase a positive amount of the good when $p \in [p_k, p_{k+1}]$. In this case, the consumption at price $p$ is given by $(\Lambda_S - G_S)^{-1}(\mathbf{a}_S - p\mathbf{1})$ and the profit of the monopolist is equal to $(p - c)\mathbf{1}^T(\Lambda_S - G_S)^{-1}(\mathbf{a}_S - p\mathbf{1})$. Thus, it follows that the maximum profit, by restricting attention to $p \in [p_k, p_{k+1}]$, is achieved either at $\hat{p} = \frac{1}{2}(\mathbf{1}^T(\Lambda_S - G_S)^{-1}(\mathbf{a}_S + c\mathbf{1}))/(\mathbf{1}^T(\Lambda_S - G_S)^{-1}\mathbf{1})$, or $\hat{p} \in \{p_k, p_{k+1}\}$, as can be seen from the first-order optimality conditions. Then, the overall optimal price is found by comparing the monopolist's profits achieved at the optimal solutions of the constrained subproblems. The complexity of the algorithm is $O(n^4)$, because there are at most $n$

such subproblems (again from Theorem 5) and each such subproblem simply involves a matrix inversion ($O(n^3)$) in computing the centrality gain and the maximum achievable profit.

### 4.3. The Case of Two Prices: Full and Discounted

In this subsection, we consider a monopolist who can offer the good in two prices, $p_L$ and $p_H$ ($p_L < p_H$). For clarity of exposition we call $p_L$ and $p_H$ the discounted and the full price, respectively. We first assume that $p_L$ and $p_H$ are exogenously specified, and analyze the allocation problem, i.e., determining the subset of agents that should be offered the discounted price, so as for the monopolist to maximize her profits. We then discuss how to optimize over the choice of $p_L$ and $p_H$.

In an analogous fashion to §§4.1 and 4.2, we start our analysis under the following assumption, which guarantees that all agents purchase a positive amount of the good at equilibrium.

Assumption 3. *The exogenous prices $p_L, p_H$ are such that $p_L, p_H < \min_{i\in\mathcal{I}} a_i$.*

Note that this assumption guarantees that all agents purchase a positive amount of the good, regardless of the consumption of their peers. Hence, under this assumption, the consumption levels satisfy $\mathbf{x} = \Lambda^{-1}(\mathbf{a} - \mathbf{p} + G\mathbf{x})$, and hence $\mathbf{x} = (\Lambda - G)^{-1}(\mathbf{a} - \mathbf{p})$. This characterization allows for expressing the monopolist's problem only in terms of prices. In particular, the monopolist's problem takes the form:

$$\text{(OPTs)} \quad \max \ (\mathbf{p} - c\mathbf{1})^T(\Lambda - G)^{-1}(\mathbf{a} - \mathbf{p}) \quad \text{s.t.} \ p_i \in \{p_L, p_H\} \quad \text{for all } i \in \mathcal{I},$$

where $\Lambda \succ 0$ is a diagonal matrix, $G$ is such that $G \geqslant 0$ (where the inequality is entrywise), $\operatorname{diag}(G) = 0$, and Assumption 1 holds.

Let $p_N \triangleq (p_H + p_L)/2$, $\delta \triangleq p_H - p_N$, $\hat{\mathbf{a}} \triangleq \mathbf{a} - p_N$, and $\hat{c} \triangleq p_N - c \geqslant \delta$. Using these variables, and noting that any feasible price allocation can be expressed as $\mathbf{p} = \delta\mathbf{y} + p_N$, where $y_i \in \{-1, 1\}$, OPT can alternatively be expressed as

$$\begin{aligned} \max \quad & (\delta\mathbf{y} + \hat{c}\mathbf{1})^T(\Lambda - G)^{-1}(\hat{\mathbf{a}} - \delta\mathbf{y}) \\ \text{s.t.} \quad & y_i \in \{-1, 1\} \quad \text{for all } i \in \mathcal{I}. \end{aligned} \tag{9}$$

We next show that OPT is NP-hard and provide an algorithm that achieves an approximately optimal solution. To obtain our results, we relate the alternative formulation of OPT in (9) to the MAX-CUT problem (see Garey and Johnson 1979, Goemans and Williamson 1995).[6]

Theorem 6. *Let Assumptions* 1, 2, *and* 3 *hold. Then, the monopolist's optimal pricing problem, i.e., problem OPT, is NP-hard.*

Theorem 7 exploits the relation of OPT to the MAX-CUT problem, and establishes that there exists an algorithm that provides a solution with a provable approximation guarantee.

Theorem 7. *Let Assumptions* 1 *and* 3 *hold and $W_{OPT}$ denote the optimal profits for the monopolist, i.e., $W_{OPT}$ is the optimal value for problem OPT. Then, there exists a randomized polynomial-time algorithm that outputs a solution with objective value $W_{ALG}$ such that $E[W_{ALG}] + m > 0.878(W_{OPT} + m)$, where*

$$m = \delta^2\mathbf{1}^T A\mathbf{1} + \delta\mathbf{1}^T|A\hat{\mathbf{a}} - A^T\hat{c}\mathbf{1}| - \hat{c}\mathbf{1}^T A\hat{\mathbf{a}} - 2\delta^2\operatorname{Trace}(A),$$

*and $A = (\Lambda - G)^{-1}$.*

Clearly, if $m \leqslant 0$, which, for instance, is the case when $\delta$ is small, this algorithm provides at least an 0.878-optimal solution of the problem. On the other hand, if $m > 0$, we obtain 0.878 optimality after a constant ($m$) addition to the objective function. This suggests that for small $m > 0$, the algorithm still provides near-optimal solutions.

Next, we provide a characterization of the optimal prices in OPT. In particular, we argue that the pricing problem faced by the monopolist is equivalent to finding the cut with maximum weight in an appropriately defined weighted graph. For simplicity, assume that $b_i = b_0$, and $a_i = a_0$ for all $i$, i.e., agents are heterogeneous only in terms of their network position (captured by the adjacency matrix $G$). In this case, the profit-maximization problem in (9) can be rewritten as:

$$\begin{aligned} \max \quad & -\delta^2\mathbf{y}^T A\mathbf{y} + \delta\mathbf{y}^T A\hat{a}\mathbf{1} - \delta\mathbf{y}^T A^T\hat{c}\mathbf{1} \\ \text{s.t.} \quad & y_i \in \{-1, 1\} \quad \text{for all } i \in \mathcal{I}, \end{aligned} \tag{10}$$

where $A = (\Lambda - G)^{-1}$ and we ignore the constant term. We can further simplify the optimization problem by noting that $y_i^2 = 1$ for all $i$ and hence, $\mathbf{y}^T A\mathbf{y} = \operatorname{Trace}(A) + \mathbf{y}^T\hat{A}\mathbf{y}$, where $\hat{A} = A - \operatorname{diag}(A)$. Because, $\operatorname{Trace}(A)$ is a constant, we can ignore it without any loss of generality. Finally, we observe that because $\mathbf{y}^T\hat{A}\mathbf{y} = \mathbf{y}^T\hat{A}^T\mathbf{y} = \mathbf{y}^T((\hat{A} + \hat{A}^T)/2)\mathbf{y}$, using the shorthand notation $\hat{Q} = (\hat{A} + \hat{A}^T)/2$, an equivalent optimization problem can be written as

$$\begin{aligned} \max \quad & [\mathbf{y}; z]^T Q[\mathbf{y}; z] \\ \text{s.t.} \quad & y_i \in \{-1, 1\} \quad \text{for all } i \in \mathcal{I}, \\ & z \in \{-1, 1\}, \end{aligned} \tag{11}$$

where

$$Q = \left[\begin{array}{c|c} -\delta^2\hat{Q} & \delta/2(\hat{a}A\mathbf{1} - \hat{c}A^T\mathbf{1}) \\ \hline \delta/2(\hat{a}\mathbf{1}^T A - \hat{c}\mathbf{1}^T A) & 0 \end{array}\right].$$

Note that if $[\mathbf{y}; z]$ is a feasible solution of (11), then so is $-[\mathbf{y}; z]$, and these solutions have the same objective value. Thus, we can assume without any loss of generality that

$z = 1$ at the optimal solution, and the equivalence of the objective values in (10) and (11) immediately follows.

It can now be seen that this optimization problem is equivalent to an instance of the MAX-CUT problem, where the weight matrix is given by $-Q$.[7] In this problem, in addition to the nodes corresponding to agents in the social network, we introduce an extra (dummy) node that always belongs to the "positive" cut-set ($z = 1$). The weights of the edges corresponding to the agents other than the dummy one are given by the off-diagonal entries of $\hat{Q}$ scaled by $\delta^2$, whereas the weights of the edges corresponding to the dummy node are a function of $\hat{a}$ and $\hat{c}$. The agents that belong to the same cut-set as the dummy node in the optimal solution of the MAX-CUT problem are charged the full price, whereas the rest are offered the discount (recall that $\mathbf{p} = \delta\mathbf{y} + p_N$; hence, $y_i = 1$ corresponds to a full price).

Observe that $(\Lambda - G)^{-1}\mathbf{1} = (1/(2b_0))(I - (1/(2b_0))G)^{-1}\mathbf{1}$; hence, the $i$th row sum of the entries of the matrix $A = (\Lambda - G)^{-1}$ is proportional to the centrality of the $i$th agent in the network. Consequently, the $(i, j)$th entry of $A$ gives a measure of how much the edge between $i$ and $j$ contributes to the centrality of agent $i$. Thus, the $(i, j)$th entry of $\hat{Q}$ captures how much the edge between agents $i$ and $j$ contributes to the nodes' total centrality. Therefore, the MAX-CUT interpretation roughly suggests that the optimal solution of the pricing problem is achieved when the monopolist tries to price discriminate agents that influence each other significantly, however, at the same time takes into account the agents' value of consumption in the absence of network effects (represented by the edges between the agents and the dummy node).

So far in this section we have assumed that the two prices, full and discounted, are exogenously given. We next describe a simple procedure for relaxing this assumption and searching over $p_L$ and $p_H$ to increase the profits. For simplicity, we assume that the monopolist seeks optimal $p_L$ and $p_H$ in $[0, p_{\max}]$, where $p_{\max} \in (0, \min_i a_i)$. Consider the set of prices $P_D = \{0, \epsilon/L\sqrt{|\mathcal{I}|}, \ldots, p_{\max}\}$, where $\epsilon > 0$ is some constant and $L$ is the Lipschitz constant of the profit function $(\mathbf{p} - c\mathbf{1})^T(\Lambda - G)^{-1}(\mathbf{a} - \mathbf{p})$. Let $\pi_D$ denote the maximum profit the monopolist can achieve if $p_L$ and $p_H$ belong to $P_D$, and $\pi_0$ denote the maximum profit if they belong to $[0, p_{\max}]$. Because $L$ is the Lipschitz constant, it follows that $\pi_D \geqslant \pi_0 - \epsilon$. For any $p_L$ and $p_H$ that belong to $P_D$, we can compute a policy that satisfies the approximate optimality condition in Theorem 7 in polynomial time. In particular, consider $p_L$ and $p_H$, which can achieve $\pi_D$ profit. If these $p_L$ and $p_H$ are known, Theorem 7 implies that in polynomial time we can find a policy that in expectation achieves profit (weakly) larger than $0.878\pi_D - (1 - 0.878)m$, where $m$ is defined as in Theorem 7. If $m \leqslant 0$, it follows that searching over all price tuples in $P_D$ (by identifying the corresonding policy Theorem 7 suggests) a policy that achieves 0.878 fraction of $\pi_D$ can be found in polynomial time in the number of agents and $Lp_{\max}/\epsilon$. Therefore, we conclude that in the case of two prices, assuming that prices are restricted to $[0, p_{\max}]$ for some $p_{\max} \in (0, \min_i a_i)$, a policy that obtains profit $\pi$, where $\pi \geqslant 0.878\pi_D \geqslant 0.878(\pi_0 - \epsilon)$ can be obtained in polynomial time for any $\epsilon > 0$. Hence, for small $\epsilon$, this procedure attains almost 0.878 optimality. Similarly, if $m > 0$, an approximate optimality condition in terms of $m$ can still be provided using Theorem 7.

We conclude this section by discussing how the results change if Assumption 3 is relaxed. Clearly, in this case, the complexity result still holds, because NP-hardness of the special case we focus on in Theorem 6 implies NP-hardness of the more general case. The approximation algorithm Theorem 7 suggests for cases where Assumption 3 holds strongly relies on representing the underlying optimization problem as a binary quadratic optimization problem as in OPT. However, when this assumption does not hold, in order to be able to write an analogous optimization formulation, we need to identify the set of agents who will purchase a positive amount of the good when the monopolist prices optimally. Unfortunately, due to the combinatorial structure of the problem, we do not have an immediate characterization of this set. Therefore, we do not have an approximation algorithm analogous to the one suggested by Theorem 7, when Assumption 3 does not hold. Additionally, due to the restriction that there are two different prices, we cannot follow a similar approach to the one in the perfect price discrimination case, reformulate a problem over the consumption variables, and obtain a solution to the pricing problem. We leave the question of finding an approximate optimal pricing policy, when the monopolist can only use two distinct prices and Assumption 3 does not hold, as an interesting future problem.

### 4.4. Comparing the Three Settings: Example

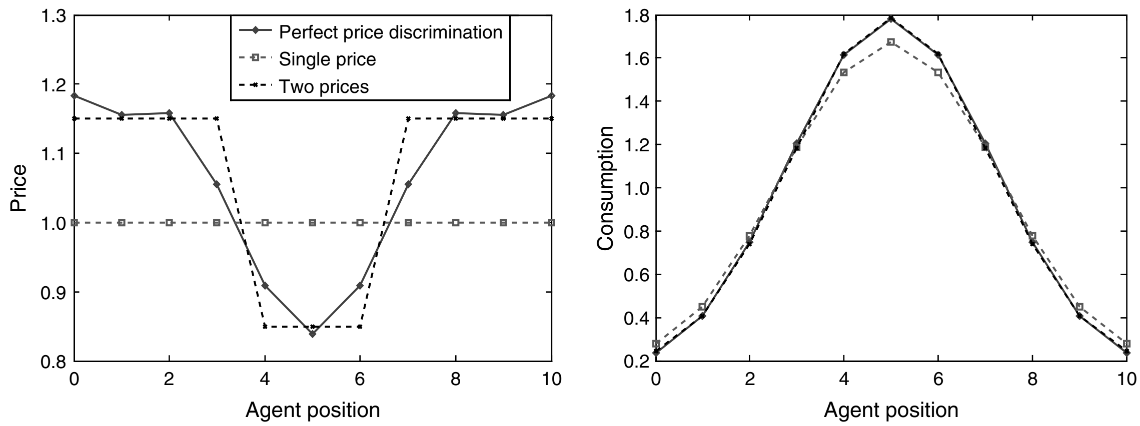
In this section, we compare the pricing rules developed in §§4.1–4.3 and obtain qualitative insights by applying them to a simple example. In particular, we consider a line network that consists of 11 nodes that are located at the integer points in $[0, 10]$. The parameters of the problem are given as follows: $c = 0$, $a_i = 2$, and $b_i = 2.5$ for all $i \in \mathcal{I}$. We also assume that the influence matrix $G$ is such that

$$g_{i+1,\,i} = 10\left(\frac{1}{4} - \left(\frac{1}{2} - \frac{i}{10}\right)^2\right) \quad \text{for } i < 10$$

$$g_{i-1,\,i} = 10\left(\frac{1}{4} - \left(\frac{1}{2} - \frac{i}{10}\right)^2\right) \quad \text{for } i > 0$$

and $g_{i,j} = 0$ for remaining $(i, j)$. That is, agent $i$, such that $0 < i < 10$, influences her neighbors with weights $10(\frac{1}{4} - (\frac{1}{2} - i/10)^2)$ (agent 0 influences only agent 1, and similarly agent 10 influences only agent 9). By this construction, agents in the center of the line influence their neighbors more than those at the end points. That is, although the agents are homogeneous in terms of their own consumption parameters, they differ with regards to their overall

**Figure 1.** Prices (left) and the corresponding consumption levels (right) for different agents.




influence because of their network position. Also, it can be easily seen that for these parameters, Assumptions 1 and 2 hold.

In this setting, if the monopolist can offer just a single price, the optimal such price should be equal to $p_0 = 1$ (this follows by applying Algorithm 1). For the two-prices case, we assume that the prices are given exogenously and are equal to $p_L = 0.85$, and $p_H = 1.15$, i.e., $\pm\%15$ from the optimal single price. Note that for these prices, Assumption 3 also holds. We then compute the optimal allocation when the monopolist can only use these two prices, and when she can perfectly price discriminate. The optimal prices and the corresponding consumption levels for all agents are given in Figure 1. This example suggests that for the three pricing rules we consider, the resulting consumption profiles are very similar. We observe that the agents who are the most influential—i.e., influence the rest of the agents more than they are influenced—consume the largest amounts of the good. Moreover, as predicted by our analysis, it is precisely these agents that are offered the most favorable prices by the monopolist. Finally, even when the monopolist is constrained to two prices, she tries again to favor those central agents, who end up getting the discounted price.

## 5. How Valuable Is It to Know the Network Structure?

Throughout our analysis, we have assumed that the monopolist has perfect knowledge of the interaction structure of the consumers and can use it optimally when choosing her pricing policy. In this section, we ask the following question: when is this information most valuable? In particular, we compare the profits generated in the following two extreme cases: (i) the monopolist prices optimally assuming that no network externalities are present, i.e., $g_{ij} = 0$ for all $i, j \in \mathcal{I}$ (however, consumers take network externalities into account when deciding their consumption levels); (ii) the monopolist has perfect knowledge of how consumers influence each other, i.e., knows the adjacency matrix $G$ and can perfectly price discriminate (as in §4.1). We will denote the profits generated in these settings by $\Pi_0$ and $\Pi_N$, respectively. The next lemma provides a closed-form expression for $\Pi_0$ and $\Pi_N$.

Lemma 3. *Under Assumptions* 1 *and* 2, *the profits* $\Pi_0$ *and* $\Pi_N$ *are given by*:

$$\Pi_0 = \left(\frac{\mathbf{a} - c\mathbf{1}}{2}\right)^T (\Lambda - G)^{-1} \left(\frac{\mathbf{a} - c\mathbf{1}}{2}\right) \tag{12}$$

*and*

$$\Pi_N = \left(\frac{\mathbf{a} - c\mathbf{1}}{2}\right)^T \left(\Lambda - \frac{G + G^T}{2}\right)^{-1} \left(\frac{\mathbf{a} - c\mathbf{1}}{2}\right). \tag{13}$$

The impact of network externalities in the profits is captured by the ratio $\Pi_0/\Pi_N$. For any problem instance, with fixed parameters $\mathbf{a}, c, \Lambda, G$, this ratio can be computed using Lemma 3. The rest of the section focuses on relating this ratio to the properties of the underlying network structure. To simplify the analysis, we make the following assumption.

Assumption 4. *The matrix* $\Lambda - G$ *is positive definite.*

Note that if $\Lambda - G$ is not symmetric, we still refer to this matrix as positive definite if $\mathbf{x}^T(\Lambda - G)\mathbf{x} > 0$ for all $\mathbf{x} \neq 0$. A sufficient condition for Assumption 4 to hold can be given in terms of the diagonal dominance of $\Lambda - G$. For instance, this assumption holds[8] if for all $i \in \mathcal{I}$, $2b_i > \sum_{j\in\mathcal{I}} g_{ij}$ and $2b_i > \sum_{j\in\mathcal{I}} g_{ji}$.

Theorem 8 provides bounds on $\Pi_N/\Pi_0$ using the spectral properties of $\Lambda - G$.

Theorem 8. *Under Assumptions* 1, 2, *and* 4,

$$0 \leqslant \frac{1}{2} + \lambda_{\min}\left(\frac{MM^{-T} + M^T M^{-1}}{4}\right) \leqslant \frac{\Pi_0}{\Pi_N} \leqslant \frac{1}{2} + \lambda_{\max}\left(\frac{MM^{-T} + M^T M^{-1}}{4}\right) \leqslant 1, \tag{14}$$

*where* $M = \Lambda - G$ *and* $\lambda_{\min}(\cdot)$, $\lambda_{\max}(\cdot)$ *denote the minimum and the maximum eigenvalues of their arguments, respectively.*

Note that when $G$ is asymmetric, so is the matrix $M = \Lambda - G$. Hence, it need not have real eigenvalues. In the proof of Theorem 8, we show that even when $G$ is asymmetric, as long as $M$ is positive definite (in the sense defined above) the eigenvalues of $(MM^{-T} + M^T M^{-1})/4$ lie between $-1/2$ and $1/2$; hence, the above bounds make sense.

If the underlying network structure is symmetric, i.e., $G = G^T$, then $MM^{-T} = M^T M^{-1} = I$ and the bounds in Theorem 8 take the following form:

$$\frac{1}{2} + \lambda_{\min}\left(\frac{MM^{-T} + M^T M^{-1}}{4}\right) = \frac{\Pi_0}{\Pi_N}$$
$$= \frac{1}{2} + \lambda_{\max}\left(\frac{MM^{-T} + M^T M^{-1}}{4}\right) = 1. \tag{15}$$

This is consistent with Corollary 1, in which we show that if the network is symmetric, then the monopolist does not gain anything by accounting for network effects. As already mentioned in the introduction, the benefit of accounting for network effects is proportional to how asymmetric the underlying interaction structure is. The minimum and maximum eigenvalues of matrix $((MM^{-T} + M^T M^{-1})/4)$ that appear in the bounds of Theorem 8 quantify this formally, because they can be viewed as a measure of the deviation from symmetric networks.

Finally, we provide a set of simulations whose goal is twofold: first, we show that the bounds of Theorem 8 are quite tight by comparing them to the actual value of the ratio of profits (which can be directly computed by Lemma 3) and, second, we illustrate that accounting for network effects can significantly boost profits, i.e., that the ratio can be much lower than 1. In all our simulations we choose the parameters so that $M = \Lambda - G$ is a positive definite matrix.

*Star Networks*. In our first set of simulations, we consider star networks with $n = 100$ agents. In particular, there is a central agent (without loss of generality agent 1) that has edges to the remaining agents, and these are the only edges in the network. Consider the following two extremes:

(1) The central agent is influenced by all her neighbors but does not influence any of them, i.e., if we denote the corresponding interaction matrix by $G^1$, then $G^1_{ij} = 1$ if $i = 1$, $j \neq i$, and $G^1_{ij} = 0$ otherwise.

(2) The central agent influences all her neighbors but is not influenced by any of them, i.e., if we denote the corresponding interaction matrix by $G^2$, then $G^2_{ij} = 1$ if $j = 1$, $j \neq i$, and $G^2_{ij} = 0$ otherwise.

We compute the ratio of profits $\Pi_0/\Pi_N$ for a class of network structures given by matrices $G^\alpha = \alpha G^1 + (1-\alpha)G^2$, where $\alpha \in [0, 1]$ ($\alpha = 1$ and $\alpha = 0$ correspond to the two extreme scenarios described above). In order to isolate the effect of the network structure, we assume that $a_i = a_1$, and $b_i = b_1$ for all $i \in \mathcal{I}$. In particular, in our first simulation we set $b_i = n/10$ and in the second simulation we set $b_i = n/20$ for all $i \in \mathcal{I}$. For both simulations, we set $a_i - c = 1$ for all $i \in \mathcal{I}$.[9]

The results are presented in Figure 2. In both simulations, the lower bound equals to the ratio $\Pi_0/\Pi_N$, implying that the bound provided in the theorem is tight. The upper bound is very close to 1 for all $\alpha$.[10] When $\alpha = \frac{1}{2}$, network effects become irrelevant, as the network is symmetric. On the other hand, for $\alpha = 0$ and $\alpha = 1$, i.e., when the star network is most "asymmetric," accounting for network effects leads to a 25% increase in profits when $b_i = n/10$ and to a 100-fold increase when $b_i = n/20$. Choosing smaller $b_i$ increases the relative significance of network effects and, therefore, the increase in profits is much higher in the second case, when $b_i = n/20$. Although star networks are extreme, this example showcases that taking network effects into consideration can lead to significant improvements in profits.

*From Asymmetric to Symmetric Networks*. In this set of simulations, we replicate the above for arbitrary asymmetric networks. Again, we consider two extreme settings: let $U$ denote a fixed upper-triangular matrix, and define the

**Figure 2.** Star networks. Left: $b_i = n/10$; right: $b_i = n/20$ for all $i \in \mathcal{I}$.

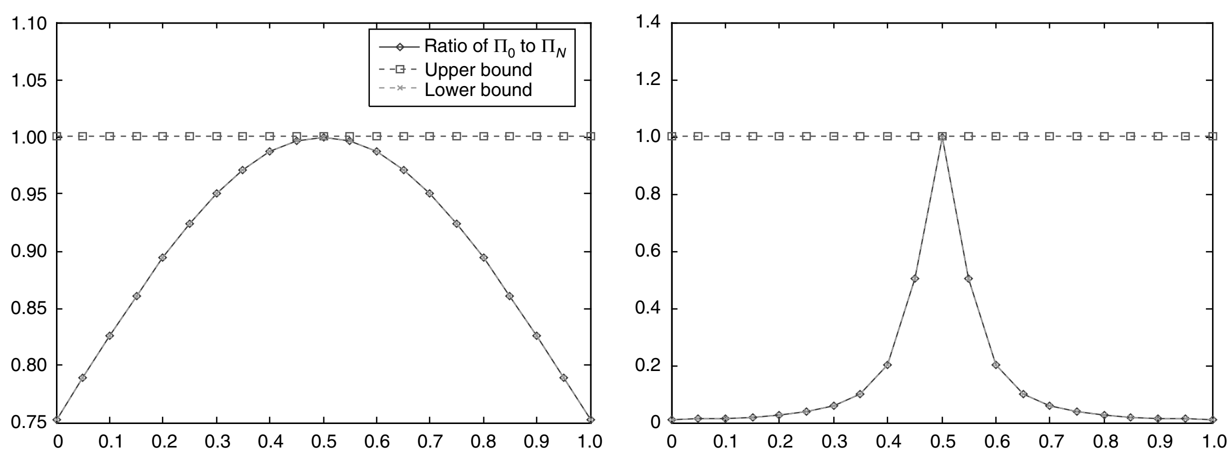
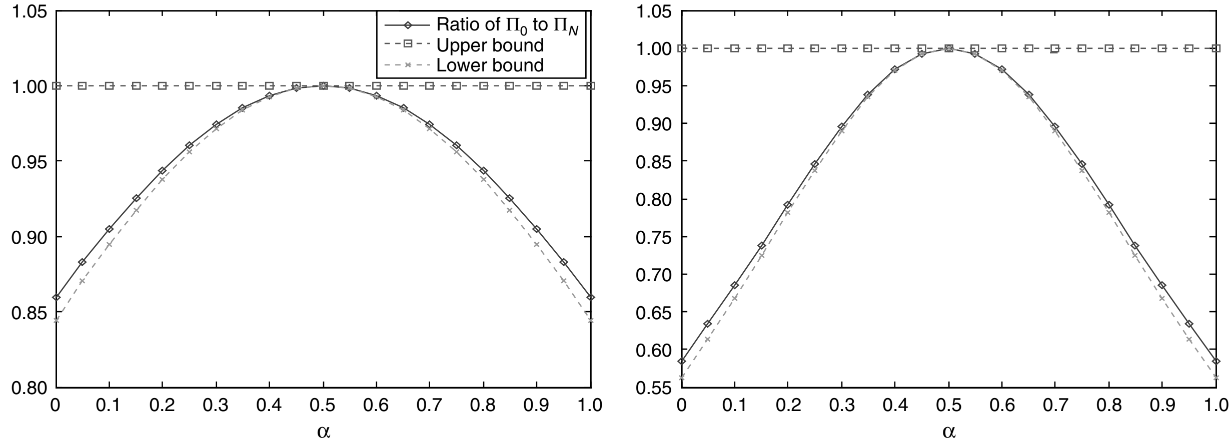


**Figure 3.** Random asymmetric matrices. Left: $b_i = n/4$; right: $b_i = n/6$ for all $i \in \mathcal{I}$.



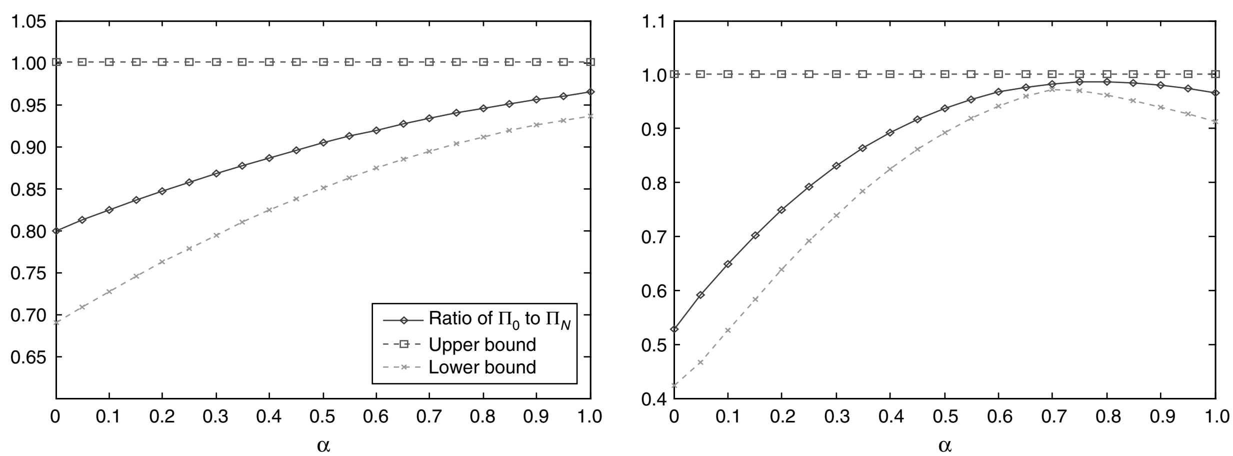

interaction matrices $G^1 = U$ and $G^2 = U^T$. The first, $G^1$, corresponds to the case where agent 1 is influenced by all her neighbors, but does not influence any other agent; and $G^2$ corresponds to the polar opposite where agent 1 influences all her neighbors. As before, we plot the ratio of profits for a class of matrices parameterized by $\alpha \in [0, 1]$, $G^\alpha = \alpha G^1 + (1-\alpha)G^2$. Specifically, we randomly generate 100 upper-triangular matrices $U$. (Each nonzero entry is an independent random variable, uniformly distributed in $[0, 1]$.) We again consider two cases: $b_i = n/4$ and $b_i = n/6$ for all $i \in \mathcal{I}$. For each of these cases and randomly generated instances, assuming $a_i - c = 1$ for all $i \in \mathcal{I}$, we obtain the ratio $\Pi_0/\Pi_N$ and the bounds as given by Theorem 8 (for the generated networks, we verify that the assumptions of the theorem hold). The plots of the corresponding averages over all randomly generated instances are given in Figure 3.

Similar to the previous set of simulations, when $\alpha = \frac{1}{2}$, i.e., the network is symmetric, there is no gain in exploiting the network effects. On the other hand, for $\alpha = 0$ and $\alpha = 1$, i.e., when the network is at the asymmetric extremes, exploiting network effects can boost profits by almost 15% or 40% depending on the value of $b_i$. Consistently with our earlier simulations, we observe that when $b_i$ is smaller, exploiting network effects leads to a more significant improvement in the profits. Note that for this network, the lower bound is not tight.

*Preferential Attachment Graphs.* Finally, we consider networks that are generated according to a preferential attachment process, which is prevalent when modeling interactions in social networks. Networks are generated according to this process as follows: initially, the network consists of two agents and at each time instant a single agent is born and she is linked to two other agents (born before her) with probability proportional to their degrees. The process terminates when the population of agents is 100.

Given a random graph generated according to the process above, consider the following two extremes: (i) only newly born agents influence agents born earlier, i.e., the influence matrix $G^1$ is such that $G^1_{ij} > 0$ for all $i, j$ that are linked in the preferential attachment graph, and $j$ is born after $i$; (ii) only older agents influence new agents, i.e., the influence matrix $G^2$ is such that $G^2_{ij} > 0$ for all $i, j$ that are

**Figure 4.** Preferential attachment network example. Left: $b_i = 1$; right: $b_i = 3/4$ for all $i \in \mathcal{I}$.



linked in the preferential attachment graph, and $j$ is born before $i$. We assume that the nonnegative entries in each row of $G^1$ are equal and such that $G^1_{ij} = 1/d_i$, where $d_i$ is the number of nonnegative entries in row $i$ (equal influence) and similarly for $G^2$ (i.e., $G^2_{ij} = 1/d_i$ whenever $G^2_{ij} > 0$). As before, we consider a family of networks parameterixed by $\alpha$: $G^\alpha = \alpha G^1 + (1-\alpha)G^2$. The interaction matrix $G^\alpha$ models the situation in which agents weigh the consumption of the agents that are "born" earlier by $1-\alpha$, and that of the new ones by $\alpha$. Note that because $G^1$ and $G^2$ are normalized separately, in this model $G^\alpha$ need not be symmetric, and in fact it turns out that for all $\alpha$ there is a profit loss due to ignoring network effects.

In this model we consider two values for $b_i$: $b_i = 1$ and $b_i = 3/4$ for all $i \in \mathcal{I}$. Also, we impose the symmetry conditions, $a_i - c = 1$ for all $i \in \mathcal{I}$. Note that by construction, each preferential attachment graph is a random graph. For each $\alpha$, we generate 100 graph instances and report the averages of $\Pi_0/\Pi_N$ and the bounds over all instances. We also numerically verify that the assumptions of Theorem 8 hold for the generated networks.

The plots (given in Figure 4) are not symmetric because, as mentioned above, $G^1$ and $G^2$ are normalized differently. Interestingly, the profit loss from ignoring network effects is larger when older agents influence agents born later ($\alpha = 0$). This can be explained by the fact that older agents are expected to have higher centrality and act as interaction hubs for the network. As before, we see a larger improvement in profits when $b_i$ is small.

## 6. Conclusions

The paper studies a stylized model of pricing of divisible goods (services) over social networks, when consumers' actions are influenced by the choices of their peers. We provide a concrete characterization of the optimal scheme for a monopolist under different restrictions on the set of allowable pricing policies when consumers behave according to the unique Nash equilibrium profile of the corresponding game. We also illustrate the value of knowing the network structure by providing an explicit bound on the profit gains enjoyed by the monopolist due to this knowledge.

Certain modeling choices, i.e., Assumptions 1, 2 and 3, were dictated by the need for tractability and were also essential for clearly illustrating our insights. For example, as discussed in §4, removing Assumptions 2 or 3 potentially leads to solutions, where a subset of the agents does not consume the good at all. Consequently, the relation between optimal prices and the network structure holds only after restricting attention to the subgraph of agents who purchase the good. Because simpler explicit expressions can be obtained when all agents purchase positive amounts of the good, for most of our analysis we make assumptions that guarantee this. However, for completeness, we also discuss how the solution changes when these assumptions are relaxed.

Throughout the paper, we consider a setting of static pricing: the monopolist first sets prices and then the consumers choose their usage levels. Moreover, the game we define is essentially of complete information, because we assume that both the monopolist and the consumers know the network structure and the utility functions of the population. Extending our analysis by introducing incomplete information is an interesting direction for future research. Concretely, consider a monopolist that introduces a new product of unknown quality to a market.[11] Agents benefit the monopolist in two ways when purchasing the product; directly by increasing her revenues, and indirectly by generating information about the product's quality and making it more attractive to the rest of the consumer pool. What is the optimal (dynamic) pricing strategy for the monopolist?

Finally, note that in the current setup we consider a single seller (monopolist) so as to focus on explicitly characterizing the optimal prices as a function of the network structure. A natural departure from this model is studying a competitive environment. The simplest such setting would involve a small number of sellers offering a perfectly substitutable good to the market. Then, pricing may be even more aggressive than in the monopolistic environment: sellers may offer even larger discounts to "central" consumers, so as to subsequently exploit the effect of their decisions to the rest of the network. Potentially one could relate the *intensity of competition* with the network structure. In particular, one would expect the competition to be less fierce when the network consists of disjoint large subnetworks, because then sellers would segment the market at equilibrium and exercise monopoly power in their respective segments.

### Electronic Companion

An electronic companion to this paper is available as part of the online version at http://dx.doi.org/10.1287/opre.1120.1066.

### Appendix

Proof of Theorem 1. The proof makes use of the following lemmas.

Lemma 4. *Under Assumption* 1, *the spectral radius of* $\Lambda^{-1}G$ *is smaller than* 1, *and the matrix* $I - \Lambda^{-1}G$ *is invertible.*

Proof. Let $v$ be an eigenvector of $\Lambda^{-1}G$, with $\lambda$ being the corresponding eigenvalue. Let $v_i$ be the largest entry of $v$ in absolute values, i.e., $|v_i| \geqslant |v_j|$ for all $j \in \mathcal{I}$. Because, $(\Lambda^{-1}G)v = \lambda v$, it follows that

$$|\lambda v_i| = |(\Lambda^{-1}G)_i v| \leqslant \sum_{j\in\mathcal{I}} (\Lambda^{-1}G)_{ij}|v_j| \leqslant \frac{1}{2b_i}|v_i| \sum_{j\in\mathcal{I}} g_{ij} < |v_i|,$$

where $(\Lambda^{-1}G)_i$ denotes the $i$th row of $(\Lambda^{-1}G)$, the first and second inequalities use the fact that $(\Lambda^{-1}G)_{ij} = g_{ij}/2b_i \geqslant 0$, and the last inequality follows from Assumption 1. Because this is true for any eigenvalue-eigenvector pair, it follows that the spectral radius of $\Lambda^{-1}G$ is strictly smaller than 1.

Note that each eigenvalue of $I - \Lambda^{-1}G$ can be written as $1-\lambda$ where $\lambda$ is an eigenvalue of $\Lambda^{-1}G$. Because the spectral radius of $\Lambda^{-1}G$ is strictly smaller than 1, it follows that none of the eigenvalues of $I-\Lambda^{-1}G$ is zero; hence, the matrix is invertible. □

LEMMA 5. *Under Assumption* 1, *the pure Nash equilibrium sets of games* $\mathcal{G} = \{\mathcal{I}, \{u_i\}_{i\in\mathcal{I}}, [0,\infty)_{i\in\mathcal{I}}\}$ *and* $\bar{\mathcal{G}} = \{\mathcal{I}, \{u_i\}_{i\in\mathcal{I}}, [0,\bar{x}]_{i\in\mathcal{I}}\}$, *where* $\bar{x} > \max_i(|a_i-p_i|)/(2b_i - \sum_{j\in\mathcal{I}} g_{ij})$, *coincide.*

PROOF. Consider a strategy profile $\mathbf{x}$ and let $i$ denote the agent with the largest consumption, i.e., $x_i \geqslant x_j$, for all $j\neq i$. Observe that the best response function (in both games) is such that

$$\beta_i(\mathbf{x_{-i}}) = \frac{a_i-p_i}{2b_i} + \frac{1}{2b_i}\sum_{j\in\mathcal{I}} g_{ij}x_j \leqslant \frac{|a_i-p_i|}{2b_i} + \frac{x_i}{2b_i}\sum_{j\in\mathcal{I}} g_{ij},$$

provided that $\beta_i(\mathbf{x_{-i}}) > 0$. Because at equilibrium $x_i = \beta_i(\mathbf{x_{-i}})$, the above inequality implies that we have $x_i \leqslant (|a_i-p_i|)/(2b_i - \sum_{j\in\mathcal{I}} g_{ij}) < \bar{x}$ at equilibria in both games, i.e., at equilibria all players consume strictly less than $\bar{x}$. Because $\mathcal{G}$ and $\bar{\mathcal{G}}$ have the same payoff functions, and the set of strategy profiles in both games contain $[0,\bar{x}]^{|\mathcal{I}|}$, the equilibrium sets of these games coincide. □

We next show that $\bar{\mathcal{G}}$ is a supermodular game. Supermodular games are games that are characterized by strategic complementarities, i.e., the strategy sets of players are lattices, and the marginal utility of increasing a player's strategy raises with increases in the other players' strategies. For details and properties of these games, see Topkis (1998).

LEMMA 6. *The game* $\bar{\mathcal{G}} = \{\mathcal{I}, \{u_i\}_{i\in\mathcal{I}}, [0,\bar{x}]_{i\in\mathcal{I}}\}$ *is supermodular.*

PROOF. It is straightforward to see that the payoff functions are continuous, the strategy sets are compact subsets of $\mathbb{R}$, and for any players $i, j\in\mathcal{I}$, $\partial^2 u_i/(\partial x_i \partial x_j) \geqslant 0$. Hence, the game is supermodular. □

Now we are ready to complete the proof of the theorem. Because the set of equilibria of games $\mathcal{G}$ and $\bar{\mathcal{G}}$ coincide, we can focus on the equilibrium set of $\bar{\mathcal{G}}$. Because $\bar{\mathcal{G}}$ is a supermodular game, the equilibrium set has a minimum and a maximum element (Topkis 1998). Let $\mathbf{x}$ denote the maximum of the equilibrium set, and let set $S$ be such that $x_i > 0$ only if $i\in S$. If $S=\varnothing$, there cannot be another equilibrium point, because $\mathbf{x}=0$ is the maximum of the equilibrium set. Thus, for the sake of contradiction, we assume that $S\neq\varnothing$ and there exists another equilibrium, $\hat{\mathbf{x}}$, of the game.

By supermodularity of the game, it follows that $x_i \geqslant \hat{x}_i$ for all $i\in\mathcal{I}$. Let $k\in\arg\max_{i\in S} x_i - \hat{x}_i$. Because $\mathbf{x}$ and $\hat{\mathbf{x}}$ are not identical and $\mathbf{x}$ is the maximum of the equilibrium set, $x_k - \hat{x}_k > 0$.

Note that at any equilibrium $\mathbf{z}$ of $\mathcal{G}$, no player has incentive to increase her consumption; thus, $(\partial u_i(y_i, \mathbf{z_{-i}}, p_i)/\partial y_i)|_{y_i=z_i} \leqslant 0$. Moreover, if $z_i > 0$, because player $i$ does not have incentive to decrease her consumption, it also follows that $(\partial u_i(y_i, \mathbf{z_{-i}}, p_i)/\partial y_i)|_{y_i=z_i} = 0$. Recall that

$$u_i(x_i, \mathbf{x_{-i}}, p_i) = a_i x_i - b_i x_i^2 + x_i \cdot \sum_{j\in\{1,\ldots,n\}} g_{ij}\cdot x_j - p_i x_i;$$

hence, from this condition it follows that ($G_k$ denotes the $k$th row of $G$) at equilibria $\mathbf{x}$ and $\hat{\mathbf{x}}$ we have

$$a_k - p_k = 2b_k x_k - G_k\mathbf{x}$$

$$a_k - p_k \leqslant 2b_k \hat{x}_k - G_k\hat{\mathbf{x}},$$

where the latter condition holds with equality if $\hat{x}_k > 0$. Using these inequalities and Assumption 1, it follows that

$$x_k - \hat{x}_k \leqslant \frac{1}{2b_k}G_k(\mathbf{x}-\hat{\mathbf{x}}) = \frac{1}{2b_k}\sum_j g_{kj}(x_j - \hat{x}_j)$$

$$\leqslant \frac{x_k-\hat{x}_k}{2b_k}\sum_j g_{kj} < x_k - \hat{x}_k.$$

We reach a contradiction; hence, both $\mathcal{G}$ and $\bar{\mathcal{G}}$ have a unique equilibrium.

PROOF OF LEMMA 1. Define price vectors $\mathbf{p}^\alpha = \alpha\hat{\mathbf{p}} + (1-\alpha)\mathbf{p}$ for $\alpha\in[0,1]$. Because $\mathbf{p}\leqslant\hat{\mathbf{p}}$ (where the inequality is entrywise), for all $\alpha$ we have $\mathbf{p}\leqslant\mathbf{p}^\alpha\leqslant\hat{\mathbf{p}}$. Let $\mathbf{x}^\alpha$ denote the consumption vector corresponding to $\mathbf{p}^\alpha$ (note that $\mathbf{x}^0$ corresponds to $\mathbf{p}$ and $\mathbf{x}^1$ corresponds to $\hat{\mathbf{p}}$). To prove the claim, it is sufficient to show that $x_i^1 \leqslant x_i^0$ for every $i$. We show this by induction on the number of agents.

First assume that there is only a single agent, say $i$. In this case there are no network effects, and it follows from the definition of the best response function given in (2) that $x_i^\alpha = \max\{(a_i - p_i^\alpha)/2b_i, 0\}$. Because $p_i^\alpha$ is increasing in $\alpha$ by construction, it immediately follows that $x_i^1 \leqslant x_i^0$.

Assume that the claim holds when there are $k$ agents. We next prove that the claim holds for a network with $|\mathcal{I}| = k+1$ agents. We consider two cases: (i) we assume that at price $\mathbf{p}^0$ only a subset $S$ of agents consumes the good and $S\neq\mathcal{I}$, and (ii) we assume that at $\mathbf{p}^0$ all agents consume the good.

We start with the case where only agents in $S$ consume the good at $\mathbf{p}^0$, and $S\neq\mathcal{I}$. Because $\mathbf{x}_{\mathcal{I}-S}=0$, the best response condition in (2) suggests that if agents in $\mathcal{I}-S$ were not part of the network, then for the remaining agents, at price vector $\mathbf{p}_S^0$ the consumption equilibrium would be given by $\mathbf{y}_S^0 = \mathbf{x}_S^0$ (where $\mathbf{p}_S^0$ and $\mathbf{x}_S^0$ denote the entries of $\mathbf{p}^0$ and $\mathbf{x}^0$ vectors corresponding to agents in $S$). Because $\mathbf{p}_S^1 \geqslant \mathbf{p}_S^0$ and $|S|\leqslant k$, using the induction hypothesis, it follows that if only the agents in $S$ were present and the price vector was set to $\mathbf{p}_S^1$, then the consumption vector $\mathbf{y}_S^1$ would be such that $\mathbf{y}_S^0 \geqslant \mathbf{y}_S^1$. We next show that when all agents in $\mathcal{I}$ are present, at the equilibrium corresponding to $\mathbf{p}^1$ the consumption levels are such that $\mathbf{x}_S^1 = \mathbf{y}_S^1$, and $\mathbf{x}_i^1 = 0$ for all $i\notin S$. Because $\mathbf{y}_S^1$ is the equilibrium corresponding to price vector $\mathbf{p}_S^1$ (when agents in $\mathcal{I}-S$ are not present), it immediately follows that agents in $i\in S$ have no incentive to change their consumption levels when all agents in $\mathcal{I}$ are present and $\mathbf{x}_i^1 = 0$, for all $i\notin S$. Also, because $\mathbf{p}^1\geqslant\mathbf{p}^0$ and $\mathbf{x}_S^1\leqslant\mathbf{x}_S^0$ by construction, an agent $i\notin S$ has less incentive to purchase the good when compared to the case price offered was $\mathbf{p}^0$ (as can be seen from (2)). Because, under $\mathbf{p}^0$ agents in $\mathcal{I}-S$ do not consume the good, it follows that they do not have incentive to consume it under $\mathbf{p}^1$ as well. Thus, we conclude that $\mathbf{x}_S^1 = \mathbf{y}_S^1$, and $\mathbf{x}_i^1 = 0$, for all $i\notin S$ is the unique equilibrium under price vector $\mathbf{p}^1$, and moreover $\mathbf{x}^1\leqslant\mathbf{x}^0$.

Finally, we consider the case where at $\mathbf{p}^0$, all agents consume the good. Observe that for $\alpha\in[0,1]$ all agents consume a positive amount of the good at the equilibrium if and only if

$$\mathbf{x}^\alpha = (\Lambda - G)^{-1}(\mathbf{a}-\mathbf{p}^\alpha) > 0, \tag{16}$$

where the necessity follows from (6) and sufficiency follows from (2). Let $\alpha_0 > 0$ be the first value of $\alpha$ for which this equation does not hold. Due to continuity of (16) in prices, we conclude there exists a subset $S$ of $\mathcal{I}$ such that $\mathbf{y}_S = ((\Lambda - G)^{-1}\cdot$

$(\mathbf{a}-\mathbf{p}^{\alpha_0}))_k > 0$ for $k \in S$ and $\mathbf{y}_{\mathcal{I}-S} = ((\Lambda - G)^{-1}(\mathbf{a}-\mathbf{p}^{\alpha_0}))_k = 0$ for $k \in \mathcal{I} - S$. It follows from (2) that the consumption vectors $\mathbf{y}_S$ and $\mathbf{y}_{\mathcal{I}-S} = 0$ satisfy the best response conditions, and hence constitute an equilibrium where only the set of agents $S$ consume the good, i.e., $\mathbf{x}_S^{\alpha_0} = \mathbf{y}_S$, and $\mathbf{x}_{\mathcal{I}-S}^{\alpha_0} = 0$.

By Lemma 4 and Assumption 1, it follows that the matrix $\Lambda - G$ is invertible and the spectral radius of $\Lambda^{-1}G$ is smaller than 1. Therefore,

$$(\Lambda - G)^{-1} = (I - \Lambda^{-1}G)^{-1}\Lambda^{-1} = \sum_{k=0}^{\infty}(\Lambda^{-1}G)^k\Lambda^{-1}, \tag{17}$$

where the last equation follows because the spectral radius of $\Lambda^{-1}G$ is smaller than 1. Observe that entries of $\Lambda^{-1}G$ and $\Lambda^{-1}$ are nonnegative. Thus, it follows from (17) that the entries of $(\Lambda - G)^{-1}$ are nonnegative. Therefore, each entry of the vector $(\Lambda - G)^{-1}(\mathbf{a}-\mathbf{p})$ is weakly decreasing in $\mathbf{p}$. This, together with the definition of $\alpha_0$, imply that $\mathbf{x}^0 \geqslant \mathbf{x}^{\alpha_0}$.

Because under consumption vector $\mathbf{x}^{\alpha_0}$ some agents do not consume the good and $\mathbf{p}^1 \geqslant \mathbf{p}^{\alpha_0}$, it follows that the consumption vectors are such that $\mathbf{x}^{\alpha_0} \geqslant \mathbf{x}^1$ (as can be seen from part (i) of the induction claim). Because we have already established $\mathbf{x}^0 \geqslant \mathbf{x}^{\alpha_0}$, it follows that $\mathbf{x}^0 \geqslant \mathbf{x}^1$. Thus, we conclude that the induction claim holds for $k+1$ agents as well, and the lemma follows. □

Proof of Theorem 2. The proof makes use of the following lemma, which states that under Assumptions 1 and 2, it is optimal for the monopolist to offer prices so that all agents purchase a positive amount of the good. □

Lemma 7. *Let Assumptions* 1 *and* 2 *hold, and let* $\mathbf{p}^*$ *denote an optimal solution of the first stage of the pricing-consumption game. At the consumption equilibrium,* $\mathbf{x}^*$, *corresponding to* $\mathbf{p}^*$, *all consumers purchase a positive amount of the good, i.e.,* $x_i^* > 0$ *for all* $i \in \mathcal{I}$.

Proof. For the sake of contradiction, let $(\mathbf{p}^*, \mathbf{x}^*)$ be such that $x_i^* = 0$ for some $i \in \mathcal{I}$. We will construct a different price vector $\mathbf{p}'$ by decreasing the price offered to player $i$ and increasing the prices offered to the rest of the agents. In our construction, we will ensure that if $\mathbf{p}'$ is used, at equilibrium, agent $i$ purchases a positive amount of the good, and the consumptions of the remaining agents do not change. This will imply that the profit of the firm increases if $\mathbf{p}'$ is used.

Consider agent $k$'s utility maximization problem. Recall that for a given price vector $\mathbf{p}$ the best response function satisfies:

$$\beta_k(\mathbf{x}_{-\mathbf{k}}) = \max\left\{\frac{a_k - p_k}{2b_k} + \frac{1}{2b_k}\sum_{j\in\mathcal{I},\, j\neq k} g_{kj}x_j, 0\right\}. \tag{18}$$

Because at equilibrium $\mathbf{x}^*$, none of the agents have incentive to unilaterally deviate, it follows that $\mathbf{x}_k^* = (a_k - p_k^*)/(2b_k) + (1/2b_k)\sum_{j\in\mathcal{I},\, j\neq k} g_{kj}x_j^*$, if $x_k^* > 0$, and $\mathbf{x}_k^* = 0 \geqslant (a_k - p_k^*)/(2b_k) + (1/2b_k)\sum_{j\in\mathcal{I},\, j\neq k} g_{kj}x_j^*$ otherwise.

Consider a price vector $\mathbf{p}'$ such that $p_i' = c + \epsilon$, where $0 < \epsilon < a_i - c$ (such an $\epsilon$ exists from Assumption 2) and

$$p_j' = p_j^* + g_{ji}\left(\frac{a_i - p_i'}{2b_i} + \frac{1}{2b_i}\sum_{k\in\mathcal{I}, k\neq i} g_{ik}x_k^*\right), \quad \text{for all } j \neq i. \tag{19}$$

Note that because $a_i > c + \epsilon = p_i'$, it follows that $p_j' > p_j^*$ from (19). Let $\{x_k'\}$ be a consumption vector such that $x_k' = x_k^*$ if $k \neq i$ and $x_i' = (a_i - p_i')/(2b_i) + (1/2b_i)\sum_{j\in\mathcal{I}, j\neq i} g_{ij}x_j^* > 0$. From (18) for all $k \in \mathcal{I}$ we have

$$\beta_k(\mathbf{x}'_{-\mathbf{k}}) = \max\left\{\frac{a_k - p_k'}{2b_k} + \frac{1}{2b_k}\sum_{j\in\mathcal{I},\, j\neq k} g_{kj}x_j', 0\right\} \tag{20}$$

under price vector $\mathbf{p}'$. It follows from (20) and the definition of the consumption vector $\{x_k'\}$ that $x_i'$ is a best response to $\mathbf{x}'_{-i}$ when price offered to agent $i$ is $p_i'$, i.e., $x_i' = \beta(\mathbf{x}'_{-i})$. Combining (19) with (20), we obtain for any agent $k \neq i$:

$$\begin{aligned}
&\beta_k(\mathbf{x}'_{-\mathbf{k}})\\
&\quad= \max\left\{\frac{a_k - p_k'}{2b_k} + \frac{1}{2b_k}\sum_{j\in\mathcal{I},\, j\neq k} g_{kj}x_j', 0\right\}\\
&\quad= \max\left\{\frac{a_k - p_k'}{2b_k} + \frac{1}{2b_k}g_{ki}x_i' + \frac{1}{2b_k}\sum_{j\in\mathcal{I},\, j\neq k,\, j\neq i} g_{kj}x_j^*, 0\right\}\\
&\quad= \max\left\{\frac{a_k - p_k^*}{2b_k} - \frac{g_{ki}}{2b_k}\left(\frac{a_i - p_i'}{2b_i} + \frac{1}{2b_i}\sum_{j\in\mathcal{I},\, j\neq i} g_{ij}x_j^*\right)\right.\\
&\qquad\qquad \left. + \frac{1}{2b_k}g_{ki}x_i' + \frac{1}{2b_k}\sum_{j\in\mathcal{I},\, j\neq k,\, j\neq i} g_{kj}x_j^*, 0\right\}\\
&\quad= \max\left\{\frac{a_k - p_k^*}{2b_k} + \frac{1}{2b_k}\sum_{j\in\mathcal{I},\, j\neq k,\, j\neq i} g_{kj}x_j^*, 0\right\},
\end{aligned}$$

where the second equality uses $x_k^* = x_k'$ for all $k \neq i$, the third equality follows by (19), and the last one follows from the definition of $x_i'$. Because $\mathbf{x}^*$ is the consumption equilibrium corresponding to $\mathbf{p}^*$ and $x_i^* = 0$ it follows that $\sum_{j\in\mathcal{I},\, j\neq k,\, j\neq i} g_{kj}x_j^* = \sum_{j\in\mathcal{I},\, j\neq k} g_{kj}x_j^*$, and hence for $k \neq i$ we obtain

$$\beta_k(\mathbf{x}'_{-\mathbf{k}}) = \max\left\{\frac{a_k - p_k^*}{2b_k} + \frac{1}{2b_k}\sum_{j\in\mathcal{I},\, j\neq k} g_{kj}x_j^*, 0\right\} = x_k^* = x_k'.$$

Because as explained earlier it is also true that $\beta_i(\mathbf{x}'_{-\mathbf{i}}) = x_i'$ we conclude that $\mathbf{x}'$ is the unique consumption equilibrium corresponding to price vector $\mathbf{p}'$.

Moreover, from (20) we obtain that $x_i' > \epsilon'$ for some $\epsilon' > 0$. Because $p_i' - c = \epsilon$, and $p_j' \geqslant p_j$ for all $j \in \mathcal{I}, j \neq i$, it follows that the monopolist increases her profits by at least $\epsilon \cdot \epsilon'$ and the lemma follows. □

Lemma 7 and the fact that $\mathbf{x}_S = (\Lambda_S - G_S)^{-1}(\mathbf{a}_S - \mathbf{p}_S)$ when a set of agents $S$ consume the product (cf. Equation (5)) imply that the optimal price $\mathbf{p}^*$ and the corresponding equilibrium vector $\mathbf{x}^*$ satisfy

$$\mathbf{a} - \Lambda\mathbf{x}^* + G\mathbf{x}^* = \mathbf{p}^*. \tag{21}$$

Thus, the problem that the monopolist faces can be rewritten as:

$$\begin{aligned}
\max_{\mathbf{p},\mathbf{x}} \quad & \sum_i p_i x_i - c x_i\\
\text{s.t.} \quad & a_i - 2b_i x_i + \sum_{j\in\mathcal{I}} g_{ij}x_j - p_i = 0, \quad \text{for every } i.\\
& x_i \geqslant 0,
\end{aligned}$$

from which we obtain by the KKT conditions (and because we have already established that $x_i^* > 0$ for all $i \in \mathcal{I}$):

$$\mathbf{a} - c\mathbf{1} = (2\Lambda - (G + G^T))\mathbf{x}^*,$$

and hence

$$\mathbf{x}^* = \left(\Lambda - \frac{G+G^T}{2}\right)^{-1} \frac{\mathbf{a}-c\mathbf{1}}{2}. \quad \square$$

Substituting $\mathbf{x}^*$ to (21) the claim follows.

Proof of Theorem 3. By Lemma 4, $(\Lambda - G)$ is nonsingular; thus, rearranging terms in (7), it follows that

$$\begin{aligned}\mathbf{p} &= \mathbf{a} - (\Lambda - G)\left(\Lambda - G - \frac{G^T - G}{2}\right)^{-1} \frac{\mathbf{a}-c\mathbf{1}}{2} \\ &= \mathbf{a} - \left(I - \frac{G^T-G}{2}(\Lambda - G)^{-1}\right)^{-1} \frac{\mathbf{a}-c\mathbf{1}}{2}. \qquad (22)\end{aligned}$$

To complete the proof, we need the matrix inversion lemma:

Lemma 8 (Matrix Inversion Lemma). *Given square matrices of appropriate size,*

$$(A - UD^{-1}V)^{-1} = A^{-1} + A^{-1}U(D - VA^{-1}U)^{-1}VA^{-1}$$

*if $A$ and $D$ are nonsingular.*

From this lemma, by setting $A, V = I$, $D = \Lambda - G$ and $U = (G^T - G)/2$, we obtain

$$\begin{aligned}&\left(I - \frac{G^T-G}{2}(\Lambda - G)^{-1}\right)^{-1} \\ &\quad = I + \frac{G^T - G}{2}\left(\Lambda - G - \frac{G^T-G}{2}\right)^{-1} \\ &\quad = I + \frac{G^T - G}{2}\left(\Lambda - \frac{G^T+G}{2}\right)^{-1}.\end{aligned}$$

Thus, from (22) it follows that

$$\mathbf{p} = \frac{\mathbf{a}+c\mathbf{1}}{2} - \frac{G^T-G}{2}\left(\Lambda - \frac{G^T+G}{2}\right)^{-1}\frac{\mathbf{a}-c\mathbf{1}}{2}. \qquad (23)$$

When $a_i = a_0$, $b_i = b_0$ for all $i \in \mathcal{I}$, substituting $\Lambda = 2b_0 I$ and $\mathbf{a} = a_0\mathbf{1}$, the vector of optimal prices can be rewritten as

$$\begin{aligned}\mathbf{p} &= \frac{a_0+c}{2}\mathbf{1} + \frac{a_0 - c}{8b_0} G\mathcal{K}\left(\frac{G+G^T}{2}, \frac{1}{2b_0}\right) \\ &\quad - \frac{a_0-c}{8b_0} G^T \mathcal{K}\left(\frac{G+G^T}{2}, \frac{1}{2b_0}\right). \quad \square\end{aligned}$$

Proof of Theorem 4. Immediate from (23) and the definition of the weighted Bonacich centrality.

Proof of Lemma 2. The weakly decreasing property of $\{x_i\}$ for each $i \in \mathcal{I}$, immediately follows from Lemma 1. The equilibrium characterization in (6) implies that if at price $p_0$ a set of agents $S$ consume a positive amount of the good, then their consumption vector is given by

$$\mathbf{x}_S(p_0) = (\Lambda_S - G_S)^{-1}(\mathbf{a}_S - p_0\mathbf{1}). \qquad (24)$$

As shown in the proof of Lemma 1, the entries of the matrix $(\Lambda_S - G_S)^{-1}$ are nonnegative. Because the matrix is invertible, none of the rows of the matrix $(\Lambda_S - G_S)^{-1}$ are identically equal to 0. Therefore, it follows that $((\Lambda_S - G_S)^{-1}\mathbf{1})_i > 0$; hence, (24) implies that $x_i$ is strictly decreasing in $p_0$ for all $i \in S$. $\square$

Proof of Theorem 5. By Lemma 2, it follows that the consumption vector at equilibrium is monotonically decreasing in $p$. Moreover, if the set of agents that purchase a positive amount of the good at equilibrium is given by $S$, then the consumption vector is as in (24). In order to prove the claim, we show that among a set of agents $S$, who purchase a positive amount of the good, the agent who first stops purchasing the good as the price increases is the one with the smallest centrality gain. Moreover, the price at which this agent stops is proportional to her centrality gain in the graph restricted to agent set $S$.

Consider a set of agents $S$ and price $p_0$ such that $p_0 < a_i$ for all $i \in S$. From (2), we obtain that all agents in $S$ have an incentive to purchase a positive amount of the good, regardless of the consumption levels of their peers. Thus, it follows that if this price is used at equilibrium, all agents in $S$ purchase a positive amount of the good. Using (24) and the definition of the weighted Bonacich centrality, the consumption vector can be rewritten as

$$\begin{aligned}\mathbf{x}_S(p_0) &= \Lambda_S^{-1}(I - G_S\Lambda_S^{-1})^{-1}(\mathbf{a}_S - p_0\mathbf{1}) \\ &= \Lambda_S^{-1}\tilde{K}(G_S, \Lambda_S^{-1}, \mathbf{a}_S) \\ &\quad - p_0\Lambda_S^{-1}\tilde{\mathcal{K}}(G_S, \Lambda_S^{-1}, \mathbf{1}). \qquad (25)\end{aligned}$$

Equivalently, for any $i \in S$, the consumption of player $i$ can be given as

$$x_i(p_0) = \frac{1}{2b_i}(\tilde{\mathcal{K}}_i(G_S, \Lambda_S^{-1}, \mathbf{a}_S) - p_0\tilde{\mathcal{K}}_i(G_S, \Lambda_S^{-1}, \mathbf{1})).$$

Therefore, it follows that when

$$p = \min_{i\in S}\frac{\tilde{\mathcal{K}}_i(G_S, \Lambda_S^{-1}, \mathbf{a}_S)}{\tilde{\mathcal{K}}_i(G_S, \Lambda_S^{-1}, \mathbf{1})} = \min_{i \in S} H_i(G_S, \Lambda_S^{-1}, \mathbf{a}_S), \qquad (26)$$

then for the first time a group of agents in $S$ stops purchasing the good.

It follows from (26) that if $p < p_1$ all agents in $\mathcal{I}$ purchase a positive amount of the good, and if the price is increased to $p_1 = \min_{i\in\mathcal{I}} H_i(G, \Lambda^{-1}, \mathbf{a})$, then agents in the set $D_1 = \arg\min_{i\in\mathcal{I}} H_i(G, \Lambda^{-1}, \mathbf{a})$, stop purchasing the good. By monotonicity, these agents do not purchase the good when the price of the good is further increased. Furthermore, monotonicity also implies that the agents in the set $\mathcal{I} - D_1$ stop purchasing the good at a higher price. Using (26) iteratively, it can be seen that the agents in $D_k$ stop purchasing the good at price $p_k$ for $k \in \{1, \dots n\}$.

Thus, the first claim follows by construction of prices $p_k$ and the monotonicity of the consumption vector. The second claim follows from the fact that if $p < p_1$, then all agents purchase a positive amount of the good, and if $p_k \leqslant p \leqslant p_{k+1}$, then only agents in $\mathcal{I} - \bigcup_{l=1}^k D_l = I_k$ purchase a positive amount of the good. Therefore, by (6), the claim follows. $\square$

Proof of Theorem 6. Recall that the MAX-CUT problem (with 0, 1 weights) is defined as follows.

Definition 5 (MAX-CUT Problem). Let $G = (V, E)$ be an undirected graph and for all $i, j \in V$, define $g_{ij}$ such that $g_{ij} = 1$ if $(i, j) \in E$ and $g_{ij} = 0$ otherwise. Find the cut with maximum size, i.e., find a partition of the agent set $V$ into $S$ and $V - S$ such that the following sum is maximized:

$$\sum_{i\in S,\, j\in V-S} g_{ij}.$$

Note that the MAX-CUT problem is equivalent to the following optimization problem:

$$\begin{aligned} \max \quad & \sum_{(i,j)\in E} W_{ij}(1-x_i x_j) \\ \text{s.t.} \quad & x_i \in \{-1,1\} \quad \text{for all } i \in V, \end{aligned}$$

where $W$ denotes the matrix of weights (we assume that $W_{ij} = W_{ji} \in \{0,1\}$). The optimal solution of the above problem corresponds to a cut as follows: let $S$ be the set of agents that were assigned value 1 in the optimal solution. Then, it is straightforward to see that the value of the objective function corresponds to the size of the cut defined by $S$ and $V-S$. We can further rewrite the optimization problem as:

$$\text{(P0)} \quad \begin{aligned} \min \quad & \mathbf{x}^T W \mathbf{x} \\ \text{s.t.} \quad & x_i \in \{-1,1\} \quad \text{for all } i \in V. \end{aligned}$$

It is well-known that this problem is NP-hard (Garey and Johnson 1979). Consider the following related problem:

$$\text{(P1)} \quad \begin{aligned} \min \quad & \mathbf{x}^T W \mathbf{x} \\ \text{s.t.} \quad & x_i \in \{-1,1\} \quad \text{for all } i \in V, \end{aligned}$$

where $W$ is a symmetric matrix with rational entries that satisfy $0 < W^T = W < 1$ (inequality is entrywise). We next show by reduction from MAX-CUT that P1 is also NP-hard.

Lemma 9. P1 *is NP-hard.*

Proof. We prove the claim by reduction from P0. Let $W$ be the weight matrix in an instance of P0. Then, let $W_\epsilon = \frac{1}{2}(\epsilon + W)$, where $\epsilon$ is a rational number such that $0 < \epsilon < 1/(2n^2)$ and $|V| = n$. Observe that for any feasible $\mathbf{x}$ in P0 or P1 it follows that

$$2\mathbf{x}^T W_\epsilon \mathbf{x} - n^2\epsilon \leqslant \mathbf{x}^T W \mathbf{x} \leqslant 2\mathbf{x}^T W_\epsilon \mathbf{x} + n^2\epsilon.$$

Because the objective value of P0 is always an integer and $n^2\epsilon < \frac{1}{2}$, it follows that the cost of P0 for any feasible vector $\mathbf{x}$ can be obtained from the cost of P1 (with $W_\epsilon$) by scaling and rounding. Hence, it can be seen that from the optimal solution of the latter, we can immediately obtain the optimal solution and the value of the former (as rounding is a monotone operation). Therefore, because P0 is NP-hard it follows that P1 is also NP-hard and the claim follows. □

Next we prove Theorem 6 by using a reduction from P1 to OPT. We consider the special instances of OPT for which we have $G = G^T$, $c = 0$, $\mathbf{a} = [a, \ldots, a]$, where $a = p_L + p_H$. Observe that under this setting, Assumptions 2 and 3 hold and $\hat{\mathbf{a}} = p_N \mathbf{1} = \hat{c}\mathbf{1}$. Hence, using (9), such instances of OPT can be rewritten as (by adding a constant and scaling the objective function)

$$\text{(OPT2)} \quad \begin{aligned} \min \quad & \mathbf{x}^T (\Lambda - G)^{-1} \mathbf{x} \\ \text{s.t.} \quad & x_i \in \{-1,1\} \quad \text{for all } i \in \mathcal{I}. \end{aligned}$$

Next we show that any instance of P1 can be transformed into an instance of OPT2 (or equivalently OPT) where $\Lambda$ and $G$ are matrices with rational entries, $G = G^T \geqslant 0$ (the inequality is entry wise), $\operatorname{diag}(G) = 0$, $\Lambda_{ij} = 0$ if $i \neq j$, $\Lambda_{kk} > 0$, $\Lambda$ and $G$ matrices satisfy Assumption 1. Note that Assumption 1 is equivalent to requiring $(\Lambda - G)_k \mathbf{1} > 0$ for all $k$, where $\mathbf{1}$ denotes the vector with all entries equal to one, and $(\Lambda - G)_k$ denotes the $k$th row of $(\Lambda - G)$. Observe that these requirements on $\Lambda$ and $G$ ensure that the corresponding instance of OPT satisfies the assumptions of the theorem.

Consider an instance of P1 with $W > 0$. Note that because $x_i^2 = 1$, P1 is equivalent to

$$\begin{aligned} \min \quad & \mathbf{x}^T (W + \gamma I)\mathbf{x} \\ \text{s.t.} \quad & x_i \in \{-1,1\} \quad \text{for all } i \in V, \end{aligned} \tag{27}$$

where we choose $\gamma$ as an integer such that $\gamma > 4\max\{\rho(W), \sum_{i,j} W_{ij}/\min_{ij} W_{ij}\}$, and $\rho(\cdot)$ denotes the spectral radius of its argument. Next, we show that this optimization problem is equivalent to an instance of OPT2 by showing that $(W + \gamma I) = (\Lambda - G)^{-1}$ for some $\Lambda$ and $G$ satisfying the requirements above.

The definition of $\gamma$ implies that the spectral radius of $W/\gamma$ is smaller than 1. Therefore, it follows that

$$(W + \gamma I)^{-1} = \frac{1}{\gamma}\left(I - \frac{W}{\gamma} + \frac{W^2}{\gamma^2} \cdots\right).$$

Observe that for all $i, j \in \{1, 2, \ldots, n\}$,

$$\begin{aligned} \left(W - \frac{W^2}{\gamma}\right)_{ij} &= W_{ij} - \frac{\sum_k W_{ik}W_{kj}}{\gamma} \geqslant W_{ij} - \frac{\sum_k W_{ik}}{\gamma} \\ &\geqslant W_{ij} - \frac{\min_{kl} W_{kl}}{4} \geqslant W_{ij} - \frac{W_{ij}}{4} > 0, \end{aligned}$$

where the first inequality follows from the fact that $0 < W < 1$, and the second inequality follows from the definition of $\gamma$. Thus, all entries of $(W - W^2/\gamma)$ are positive. Rewriting $(W + \gamma I)^{-1}$ as

$$(W+\gamma I)^{-1} = \frac{1}{\gamma}\left(I - \frac{1}{\gamma}\left(W - \frac{W^2}{\gamma}\right) - \frac{W^2}{\gamma^3}\left(W - \frac{W^2}{\gamma}\right)\cdots\right)$$

and noting that all entries of $W$ and $(W - (W^2/\gamma))$ are positive, the above equality implies that the off-diagonal entries of $(W + \gamma I)^{-1}$ are negative. Thus, $(W + \gamma I)^{-1} = (\Lambda - G)$ for some diagonal matrix $\Lambda$ and for some $G \geqslant 0$ with $\operatorname{diag}(G) = 0$. Moreover, because $W = W^T$, $G$ is also a symmetric matrix. Note that because the spectral radius of $W/\gamma$ is smaller than 1, it also follows that

$$\begin{aligned} &((\Lambda - G)\mathbf{1})_k \\ &\quad = ((W + \gamma I)^{-1}\mathbf{1})_k = \left(\frac{1}{\gamma}\left(I - \frac{W}{\gamma} + \frac{W^2}{\gamma^2} \cdots\right)\mathbf{1}\right)_k. \end{aligned}$$

Because $W > 0$, it can be seen that $W^l > 0$ for all $l \in \mathbb{Z}_+$. Using this observation, we obtain

$$((\Lambda - G)\mathbf{1})_k \geqslant \frac{1}{\gamma}\left(1 + \left(-\frac{W\mathbf{1}}{\gamma} - \frac{W^2\mathbf{1}}{\gamma^2} \cdots\right)_k\right).$$

By the definition of $\gamma$, it follows that $(W\mathbf{1})/\gamma \leqslant ((\sum_{ij} W_{ij})/\gamma)\mathbf{1} \leqslant \frac{1}{4}\mathbf{1}$. Therefore, the above inequality implies that

$$((\Lambda - G)\mathbf{1})_k \geqslant \frac{1}{\gamma}\left(1 - \frac{1}{4}\left(\sum_{l=0}^{\infty}\left(\frac{1}{4}\right)^l\right)\right) = \frac{1}{\gamma}\left(\frac{2}{3}\right) > 0.$$

Thus, Assumption 1 holds for the game defined with the matrices $\Lambda$ and $G$. Note that because the off-diagonal entries of $\Lambda - G$ are nonpositive, Assumption 1 implies that the diagonal entries of $\Lambda$ are positive.

Therefore, problem P1 can be reduced to an instance of OPT2 by defining $\Lambda$ and $G$ according to $(W+\gamma I)^{-1} = (\Lambda - G)$. Thus, it follows that OPT2, and hence OPT, are NP-hard.

Proof of Theorem 7. First, we describe a semidefinite programming (SDP) relaxation for the following optimization problem:

$$\begin{aligned} \max \quad & \tfrac{1}{4}\sum_{i,j} w_{ij}(1 - x_i x_j) \\ \text{s.t.} \quad & x_i \in \{-1, 1\} \quad \text{for all } i \in V. \end{aligned} \tag{28}$$

Note that (28) can be relaxed to

$$\begin{aligned} \max \quad & \tfrac{1}{4}\sum_{i,j} w_{ij}(1 - \nu_i \cdot \nu_j) \\ \text{s.t.} \quad & \nu_i \in S_n \quad \text{for all } i \in V \end{aligned} \tag{29}$$

where $\nu_i \cdot \nu_j$, denotes the regular inner product of vectors $\nu_i, \nu_j \in \mathbb{R}^n$, and $S_n$ denotes the $n$-dimensional unit sphere, i.e., $S_n = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x}\cdot\mathbf{x} = 1\}$. We next show that (29) leads to a semidefinite program.

Consider a collection of vectors $\{\nu_1, \ldots, \nu_n\}$ such that $\nu_i \in S_n$. Define a symmetric matrix $Y \in R^{n\times n}$, such that $Y_{ij} = \nu_i \cdot \nu_j$ and $Y_{ii} = 1$. It can be seen that $Y = F^T F$, where $F \in \mathbb{R}^{n\times n}$ is such that $F = [\nu_1, \nu_2, \ldots, \nu_n]$. This implies that $Y \succeq 0$. Conversely, consider a positive semidefinite matrix $Y \in \mathbb{R}^{n\times n}$, such that $Y_{ii} = 1$. Because $Y$ is positive semidefinite, there exists $F \in \mathbb{R}^{n\times n}$ (which can be obtained through the Cholesky factorization of the original matrix) such that $Y = F^T F$. Denote the columns of $F$ by $\nu_i$, i.e., $F = [\nu_1, \nu_2, \ldots, \nu_n]$. Because $Y_{ii} = 1$ and $Y = F^T F$, it follows that $\nu_i \cdot \nu_i = 1$. These arguments imply that the feasible set in (29) can equivalently be defined in terms of positive semidefinite matrices. Hence, it follows that the optimization problem in (29) can be equivalently written as

$$\begin{aligned} \max \quad & \tfrac{1}{4}\sum_{i,j} w_{ij}(1 - Y_{ij}) \\ \text{s.t.} \quad & Y_{i,i} = 1 \quad \text{for all } i \in V, \\ & Y \succeq 0. \end{aligned} \tag{30}$$

Next, we show how to obtain a provable approximation guarantee for binary quadratic optimization problems of the form:

$$\begin{aligned} \max \quad & \mathbf{x}^T Q \mathbf{x} + 2\mathbf{d}^T \mathbf{x} + z \\ \text{s.t.} \quad & x_i \in \{-1, 1\}, \quad i \in \{1, \ldots, n\}, \end{aligned} \tag{31}$$

where $Q$, $\mathbf{d}$, and $z$ have rational entries, i.e., $Q \in \mathbb{Q}^{n\times n}$, $\mathbf{d} \in \mathbb{Q}^n$, and $z \in \mathbb{Q}$. Observe that $\mathbf{x}^T Q \mathbf{x} = \mathrm{Trace}(Q) + \mathbf{x}^T \tilde{Q}\mathbf{x}$, where $\tilde{Q} = Q - \mathrm{diag}(Q)$ and $x_i \in \{-1, 1\}$. Thus, the diagonal entries of the $Q$ matrix can be expressed as a part of the constant term, and thus, we can assume that $\mathrm{diag}(Q) = 0$ without any loss of generality. Also, again without loss of generality, we can assume that the matrix $Q$ is symmetric, because $\mathbf{x}^T Q \mathbf{x} = \mathbf{x}^T Q^T \mathbf{x} = \mathbf{x}^T (Q + Q^T)/2\mathbf{x}$.

Consider the following optimization problem

$$\begin{aligned} \max \quad & [\mathbf{x}; y]^T \hat{Q} [\mathbf{x}; y] + z \\ \text{s.t.} \quad & x_i \in \{-1, 1\}, \quad i \in \{1, \ldots, n\}, \\ & y \in \{-1, 1\}, \end{aligned} \tag{32}$$

where

$$\hat{Q} = \left[\begin{array}{c|c} Q & \mathbf{d} \\ \hline \mathbf{d}^T & 0 \end{array}\right]. \tag{33}$$

Note that given a feasible solution $[\mathbf{x}; y]$ of (32), another feasible solution with the same objective value is $-[\mathbf{x}; y]$. Therefore, given an optimal solution of (32), another optimal solution where $y = 1$ can be obtained. Because by construction $[\mathbf{x}; y]^T \hat{Q} [\mathbf{x}; y] = \mathbf{x}^T Q \mathbf{x} + 2y\mathbf{d}^T \mathbf{x}$, it follows that the $\mathbf{x}$ corresponding to such an optimal solution of (32) is also optimal for (31) and the optimal objective values for the two problems are equal. Therefore, instead of solving (31), we focus on (32).

Following (28), and (29), we can relax (32) to:

$$\begin{aligned} \max \quad & \sum_{ij} \nu_i \cdot \nu_j \hat{Q}_{ij} + z \\ \text{s.t.} \quad & \nu_i \in S_{n+1}, \quad i \in \{1, \ldots, n, n+1\}, \end{aligned} \tag{34}$$

and obtain an equivalent SDP (by defining $Y_{ij} = \nu_i \cdot \nu_j$) as follows:

$$\begin{aligned} \max \quad & \sum_{ij} Y_{ij} \hat{Q}_{ij} + z \\ \text{s.t.} \quad & Y_{ii} = 1 \quad i \in \{1, \ldots, n, n+1\}, \\ & Y \succeq 0. \end{aligned} \tag{35}$$

Using this SDP relaxation, Algorithm 2 provides an approximate solution to the original problem. We prove this, using a similar approach to Goemans and Williamson (1995).

**Algorithm 2** (Compute $\{x_1, \ldots, x_n\}$, which is an approximate solution of (31))

*Step* 1. Solve the SDP relaxation in (35), find an optimal $Y$.
*Step* 2. Obtain the Cholesky factorization of $Y$, i.e., find $F$ such that $Y = F^T F$. Denote the $i$th column of $F$ by $\nu_i$. Denote by $\nu_{n+1}$ the vector corresponding to the variable $y$ in (32).
*Step* 3. Let $r$ be a vector uniformly distributed on the unit sphere $S_{n+1}$.
*Step* 4. Let $S = \{i \mid r \cdot \nu_i \geqslant 0\}$. If $n+1 \in S$, then set $x_i = 1$ for all $i \in S \cap \{1, \ldots, n\}$ and set the remaining $x_i$ to $-1$. Else, if $n+1 \notin S$, then set $x_i = -1$ for all $i \in S \cap \{1, \ldots, n\}$ and set the remaining $x_i$ to $-1$.
**Output:** $\{x_1, \ldots, x_n\}$.

Proposition 1. *Let* $z \geqslant \sum_{i,j} |\hat{Q}_{ij}|$. *Then, a solution given by Algorithm* 2, *in expectation achieves at least* 0.878 *times the optimal objective value of the original problem in* (31).

Proof. Let $W$ denote the objective value of a solution the algorithm provides, $W_M$ denote the optimal solution of the underlying quadratic optimization problem (31), and $W_P$ denote the optimal value of the SDP relaxation. Let $\{\nu_i\}$ denote the solution of SDP relaxation, then the corresponding optimal value can be given as

$$W_P = \sum_{i,j} \hat{Q}_{ij} \nu_i \cdot \nu_j + z.$$

It can be seen that for solutions the algorithm provides, the probability[12] that agents $i$ and $j$ have opposite signs is $(\arccos(\nu_i \cdot \nu_j))/\pi$, and similarly the probability that agents have the same sign is $1 - (\arccos(\nu_i \cdot \nu_j))/\pi$ (see Goemans and

Williamson 1995). Thus, the expected contribution of this pair of agents to the objective function is given by $\hat{Q}_{ij}(1-2((\arccos(\nu_i\cdot\nu_j))/\pi))$. Hence, it follows that the expected value of a solution the algorithm provides is given by

$$E[W]=\sum_{i,j}\left(1-2\frac{\arccos(\nu_i\cdot\nu_j)}{\pi}\right)\hat{Q}_{i,j}+z.$$

Because $z\geqslant\sum_{i,j}|\hat{Q}_{ij}|$, it follows that both $W_M$ and $E[W]$ are nonnegative; also, because $W_P$ corresponds to the optimal solution of the relaxation, it follows that $W_P\geqslant W_M$. Using these, it follows that

$$W_P=\sum_{i,j;\hat{Q}_{ij}>0}\hat{Q}_{ij}(1+\nu_i\cdot\nu_j)+\sum_{i,j;\hat{Q}_{ij}<0}|\hat{Q}_{ij}|(1-\nu_i\cdot\nu_j)+z_2$$

and

$$E[W]=\sum_{i,j;\hat{Q}_{ij}>0}\hat{Q}_{ij}\left(2-2\frac{\arccos(\nu_i\cdot\nu_j)}{\pi}\right)+\sum_{i,j;\hat{Q}_{ij}<0}|\hat{Q}_{i,j}|2\frac{\arccos(\nu_i\cdot\nu_j)}{\pi}+z_2,$$

where $z_2=z-\sum_{i,j}|\hat{Q}_{ij}|\geqslant 0$. Because the arccos function satisfies $\arccos x/\pi\geqslant(\alpha/2)(1-x)$ and $1-\arccos x/\pi\geqslant(\alpha/2)(1+x)$ for all $x\in[-1,1]$, where $\alpha\approx 0.878$ (see Goemans and Williamson 1995), it follows that $E[W]>0.878W_P\geqslant 0.878W_M$. $\square$

This result can be extended by relaxing the condition $z\geqslant\sum_{i,j}|Q_{ij}|$. To see this, we first add a positive constant to the objective function of the original problem, ensuring that the modified problem satisfies this condition, and then provide an approximation to this new problem. Note that the constant change in the objective function does not affect the output of the algorithm. The following corollary summarizes this result.

Corollary 2. *Let $W$ denote the objective value of a solution output by Algorithm 2 and $W_M$ denote the optimal solution of the underlying quadratic optimization problem. Then,* $E[W]+\sum_{ij}|\hat{Q}_{ij}|-z>0.878(W_M+\sum_{ij}|\hat{Q}_{ij}|-z)$.

Finally, when Assumption 3 holds, using (9), and $A=(\Lambda-G)^{-1}$, the pricing problem of the firm can be expressed as

$$\begin{aligned}\max\quad&\delta^2\mathbf{y}^TA\mathbf{y}+\delta(\hat{\mathbf{a}}^TA^T-\hat{c}\mathbf{1}^TA)\mathbf{y}+\hat{c}\mathbf{1}^TA\hat{\mathbf{a}}\\ \text{s.t.}\quad&y_i\in\{-1,1\}\quad\text{for all } i\in\mathcal{I}.\end{aligned}\tag{36}$$

Using Lemma 1, it can be seen that Assumption 1 and nonnegativity of entries of $\Lambda$ and $G$ imply that $A=(\Lambda-G)^{-1}=\Lambda^{-1}\sum_{l=0}^{\infty}(G\Lambda^{-1})^l$ is a matrix with nonnegative entries. Therefore, Theorem 7 follows by using the formulation (36), rewriting the pricing problem of the monopolist in the form of (31), and applying Corollary 2.

Proof of Lemma 3. Note that ignoring the network effects is equivalent to assuming that $G=0$. Thus, the optimal prices for the setting described in the statement of the lemma, denoted by $\mathbf{p}_0$ and $\mathbf{p}_N$, respectively, are given by (as can be seen from Theorem 2):

$$\mathbf{p}_0=\frac{\mathbf{a}+c\mathbf{1}}{2}\tag{37}$$

and

$$\mathbf{p}_N=\mathbf{a}-(\Lambda-G)\left(\Lambda-\frac{G+G^T}{2}\right)^{-1}\frac{\mathbf{a}-c\mathbf{1}}{2}.\tag{38}$$

By Lemma 7, under the price vector $\mathbf{p}_N$ all agents purchase a positive amount of the good. Assumption 2 implies that $a_i>c$ for all $i\in\mathcal{I}$. Thus, under the price vector $\mathbf{p}_0$, $a_i$ is greater than the price offered to agent $i$, and agents still purchase a positive amount of the good. The corresponding consumption vectors (denoted by $\mathbf{x}_0$ and $\mathbf{x}_N$) are given by (cf. Equation (5))

$$\mathbf{x}_0=(\Lambda-G)^{-1}(\mathbf{a}-\mathbf{p}_0)=(\Lambda-G)^{-1}\frac{\mathbf{a}-c\mathbf{1}}{2}\tag{39}$$

and

$$\begin{aligned}\mathbf{x}_N&=(\Lambda-G)^{-1}(\mathbf{a}-\mathbf{p}_N)\\&=\left(\Lambda-\frac{G+G^T}{2}\right)^{-1}\frac{\mathbf{a}-c\mathbf{1}}{2}.\end{aligned}\tag{40}$$

It follows that

$$\Pi_0=(\mathbf{p}_0-c\mathbf{1})^T\mathbf{x}_0=\frac{\mathbf{a}-c\mathbf{1}}{2}(\Lambda-G)^{-1}\frac{\mathbf{a}-c\mathbf{1}}{2},\tag{41}$$

and if we let $M=\Lambda-G$, Equations (38) and (40) imply that

$$\begin{aligned}\Pi_N&=(\mathbf{p}_N-c\mathbf{1})^T\mathbf{x}_N\\&=\left(\mathbf{a}-c\mathbf{1}-M\left(\frac{M+M^T}{2}\right)^{-1}\frac{\mathbf{a}-c\mathbf{1}}{2}\right)^T\cdot\left(\left(\frac{M+M^T}{2}\right)^{-1}\frac{\mathbf{a}-c\mathbf{1}}{2}\right)\\&=\left(\frac{\mathbf{a}-c\mathbf{1}}{2}\right)^T\left(2I-M\left(\frac{M+M^T}{2}\right)^{-1}\right)^T\cdot\left(\frac{M+M^T}{2}\right)^{-1}\left(\frac{\mathbf{a}-c\mathbf{1}}{2}\right)\\&=2\left(\frac{\mathbf{a}-c\mathbf{1}}{2}\right)^T\left(\frac{M+M^T}{2}\right)^{-1}\left(\frac{\mathbf{a}-c\mathbf{1}}{2}\right)\\&\quad-\left(\frac{\mathbf{a}-c\mathbf{1}}{2}\right)^T\left(\frac{M+M^T}{2}\right)^{-T}M^T\left(\frac{M+M^T}{2}\right)^{-1}\left(\frac{\mathbf{a}-c\mathbf{1}}{2}\right).\end{aligned}\tag{42}$$

Note that for a matrix $A$ and vector $\mathbf{x}$, $\mathbf{x}^TA\mathbf{x}=\mathbf{x}^T((A+A^T)/2)\mathbf{x}$; thus, it follows that

$$\begin{aligned}&\left(\frac{\mathbf{a}-c\mathbf{1}}{2}\right)^T\left(\frac{M+M^T}{2}\right)^{-T}M^T\left(\frac{M+M^T}{2}\right)^{-1}\left(\frac{\mathbf{a}-c\mathbf{1}}{2}\right)\\&\quad=\left(\frac{\mathbf{a}-c\mathbf{1}}{2}\right)^T\left(\frac{M+M^T}{2}\right)^{-1}\left(\frac{\mathbf{a}-c\mathbf{1}}{2}\right).\end{aligned}$$

Thus, from (42) we can rewrite $\Pi_N$ as

$$\Pi_N=\left(\frac{\mathbf{a}-c\mathbf{1}}{2}\right)^T\left(\frac{M+M^T}{2}\right)^{-1}\left(\frac{\mathbf{a}-c\mathbf{1}}{2}\right).\tag{43}$$

The claim follows noting that $M=\Lambda-G$.

Proof of Theorem 8. To simplify the notation we denote $\Lambda-G$ by $M$. Note that by the assumption of the theorem, $M$ is positive definite. We state some useful properties of this matrix in Lemmas 10 and 11, and then prove the claim using these properties. The proofs of these lemmas can be found at the end of this proof.

Lemma 10. *If $M$ is positive definite, then the following matrices are also positive definite*: $M^{-1}$, $(M+M^T)/2$, $((M+M^T)/2)^{-1}$, $((M^{-1}+M^{-T})/2)$, *and* $((M^{-1}+M^{-T})/2)^{-1}$.

Lemma 11. *Let $M$ be positive definite and $\lambda$ be an eigenvalue of $MM^{-T}=(\Lambda-G)(\Lambda-G)^{-T}$; then,* (i) *$|\lambda|=1$ and the real part of $\lambda$ satisfies $\Re(\lambda)>-1$.* (ii) *The eigenvalues of $MM^{-T}+M^TM^{-1}$ are real and they belong to* $(-2,2]$.

Let $\mathbf{v}=((\mathbf{a}-c\mathbf{1})/2)$. Lemma 3 implies that

$$\frac{\Pi_N}{\Pi_0}=\frac{\mathbf{v}^T(\Lambda-(((G+G^T)/2)^{-1})\mathbf{v}}{\mathbf{v}^T(\Lambda-G)^{-1}\mathbf{v}} \leqslant \max_{\|\mathbf{x}\|=1}\frac{\mathbf{x}^T(((M+M^T)/2)^{-1})\mathbf{x}}{\mathbf{x}^TM^{-1}\mathbf{x}} = \max_{\|\mathbf{x}\|=1}\frac{\mathbf{x}^T(((M+M^T)/2)^{-1})\mathbf{x}}{\mathbf{x}^T((M^{-1}+M^{-T})/2)\mathbf{x}}, \tag{44}$$

and similarly

$$\frac{\Pi_0}{\Pi_N}\leqslant \max_{\|\mathbf{x}\|=1}\frac{\mathbf{x}^TM^{-1}\mathbf{x}}{\mathbf{x}^T(((M+M^T)/2)^{-1})\mathbf{x}} = \max_{\|\mathbf{x}\|=1}\frac{\mathbf{x}^T((M^{-1}+M^{-T})/2)\mathbf{x}}{\mathbf{x}^T(((M+M^T)/2)^{-1})\mathbf{x}}. \tag{45}$$

Because $(M^{-T}+M^{-1})/2$ and $(M^T+M^1)/2$ are symmetric positive definite matrices, the matrices $((M^{-T}+M^{-1})/2)^{1/2}$ and $((M^T+M^1)/2)^{1/2}$ are well defined. Consequently, we obtain

$$\begin{aligned}&\max_{\|\mathbf{x}\|=1}\frac{\mathbf{x}^T(((M+M^T)/2)^{-1})\mathbf{x}}{\mathbf{x}^T((M^{-1}+M^{-T})/2)\mathbf{x}}\\ &=\max_{\|\mathbf{x}\|=1}\frac{\mathbf{x}^T(((M+M^T)/2)^{-1})\mathbf{x}}{\mathbf{x}^T(((M^{-1}+M^{-T})/2)^{1/2})(((M^{-1}+M^{-T})/2)^{1/2})\mathbf{x}}\\ &=\max_{\|\mathbf{y}\|=1}\mathbf{y}^T\left(\frac{M^{-1}+M^{-T}}{2}\right)^{-1/2}\left(\frac{M+M^T}{2}\right)^{-1}\left(\frac{M^{-1}+M^{-T}}{2}\right)^{-1/2}\mathbf{y}\\ &=\lambda_{\max}\left(\left(\frac{M^{-1}+M^{-T}}{2}\right)^{-1/2}\left(\frac{M+M^T}{2}\right)^{-1}\cdot\left(\frac{M^{-1}+M^{-T}}{2}\right)^{-1/2}\right),\end{aligned} \tag{46}$$

where the second line follows by defining

$$\mathbf{z}\triangleq\left(\frac{M^{-1}+M^{-T}}{2}\right)^{1/2}\mathbf{x},$$

rewriting the first line in terms of $\mathbf{z}$ and setting $\mathbf{y}=\mathbf{z}/\|\mathbf{z}\|$, and the third line follows from the Rayleigh-Ritz Theorem (Horn and Johnson 2005). Similarly, we have,

$$\max_{\|\mathbf{x}\|=1}\frac{\mathbf{x}^T((M^{-1}+M^{-T})/2)\mathbf{x}}{\mathbf{x}^T(((M+M^T)/2)^{-1})\mathbf{x}} =\lambda_{\max}\left(\left(\frac{M+M^T}{2}\right)^{1/2}\left(\frac{M^{-1}+M^{-T}}{2}\right)\left(\frac{M+M^T}{2}\right)^{1/2}\right). \tag{47}$$

Note that for a real matrix $A$ and invertible real matrix $B$ the eigenvalues of $A$ and $B^{-1}AB$ are identical (similarity transformation). Therefore, it follows from the above equations that

$$\max_{\|\mathbf{x}\|=1}\frac{\mathbf{x}^T(((M+M^T)/2)^{-1})\mathbf{x}}{\mathbf{x}^T((M^{-1}+M^{-T})/2)\mathbf{x}} =\lambda_{\max}\left(\left(\frac{M^{-1}+M^{-T}}{2}\right)^{-1}\left(\frac{M+M^T}{2}\right)^{-1}\right) =\lambda_{\max}\left(\left(\frac{2I+MM^{-T}+M^TM^{-1}}{4}\right)^{-1}\right). \tag{48}$$

Lemma 11 implies that the eigenvalues of $MM^{-T}+M^TM^{-1}$ are real and belong to $(-2,2]$. Thus, it follows that the eigenvalues of $((2I+MM^{-T}+M^TM^{-1})/4)$ are positive and

$$\lambda_{\max}\left(\left(\frac{2I+MM^{-T}+M^TM^{-1}}{4}\right)^{-1}\right) =1/\lambda_{\min}\left(\left(\frac{2I+MM^{-T}+M^TM^{-1}}{4}\right)\right). \tag{49}$$

Similarly, we obtain,

$$\max_{\|\mathbf{x}\|=1}\frac{\mathbf{x}^T((M^{-1}+M^{-T})/2)\mathbf{x}}{\mathbf{x}^T(((M+M^T)/2)^{-1})\mathbf{x}} =\lambda_{\max}\left(\left(\frac{M+M^T}{2}\right)\left(\frac{M^{-1}+M^{-T}}{2}\right)\right) =\lambda_{\max}\left(\frac{2I+MM^{-T}+M^TM^{-1}}{4}\right). \tag{50}$$

Thus, it follows from (44) and (45) that

$$\lambda_{\min}\left(\frac{2I+MM^{-T}+M^TM^{-1}}{4}\right) \leqslant\frac{\Pi_0}{\Pi_N}\leqslant\lambda_{\max}\left(\frac{2I+MM^{-T}+M^TM^{-1}}{4}\right), \tag{51}$$

or equivalently

$$\frac{1}{2}+\lambda_{\min}\left(\frac{MM^{-T}+M^TM^{-1}}{4}\right) \leqslant\frac{\Pi_0}{\Pi_N}\leqslant\frac{1}{2}+\lambda_{\max}\left(\frac{MM^{-T}+M^TM^{-1}}{4}\right). \tag{52}$$

The claim follows because the eigenvalues of $MM^{-T}+M^TM^{-1}$ belong to $(-2,2]$.

Proof of Lemma 10. Note that because

$$\mathbf{x}^TM\mathbf{x}=\mathbf{x}^T\left(\frac{M+M^T}{2}\right)\mathbf{x},$$

it immediately follows that $(M+M^T)/2$ is positive definite. For $\mathbf{y}=M\mathbf{x}$, $\mathbf{x}^TM\mathbf{x}=\mathbf{x}^TM^T\mathbf{x}=\mathbf{y}^TM^{-1}\mathbf{y}$. Thus, it follows that $\mathbf{y}^TM^{-1}\mathbf{y}=\mathbf{y}^T((M^{-T}+M^{-1})/2)\mathbf{y}>0$ for all real vectors $\mathbf{y}\neq 0$. Hence, $M^{-1}$ and $(M^{-T}+M^{-1})/2$ are also positive definite.

Finally, note that if $A$ is a symmetric positive definite matrix, then so is $A^{-1}$. Therefore, positive definiteness of

$$\left(\frac{M+M^T}{2}\right)^{-1}\quad\text{and}\quad\left(\frac{M^{-1}+M^{-T}}{2}\right)^{-1}$$

follows directly from the fact that

$$\frac{M+M^T}{2}\quad\text{and}\quad\frac{M^{-1}+M^{-T}}{2}$$

are positive definite.

Proof of Lemma 11. Assume that $\mathbf{x}$ is a left eigenvector of $MM^{-T}$ corresponding to the eigenvalue $\lambda$, i.e., $\mathbf{x}^TMM^{-T}=\lambda\mathbf{x}^T$. Then, $(\lambda,\mathbf{x})$ satisfies $\mathbf{x}^TM=\lambda\mathbf{x}^TM^T$, or equivalently,

$$M^T\mathbf{x}=\lambda M\mathbf{x}. \tag{53}$$

Because $MM^{-T}$ need not be a symmetric matrix, $\lambda$ and $\mathbf{x}$ are not necessarily real. Let $x = x_1 + ix_2$, and $\mathbf{x}^*$ denote the conjugate transpose of $\mathbf{x}$, i.e., $\mathbf{x}^* = x_1^T - ix_2^T$. Note that

$$\mathbf{x}^* M^T \mathbf{x} = x_1^T M^T x_1 + x_2^T M^T x_2 + i(x_1^T M^T x_2 - x_2^T M^T x_1), \qquad (54)$$

and

$$\mathbf{x}^* M \mathbf{x} = x_1^T M x_1 + x_2^T M x_2 + i(x_1^T M x_2 - x_2^T M x_1). \qquad (55)$$

Because $M$ and $M^T$ are real and positive definite, $\Re(\mathbf{x}^* M\mathbf{x}) = \Re(\mathbf{x}^* M^T \mathbf{x}) = x_1^T M x_1 + x_2^T M x_2 > 0$. Additionally, taking the transpose, it can be seen that $x_1^T M^T x_2 = x_2^T M x_1$ and $x_1^T M x_2 = x_2^T M^T x_1$, and consequently $\Im(\mathbf{x}^* M\mathbf{x}) = -\Im(\mathbf{x}^* M^T \mathbf{x})$. Thus, from (53) it follows that

$$|\lambda| = \left| \frac{\mathbf{x}^* M^T \mathbf{x}}{\mathbf{x}^* M \mathbf{x}} \right| = 1. \qquad (56)$$

It follows from (54) and (55) that $\lambda = -1$ only when $x_1^T M^T x_1$ and $x_2^T M^T x_2$ equal to zero. However, because $M$ is positive definite this can never happen. Using $|\lambda| = 1$, this implies that $\Re(\lambda) > -1$.

Let the Jordan normal form of $MM^{-T}$ be $P^{-1}JP$, i.e.,

$$MM^{-T} = P^{-1}JP, \qquad (57)$$

where $J$ is an upper-triangular block diagonal matrix, and $P$ is an invertible matrix. Eigenvalues of $MM^{-T}$ correspond to the diagonal entries of $J$. Observing that $(MM^{-T})^{-1} = M^T M^{-1}$, it follows that

$$MM^{-T} + M^T M^{-1} = P^{-1}JP + P^{-1}J^{-1}P = P^{-1}(J + J^{-1})P. \qquad (58)$$

The inverse of an upper-triangular block diagonal matrix is upper-triangular block diagonal. Thus, it follows that $P^{-1}(J + J^{-1})P$ is a Jordan normal form for $MM^{-T} + M^T M^{-1}$. Also note that because $J$ and $J^{-1}$ are upper triangular, if $J_{ii} = \lambda$, then $J_{ii}^{-1} = 1/\lambda$. Consequently, the diagonal entries of $(J + J^{-1})$ take the form $\lambda + (1/\lambda)$, where $\lambda$ is a diagonal entry of $J$. Because $P^{-1}JP$ and $P^{-1}JP + P^{-1}J^{-1}P = P^{-1}(J + J^{-1})P$ are the Jordan normal forms of $MM^{-T}$ and $MM^{-T} + M^T M^{-1}$, we conclude that $\lambda$ is an eigenvalue of $MM^{-T}$ if and only if $\lambda + 1/\lambda$ is an eigenvalue of $MM^{-T} + M^T M^{-1}$.

From part (i), it follows that the eigenvalues of $MM^{-T}$ take the form $e^{i\omega}$ for some $\omega \in [0, \pi) \cup (\pi, 2\pi)$. Thus, the eigenvalues of $MM^{-T} + M^T M^{-1}$ are given by $e^{i\omega} + e^{-i\omega} = 2\cos(\omega)$ for some $\omega \in [0, \pi) \cup (\pi, 2\pi)$. Thus, we conclude that the eigenvalues of $MM^{-T} + M^T M^{-1}$ are real and they belong to $(-2, 2]$.

## Endnotes

1. We use the terms "agent" and "consumer" interchangeably.
2. The hardness result can be extended to the case of more than two prices.
3. The utility function specified in (1) is compatible with standard consumer theory. Typically, in consumer theory there is an additional constraint regarding the income levels of the economic agents. In our approach, we choose not to include such a constraint, implicitly assuming large income levels, because this allows for a cleaner characterization of the equilibrium strategies of agents. However, note that Assumption 1 is no longer necessary to ensure bounded consumption in the presence of an income constraint.
4. Previous results in the literature (in particular, in Ballester et al. 2006, Theorem 1; Bramoullé et al. 2012, Proposition 2) also establish uniqueness in a model related to ours. However, these results are not immediately applicable in our setting because they assume the symmetry of the adjacency matrix $G$. On the other hand, a contraction argument similar to the one given in (Bramoullé et al. 2012, p. 32) can be used to establish uniqueness of equilibrium in our setting. Here, we do not impose any symmetry condition on the adjacency matrix, and offer a novel proof of uniqueness, which also illustrates how the strategic complementarity condition can be used to establish the uniqueness of equilibria.
5. Recall that the Bonacich centrality of a node is proportional to the (discounted) number of times a random walk defined over the nodes of the graph (with uniform initial distribution) visits this node. Centrality gain captures the change in the expected number of visits when the initial distribution is not uniform.
6. Informally, the (weighted) MAX-CUT problem is to find a subset $S$ of the vertex set such that the total weight of the edges between the set $S$ and its complement is maximized.
7. Instances of MAX-CUT with weight matrix $W$ can be expressed as $\max_{x_i \in \{-1,1\}} \sum_{i,j} (1 - x_i x_j) W_{ij}$. Omitting the constant term, the relation between (11) and MAX-CUT is now clear. See the appendix for details.
8. This claim immediately follows from the Gershgorin circle theorem (see Golub and Loan 1996).
9. We note that for this choice of parameters Assumptions 2 and 4 hold. However, Assumption 1 may be violated for the center node for some values of $\alpha$. Despite that, we numerically establish that there is a unique equilibrium, and all agents consume positive amounts of the good at this equilibrium. Thus, it follows that both Lemma 3 and Theorem 8 hold for these parameters.
10. It can be verified numerically that for graphs of smaller size, the upper bound can be significantly different than 1. However, for the graphs considered in this section, the upper bound turned out to be very close to 1.
11. Pricing policies for profit maximization, in the presence of incomplete information about agents' private valuations, have been considered in the literature (see e.g., Hartline et al. 2008, Akhlaghpour et al. 2010). However, the problem of learning quality, or the match between the product and the buyers, differs from this line of work in that in the latter setting the monopolist uses pricing also as a means to learn a common parameter (quality).
12. We assume that the range of the arccos function is $[0, \pi]$.

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