CURVES OF GENERAL PURSUIT

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OUR discussion of pursuit curves has considered: [A] Achilles' one-dimensional pursuit of Zeno's tortoise; [B] Bouguer's *lignes de poursuite* for a linear track; and [C] Circular track as studied by Hathaway and others. We conclude with a glimpse at [D] Differential equations valid for arbitrary track and variable speeds, and [E] Extraneous problems sometimes confused with the pure pursuit problem.

[D1] Pursuit with Variable Speed. As early as 1732 Pierre-Louis Moreau de Maupertuis had derived the differential equations for arbitrary pursuit ("Sur les courbes de poursuite," *Memoires de l'Academie Royale des Sciences*, p. 15–16) showing that they applied not only to Bouguer's particular case (*Ibid.*, p. 1–14) but also where the track of the pursued is an arbitrary curve and where the ratio of speeds is not necessarily constant.

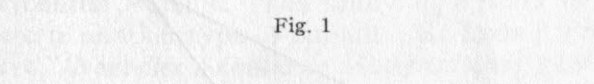

Fig. 1

The point Q traverses an arbitrary track $Q(w)$ with speed $n = dw/dt$ and is pursued by the point P along the curve $P(s)$ with speed $m = ds/dt$. The ratio of corresponding arc lengths $ds/dw = m/n$ is arbitrary, but the velocity of pursuit dP/dt has always the same direction as the separation vector PQ.

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Using rectangular coordinates, $P = (x,y)$ and $Q = (u,v)$, these differential conditions may be written

$$\frac{dx}{u - x} = \frac{dy}{v - y} = \frac{ds}{r} = \frac{m\,dw}{nr}, \tag{1}$$

where $r^2 = (x - u)^2 + (y - v)^2$. For *uniform* pursuit (m/n = constant) Maupertuis suggests an equivalent geometric formulation: The curve QQ' is given; find the curve PP' such that any two of its tangents PQ and $P'Q'$ intercept an arc QQ' proportional to the arc PP'.

[D2] **Curvature.** We have seen how Dubois-Ayme and Saint-Laurent aroused interest in the radius of curvature. A contribution by Maurice d'Ocagne, "On the center of curvature of curves of pursuit," [*Bulletin de la Societe Mathematique de France,* v. 11 (1883), p. 134–135] is valid for general pursuit. The new result involves a simple geometric construction:

Lay off $QT = PQn/m$ tangent to the track so that if the velocities remained constant at their instantaneous values, then Q moving at speed n would reach T by the time P moving at speed m had reached the initial position of Q. Then PQT is the triangle of velocities drawn to the scale $PQ = m$, $QT = n$, and such that PT represents the resultant velocity.

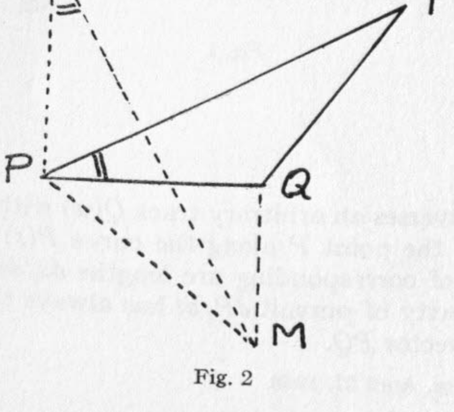

Fig. 2

At Q draw QM perpendicular to PQ, and at P draw PM perpendicular to QT, intersecting at M. The perpendicular to PT from M meets the perpendicular to PQ from P at some point C, which is the *center of curvature of the pursuit curve* $P(s)$.

For, $ds:dw = ds/dt:dw/dt = m:n = PQ:QT$. Since triangles PQT and CPM are similar, the ratios $PQ:QT$ and $CP:PM$ are equal. Thus $ds:dw = CP:PM$, and $ds/CP = dw/PM$. Now the latter ratio may be written

$$\frac{dw}{PM} = \frac{dw \sin TQP}{PM \sin MPC} = \frac{dw'}{PQ},$$

where dw' is the component of dw which is transverse to PQ. But the tangent PQ to $P(s)$ turns through this angle $dw'/PQ = ds/\rho$, where ρ is the radius of curvature. This shows that $\rho = CP$ and C is the center of curvature.

This geometric theorem of Ocagne provides a neat formula for the radius of curvature ρ. Let $r = PQ$ and let ϕ be the angle between the two velocities. From similar triangles

$$CP/PM = PQ/QT = m/n,$$

hence $\rho = PM\ m/n$. But angle QMP is ϕ so that $\sin\phi = PQ/PM$, and $PM = r/\sin\phi$. This yields

$$\rho = \frac{mr}{n \sin\phi}. \tag{2}$$

Equation (2) has been obtained for the special case of linear track [B9, eq. (23)] but Ocagne's theorem shows that it holds for arbitrary track and for variable speeds. Ocagne mentions particularly that the curvature of $P(s)$ does not depend on the curvature of $Q(w)$ but only on the relative velocities.

[**D3**] **Barycentric Pursuit.** The study of centers of gravity led Ernest Cesaro to another type of pursuit. He finds ["Properties of a Pursuit Curve," *Nouvelles Annales de Mathematiques*, (1883) p. 85–89] that the center of gravity of an arc with one end fixed pursues the moving end. This provides a complete answer to an earlier question he posed in *Nouvelles Correspondance* [v. 5 (1879) p. 110].

Employing the notation of [D1] the centroid $P(x,y)$ of the arc $w = t$ generated by $Q(u,v)$ is determined by the first moments $tx = \int u dt$ and $ty \int v dt$. On differentiating, $tdx = (u - x)dt$ and $tdy = (v - y)dt$, we find that P pursues Q according to eqs. (1). The simplification $w = t$ implies that Q moves with unit speed, $n = 1$. If m is constant then the pursuit is uniform, but for barycentric pursuit Cesaro derives the condition

$$m = \frac{ds}{dt} = \frac{r}{t} \,. \tag{3}$$

Equation (3) follows from (1) and the relation $ds^2 = dx^2 + dy^2$ which defines the differential of arc. Differentiating the relation which defines $r = PQ$, Cesaro finds the radial velocity

$$\frac{dr}{dt} = -m + \cos\phi \,. \tag{4}$$

Since his derivation of (4) is based on the general pursuit relations (1) without the restriction (3), it holds for arbitrary pursuit. The same formula was obtained [B5, eq. (10)] for uniform pursuit. Cesaro explicitly recognizes the generality of (4) in a later paper [see D4].

But (3) permits another computation for the radial velocity. From $r = mt$ we have $dr/dt = d(mt)/dt = m + tdm/dt$. Combining this result with (4) Cesaro writes

$$\cos\phi = 2m + t\,\frac{dm}{dt} \,. \tag{5}$$

Can barycentric pursuit also be uniform? Holding m constant in (5) we obtain the necessary condition $\cos\phi = 2m$, which shows that ϕ must remain constant. Discarding the trivial case of linear track, $\phi = 0$, Cesaro examines the case of a logarithmic spiral.

Introducing polar coordinates R and A, let the point Q track the spiral

$$u = R\cos A, \qquad v = R\sin A,$$

where

$$\log R = \log C + A \cot\phi/2. \tag{6}$$

The radius R assumes the value C when the polar angle A vanishes.

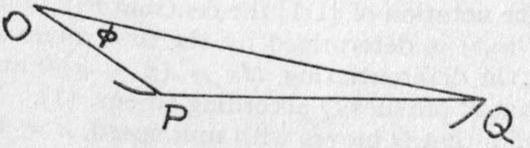

Fig. 3

Measuring the arc OQ along (6) from the pole O, the rectangular coordinates of its centroid $P(x,y)$ are given by

$$x = mR \cos (A - \phi),$$
$$y = mR \sin (A - \phi), \tag{7}$$

where $m = {}^1/_2 \cos \phi$. Equations (7) show that the ratio $OP/OQ = m$ and the angle $POQ = \phi$ are both constant. Hence triangle POQ retains the same shape, so that P and Q move on *similar* spirals with a common pole. Therefore the track (6) and pursuit curve (7) are *congruent* logarithmic spirals, differing by the rotation $\phi + 2 \tan \phi \log m$.

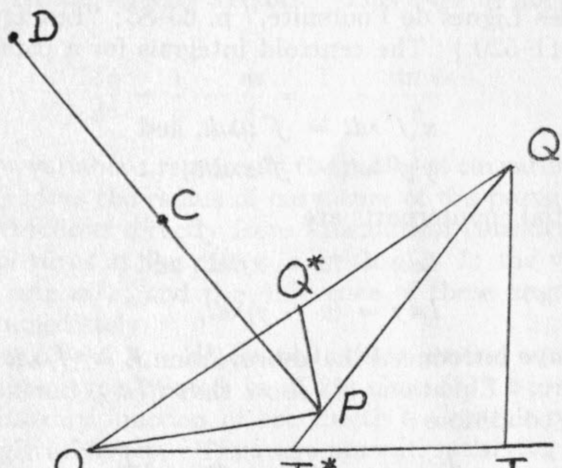

Fig. 4

Let T be the foot of the perpendicular from the pole O to the line QT which is tangent at Q to the track. Let Q^* and T^* be the midpoints of OQ and OT, respectively. Then, *the centroid P of the arc OQ lies on the median line QT^* of right triangle OTQ.* For, the logarithmic derivative $d(\log R)/dA$ of (6) is $\cot \phi/2$, and, as this is $\cot OQT$, we have

$$\tan OQT = 2 \tan \phi = 2 \tan PQT. \tag{8}$$

But, $\tan T^*QT/\tan OQT = T^*T/OT = {}^1/_2$, by construction. Accordingly $\tan T^*QT = \tan \phi = \tan PQT$, which means that P, Q, and T^* are collinear. (Incidentally, P is *between* T^* and Q whenever $\cos^2 \phi$ *exceeds* ${}^2/_3$.) Since $OP = MOQ = {}^1/_2 OQ \cos \phi = OQ^* \cos \phi$, it follows that OPQ^* is a right angle, and P *lies on the circle* with diameter OQ^*. Recapitulating, angle $POQ = \phi =$ angle PQT, and angle $Q^*PO = {}^1/_2\pi$. Given triangle OTQ we have three loci any two of which will determine P.

Cesaro concludes his first paper on barycentric pursuit by giving the location of the centers of curvature C and D of the P and Q spirals, respectively. The point D is diametrically opposite Q on the circle OPQ, while C is the midpoint of the chord PD.

[D3] Generalized Barycentric Pursuit. Three years after his first barycentric study of curves Cesaro considered the generalization involving a variable linear mass density λ. [*Nouv. Ann. Math.*, v. 5 (1886), "Sur les Lignes de Poursuite," p. 65–83; "Les lignes barycentriques," p. 511–520.] The centroid integrals for a plane curve $Q(t)$ become

$$\begin{aligned} x \textstyle\int \lambda dt &= \textstyle\int u\lambda dt, \text{ and} \\ y \textstyle\int \lambda dt &= \textstyle\int v\lambda dt. \end{aligned} \tag{9i}$$

Their differential counterparts are

$$\begin{aligned} Ldx &= (u - x)\lambda dt, \text{ and} \\ Ldy &= (v - y)\lambda dt, \end{aligned} \tag{9d}$$

in which we have introduced the abbreviation $L = \int \lambda dt$ for the total mass of the arc. Equations (9) show that $P(x,y)$ pursues $Q(u,v)$ according to the conditions

$$\frac{dx}{u - x} = \frac{dy}{v - y} = \frac{ds}{r} = \frac{\lambda dt}{L}. \tag{10}$$

Comparison of (10) with (1) yields the relative speed m of generalized barycentric pursuit,

$$m = \frac{ds}{dt} = \frac{\lambda r}{L}. \tag{11}$$

Substituting $r = Lm/\lambda$ in (4) we obtain the generalization of (5), namely

$$\cos \phi = 2m + L\frac{d}{dt}\left(\frac{m}{\lambda}\right). \tag{12}$$

From eq. (11) we have $dL/L = mdt/r = ds/r$. If r is given as a function of s, then L may be computed by an integration, thus

$$\log L = \int \frac{ds}{r}. \tag{13}$$

Considering infinitesimal displacements $PP' = ds$ and $QQ' = dt$ and arbitrary pursuit speed $m = ds/dt$, Cesaro writes the law of sines for triangle PQQ' in the form

$$\frac{dt}{ds/\rho} = \frac{r}{\sin(\phi - ds/\rho)} = \frac{r + dr + ds}{\sin(\pi - \phi)}.$$

The first equality yields the radius of curvature (2), and the second equality yields $\cos\phi = d(r + s)/dt$ which is equivalent to the familiar formula (4) for the radial velocity. But he is specially interested in the angle ϕ between the velocity vectors. This *angle of attack* changes at the rate

$$\frac{d\phi}{dt} = \frac{1}{a} - \frac{m}{\rho} = \frac{1}{a} - \frac{\sin\phi}{r}, \tag{14}$$

where the new variable a represents the radius of curvature of the track $Q(t)$, just as ρ gives the radius of curvature of the pursuit curve $P(s)$. Equation (14) follows directly from kinematical considerations. The velocity dQ/dt turns at the rate n/a, with $n = 1$; the velocity dP/dt turns at the rate m/ρ; and the difference of these angular velocities gives $d\phi/dt$ immediately.

On choosing $L = t$, $\lambda = dL/dt = 1$, these general equations reduce to the particular type treated in [D2], since (11) becomes (3); but with L an arbitrary function of arc length t, Cesaro's pursuit (10) is correspondingly arbitrary. Thus any pursuit, satisfying (1) by definition, may be interpreted via (11) as generalized barycentric pursuit. For example, if $d(\log L)/dt$ is inversely proportional to r, then the pursuit is uniform, m being the proportionality constant.

[D4] **Attack Curves.** One type of pursuit which Cesaro examines particularly is attack from a constant angle ϕ, where $d\phi/dt$ vanishes, hence

$$\rho = ma \quad \text{and} \quad r = a \sin\phi. \tag{15}$$

Equation (15) means geometrically that P lies on an *attack circle* which is independent of the particular attack angle. It is tangent to the track at Q, and the diameter QD of length a terminates at the center of curvature D. Cesaro says this result is due to Réamur.

Among the attack curves [développoïdes] associated with a given tract $Q(t)$ the *evolute* [développée] corresponds to the case $\phi = {}^1/_2\pi$. Otherwise said, the center of curvature attacks any curve at right angles. The relation $r = a \sin \phi$ yields $r = a$; while the differential equation $dr/dt = \cos \phi - m$ reduces to $\frac{d}{dt}(r + s) = 0$. Integration yields $s = -r +$ constant, or simply $s^* = -r$, if we measure the arc s^* from the point where $r = 0$. Again, the corresponding radius ρ^* is given by $\rho^* = am = ads/dt = -ada/dt$. Given the functional relation between the track length t and its curvature a, the equations

$$s^* = -a, \text{ and}$$
$$\rho^* = -a\frac{da}{dt} \tag{16*}$$

express parametrically the relation between the intrinsic coordinates, s^* and, of its evolute.

The mass L^* and linear density λ^* which interpret any evolute as a pursuit curve (9) may be written

$$L^* = 1/a, \qquad \text{and} \qquad \lambda^* = \rho^*/a^3. \tag{17*}$$

For eq. (13) yields $\log L^* = -da/a = -\log a$, so that the mass L^* is inversely proportional to the radius a. Then $\lambda^* = \frac{dL^*}{dt} = \frac{-1}{a^2}\frac{da}{dt}$ which combines with (16*) to give (17*).

More generally, let ϕ be constant but not necessarily a right angle. Then eqs. (16*) are replaced by

$$s = t \cos \phi - a \sin \phi, \text{ and}$$
$$\rho = a \cos \phi + \rho^* \sin \phi, \tag{16}$$

where ρ^* is the value of ρ for $\phi = {}^1/_2\pi$ as given in (16*). We shall call the P locus defined by (16) the *attack curve* at the angle ϕ of the Q locus.

Equation (16) for ρ means geometrically that the center of curvature C for each attack curve lies on the circle of diameter DE, where D and E are the centers of curvature of the track $Q(t)$ and its evolute $P(s^*)$, respectively. For $PD = a \cos \phi$ and $DC = \rho^* \sin \phi$. Cesaro credits Minich and Habich with discovering this result independently.

Letting $D(\phi)Q$ symbolize the transformation (16), the accompanying figure shows that $D(\phi)D({}^1/_2\pi)Q = C = D({}^1/_2\pi)D(\phi)Q$. In words, the attack curve of the evolute is the evolute of the attack curve. This is only a particular case of a theorem by Lancret, which states that

$D(\phi_1)D(\phi_2)Q = D(\phi_2)D(\phi_1)Q$; so that, in a succession of attacks at varying angles, the operators $D(\phi)$ commute.

If the track curvature $1/a$ is always $\tan \phi/t$, then s and the attack speed m vanish, and P is a fixed point. This occurs when the track is a logarithmic spiral crossing lines issuing from pole P at the constant angle ϕ.

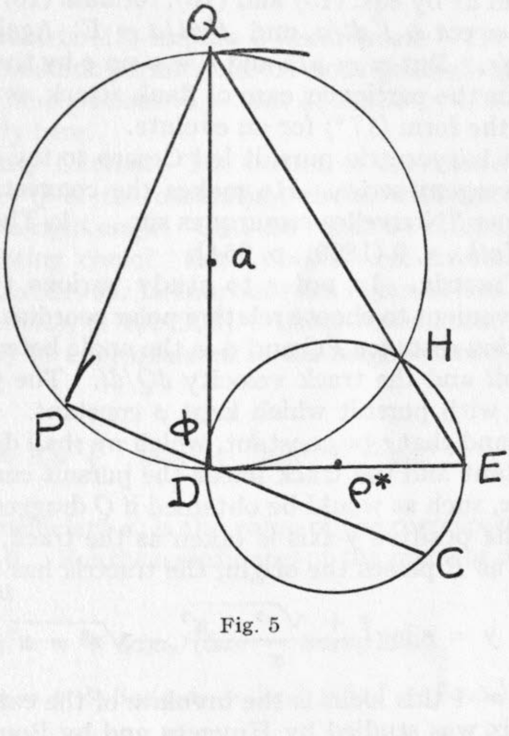

Fig. 5

Let the attack circles DPQ and DCE of the track and its evolute intersect again in point H. This poing (point de rebroussement) is the foot of the altitude to the hypotenuse in right triangle QDE, whose legs are the radii of curvature *in position*. The points Q and H separate the points on the attack circle into two arcs which distinguish attack proper from flight. For, the pursuit speed $m = \rho/a$ becomes negative when the curvatures have opposite sense, indicating flight. At the transition point H the speed m and curvature ρ both vanish. Among other interesting results Cesaro proves (p. 73) that the tangents to the loci of Q and H form the sides of an isosceles triangle with base QH. Accordingly, both are tangent to the attack circle.

We conclude this section by setting down the generalization of eqs. (17*), namely

$$L = e^{E \cot \phi}/a, \qquad \text{and} \qquad \lambda = \rho L/a^2 \sin \phi, \tag{17}$$

where E indicates the total angular deviation of the track. For, evaluating r and ds by eqs. (15) and (16), formula (13) may be written $\log L + \log a = \cot \phi \int dt/a$, and $\int dt/a = E$. Again, from (11) we have $\lambda/L = m/r$. But $m = \rho/a$ and $r = a \sin \phi$ by the conditions (15) for attack. For the particular case of flank attack, $\phi = {}^1/_2\pi$, the eqs. (17) reduce to the form (17*) for an evolute.

The study of barycentric pursuit led Cesaro to the concept of summability of divergent series. He makes the connection of ideas explicit in his paper "Nouvelles remarques sur . . . la Théorie des séries" [*Nouv. Ann. Math.*, v. 9 (1890), p. 364].

[D5] The Tractrix. In order to study various types of pursuit curves it is convenient to choose relative polar coordinates (r, ϕ), where r is the separation distance PQ and ϕ is the angle between the velocity of pursuit dP/dt and the track velocity dQ/dt. The previous section was concerned with pursuit which kept ϕ constant. It is intuitively simpler to demand that r be constant, which we shall do in this section.

With r constant and the track linear the pursuit curve becomes the familiar *tractrix*, such as would be obtained if Q dragged P by a rope of length r. If the positive y-axis is taken as the track, Q being on the x-axis at $(r, 0)$ as P passes the origin, the tractrix has the equation

$$y = r \log \frac{r + \sqrt{r^2 - x^2}}{x} - \sqrt{r^2 - x^2}\,. \tag{18}$$

On the scale $r = 1$ this locus is the involute of the catenary $x = \cosh y$. The tractrix was studied by Huygens and by Bourie ["Propriétés de la tractrice," *Mèmoires de l'Academie Royale des Sciences* (1712), p. 215–225]. Professor C. A. Scott in *Encyclopedia Americana* mentions the use of this curve by A. F. Beltrami for an interpretation of non-Euclidean geometry ["*Saggio d'interpretazione della geometria non-euclidea*," (1868); *Ann. de l'Ec. Norm.*, v. 6 (1869) p. 251]. See also a paper by E. Cesaro, "Sur la tractrice," (*Mathesis*, v. 2, p. 217–219) for its duality with the logarithmic spiral.

During the eighteenth century the tractrix was called "curve of pursuit," a term which also applied to the *courbe du chien* studied by Bouguer and Dubois-Aymé [B7]. This ambiguous terminology has caused Niels Nielsen to confuse them in an off-the-cuff comment in *Gèomètres Francais*. "We remark in passing that the tractrix is nothing else but the curve of pursuit, studied later by Bouguer and by Maupertuis; we are able to add that neither Bouguer nor Maupertuis mentioned their precedessors." Our exposition of pursuit has included many examples of the truism that problems with strong intuitive appeal are often solved more than once. But the tractrix, with r = constant, and the courbe du chien, with m = constant, are two different problems.

The simple tractrix (18) implies a linear track. The pursuit restriction that r be constant permits the obvious generalization to an arbitrary track. The equations $m = \cos\phi$, $d\phi/dt = 1/a - \sin\phi/r$, and $\rho = r\cot\phi$ apply here.

[D6] Aberrancy Curves. The center of curvature D pursues the point of contact Q of the osculating circle, with an angle of attack which is always a right angle. Can the result be extended to the center A of the osculating conic? Such conics were investigated by Abel Transon, "Recherches sur la courbure des lignes et des surface" [*Jour. Liouville*, v. 6 (1841), p. 191–208]. Using rectangular coordinates the track $Q(u,v)$ may be approximated in the neighborhood of the origin by the series

$$v = a_1u + {}^1\!/_2 a_2u^2 + \frac{1}{6}a_3u^3 + \frac{1}{24}a_4u^4 + \dots,$$

in which each coefficient a_n is the value of the corresponding derivative d^nv/du^n at $u = 0$. For the coordinates of the *center of aberrancy* $A(x,y)$ Transon obtains

$$\begin{aligned} x &= u + 3a_2a_3/(5a_3^2 - 3a_2a_4), \text{ and} \\ y &= v + 3a_2(a_1a_3 - 3a_2^2)/(5a_3^2 - 3a_2a_4). \end{aligned} \tag{19}$$

The quotient $(v - y)/(u - x) = a_1 - 3a_2^2/a_3$ gives the slope of the *axis of aberrancy* QA, which deviates from the normal QD by an angle AQD such that

$$\tan AQD = a_1 - (1 + a_1^2)a_3/3a_2^2 = \rho^*/3a, \tag{20}$$

where a and ρ^* indicate the radii of curvature for the track and its evolute, respectively. The *radius of aberrancy* QA is given by

$$r = \frac{3a_2\sqrt{a_3{}^2 + 9a_2{}^4}}{3a_2a_4 - 5a_3{}^2}. \tag{21}$$

This radius becomes infinite whenever the osculating conic is a parabola.

Choosing axes so the slope a_1 vanishes, and choosing the scale so $a_2 = 2$, the conic has the equation

$$72y = 72x^2 + 12a_3xy + (3a_4 - 2a_3{}^2)y^2.$$

Using Transon's formulas (19) it is easy to verify that the center of aberrancy does pursue the point of osculation. For, $dy/dx = (v - y)/(u - x)$, in agreement with eqs. (1) of section [D1]. But the angle of attack, which is the complement of angle AQD, generally varies as $Q(u,v)$ moves along its track, being constant only on a logarithmic spiral.

[D7] Miscellaneous Pursuit Problems. The eqs. (1) are so general that any two plane curves may be interpreted as track and pursuit, so long as P and Q are properly coupled. From each point Q on the first curve, we draw any tangent QP to the other. If there happens to be several tangents we obtain several points P which pursue Q along the same course. Conversely, at each point P on one curve we may consider the tangent line and its intersection Q with the other curve. If there are several intersections each generates the same track. If the loci of both P and Q are given the problem would be to determine the coupling, and the corresponding ratio of speeds.

Normally the Q track is given and we seek to find the P locus by integrating (1) subject to some auxiliary condition which implies a definition of the speed ratio. Thus Bouguer [B] and Hathaway [C] suppose explicitly that the speeds are uniform, with linear and circular tracks, respectively. The tractrix [D5] with fixed separation, and the attack curves [D4] with fixed angle of approach imply speeds $m = \cos \phi$ and $m = \rho/a$, respectively. The barycenter [D3], the center of curvature [D3], and the center of aberrancy [D6] each generate a curve of pursuit.

The examples of pursuit discussed here are by no means comprehensive. Cantor in *Geschichte der Mathematik* (v. 4, p. 506) mentions the early interest of Lambert (1769) and Euler (1775). Beginning with a paper by C. Sturm, "Extension du problème des courbes des poursuite," the *Annales de Mathematique* [v. 13, (1822–23)] carries suggestions by Saint Laurent, Querret, Tédénat, and others.

We would like to mention two papers by V. Nobile. In the first, "Sullo studio intrinseco della curva di caccia," [*Palermo Rend.*, v. 20

(1905), p. 73–82] he obtains the differential equations in intrinsic coordinates, s and ρ, and integrates them for a number of cases. In the second, "Sul problema della curva di caccia," [*Batt. Giorn.*, v. 46 (1908), p. 135–143] he introduces an auxiliary curve for the geometric determination of the pursuit curve.

The drama of pursuit may be played on a three-dimensional stage. But if the track is linear the curve of pursuit lies in the plane determined by this straight line and the initial position of the pursuer. Thus C. C. Puckette in "The Curve of Pursuit" [*Math. Gazette*, v. 37 (1953), p. 256–260] shows that the space curve formulas of Frenet reduce to the differential eqs. [B] for a plane curve. Bouguer's 1732 solution for constant speeds is more recently packaged with tables and graphs for the Office of Scientific Research and Development ["Pursuit Curve Characteristics," *O. S. R. D.* No. 3721 (February, 1944)]. In the pursuit of a bomber flying horizontally with uniform velocity by a fighter the kinematical problem is still two dimensional if the pursuit speed m, distance r, or attack angle ϕ is constant. But if m varies with vertical altitude the plane formulation requires that m be given as a function of the distance from P to the linear track. Again, if the speed of the bomber is comparable with that of the bullets from the pursuing fighter P, it is necessary to aim the guns at a point Q' ahead of the moving target Q. *Lead pursuit* satisfies the tangential requirement of Maupertuis [D1] if we replace Q by its anticipated position Q'. But then the track speed dQ'/dt depends in a complicated way on the instantaneous triangle PQQ'. It seems simpler to deal with P and Q and to regard the angle QPQ' as a perturbation from *pure pursuit*.

The psychology of pursuit assumes that the motion of the quarry Q along its track is independent of the proximity of the pursuer P. A new species of problem is posed if the track of Q is to be determined in order to achieve some object, such as to enable Q to *escape* to some point of refuge Q^*, or to *decoy* P to a neutral zone, say $PQ^* > d$, or into a trap at P^*. Such problems have more than academic interest, but seem to have been neglected.

In this category is the concluding essay "Lion and Man" given by J. E. Littlewood in *A Mathematician's Miscellany* (London, 1953). "A lion and a man in a closed arena have equal maximum speeds. What tactics should the lion employ to be sure of his meal?" The question was invented by R. Rado. If the lion chooses *pursuit*, Littlewood remarks that capture takes an infinite time provided the man runs in a circle. The lion P can defeat the circular strategy by keeping on the radius OQ from some fixed point O to the instantaneous position Q of the man. But if the man alters his track to a certain broken line spiral, A. S. Besicovitch discovered that the lion again dies of starvation if he stubbornly keeps to the radius. Littlewood argues that the man can avoid capture whatever the lion does. He urges the man at the nth stage to travel a route Q_nQ_{n+1} perpendicular to the displacement vector P_nQ_n from the position of the lion P_n and man Q_n at the beginning of that stage, continuing in that direction until Q_{n+1} is *beyond* its closest approach O_n to a fixed point O by the amount $1_n = O_nQ_{n+1} = 1_1n^{-3/4}$. He says that for a suitable choice of the constant 1_1 this infinitely long path will stay inside a circle with center O inside the arena. We are not convinced that this disposes of the general case, however, for the lion might be so disagreeable as to run parallel to the man, so that the man must either travel along a straight course *or alter his strategy*. This seems to be a typical *theory of games* situation. If either player knows the other's fixed strategy he may choose his own to better advantage. At any rate the lion's meal, like happiness, is not to be gained by pursuit.

The history of pursuit curves has attained international stature. Thus André Clarinval wrote "Esquisse historique de la courbe de poursuite" for UNESCO [*Archives internationales d'Histoire des Sciences*, v. 10, 38 (1957), p. 25–37]. He devotes four pages to the solution of Bouguer, but a typographical error mars the integral (p. 28). He condemns the commendable solution of Dubois-Ayme [B7] as "replete with errors," and substitutes the "corrected" formula of Saint-Laurent (again with typographical errors, some old, some new). This historical sketch is chronological, and includes several references to the case of circular track [C], with precise quotations of various published problems, and with brief but accurate reviews of the longer papers devoted to their solution. (American readers may smile at his assumption that F. E. Hackett was named Johns Hopkins.) Clarinval accuses Samuel Jones (*Appendix to the gentleman's Diary*, 1839; reprinted in *Math. Gazette*, v. 15) of "deliberately ignoring" French contributors, and he concludes with a number of references that cross national barriers. Among these are R. Hoseman, *Fiat Review of German Science, 1939–1946*, Applied Mathematics, part I, p. 269; Luke Chia-Liu Yuan, "Homing and Navigational Courses of Automatic Target-Seeking Devices," *J. App. Phys.*, v. 19 (1948), p. 1122–1128; Ralph Hoyt Bacon, "The pursuit course," *J. App. Phys.*, v. 21 (1950), p. 1065–1066; and Pedrazzini, *Period. Mat. Ital.*, n°2 (1949), p. 99–103.

[E] **Extraneous Problems.** We say "*P pursues Q*" if, and only if, the displacement vector PQ is tangent to the path of P, in accordance with eqs. (1). In non-technical usage "pursuit" may have other meanings, which are extraneous to our definition. We list a few examples.

[E1] **Navigation.** Does one swimmer P pursue another Q when his *course* is toward Q though his *heading* is somewhat upstream? If P swims through the water medium at speed e, and the current flows with speed f at an angle ϕ with the desired course PQ, then P must head off course by a correction angle ϵ in order to make good his course.

Applying the law of sines to the triangle of velocities, we obtain $e/\sin \phi = f/\sin \epsilon = g/\sin (\phi + \epsilon)$, where g is the resultant speed. Therefore $\epsilon = \arcsin (f \sin \phi/e)$ and $g = f \sin (\phi + \epsilon)/\sin \epsilon$. Incidentally, the magnitude of ϵ is unchanged if either the current or the course is reversed. These concepts apply to air navigation where e is airspeed, f is wind speed, and g is ground speed. In case the course is at right angles to the current, $\epsilon = \arcsin (f/e)$ and $g = \sqrt{e^2 - f^2}$. Such formulas are crucial in interpreting the interferometer ether drift experiments of Michelson and Morley. In general, P does not pursue Q *with speed e*. Using a ground fixed reference frame, P pursues Q with speed g; but with axes moving with the current, P does not pursue Q. Literally speaking, P pursues Q along-the-ground but not through-the-water!

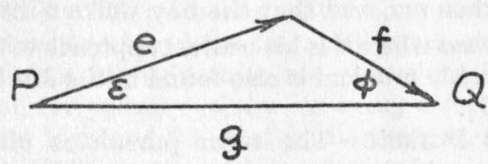

Fig. 6

Exploiting the ambiguity in what is meant by swimming "directly toward" an object is a pamphlet by L. T. Houghton entitled "A Common Sense Solution of a Pursuit Problem That Has Been Considered Unsolvable by Many Eminent Mathematicians." [This reference, among others, is supplied by R. C. Archibald and H. P. Manning, *Am. Math. Mo.*, v. 28 (1921), p. 93.]

A similar problem was proposed by Thomas de Saint-Laurent [*Annales de Gergonne*, v. 13 (1822), p. 289]. A dog swims a canal headed toward his master who walks along the other bank. In the same volume Querret and Tedenat showed that the problem may be interpreted as pursuit (p. 392) by choosing axes which move with the current.

Howard Eves and E. P. Starke selected two pursuit problems as among the 400 best problems appearing in the *Mathematical Monthly* from 1918 to 1950 for inclusion in the *Otto Dunkel Memorial Problem Book* [*Am. Math. Mo.*, v. 64 (1957)]. One by H. E. Tester (#3942) is Bouguer's linear track, with $m = 2n$. The other (#3696) by J. B. Reynolds is more involved. "A dog directly opposite his master on the banks of a stream, flowing with uniform speed, swims at a still-water speed of two miles per hour heading directly toward his master at all times. The man notes that the dog does not stop drifting down stream until he is two-thirds across measured perpendicularly to the banks, and that it takes five minutes longer to make the trip than if the water had been still. How wide is the stream?"

Artemas Martin proposed the following problem in *Educational Times*. A boy walked across a horizontal turntable while it was in motion at a uniform rate of speed, keeping all the time in the same vertical plane. The boy's velocity is uniform with respect to the table, and equal to m times the velocity of a point in the circumference of the table. James McMahon showed [*Math. Questions from E. T.*, v. 51 (1889), p. 158] that the curve described on the table of radius a is another circle of radius ma, and that the boy walks a distance $s = 2ma$ arc sin $\sqrt{a^2 - d^2}/ma$ where d is his nearest approach to the center of the table. The turntable problem is also found in the *Mathematics Visitor* [v. 1 (1878), p. 37].

[E2] **Dynamic Pursuit.** The term pursuit is often applied to motions where the physical forces must be considered. In such dynamic situations the *velocity* dP/dt no longer has the direction PQ. Thus in Keplerian motion it is the acceleration of the planet (not its velocity) which points toward the sun.

George H. Handelman wrote his doctoral dissertation on "Aerodynamic Pursuit Curves for Overhead Attacks" [*Jour. Franklin Institute*, v. 247 (1949), p. 205–221], and amplified the discussion in reports to the Office of Scientific Research and Development with W. Prager [106.1R] and with W. R. Heller [106.2R]. This thesis includes four graphs which show how the aerodynamic pursuit curve deviates from the pure pursuit curve. An elegant extension to three dimensions was made by L. W. Cohen in "Equations for Aerodynamic Lead Pursuit Curves" (*Applied Math. Panel Report* 153.1R).

Pursuit curves were conceived with piracy on the high seas [B1] and doubtless much of the recent research is classified as a military secret.

But man does not live by aerial dogfights alone. While one member of Napoleon's expedition to Egypt discovered the Rosetta stone, another observed the first *courbe du chien* in the sand. But, if life began in the sea, long before this trace launched a thousand other spoors, there must have been countless examples of dynamic pursuit in the ocean depths. The following query was published in *L'Intermediaire des Mathematiciens,* v. 1 (1894), p. 183. "Supposing that the light of a star takes twelve hours to reach the center of the circle which the star traverses in twenty-four, what is the path of a fish which, starting from the center, moves with a given speed constantly directly to the point where it sees the star" [quoted by R. C. Archibald and H. P. Manning in *Amer. Math. Mo.*, v. 28 (1921), p. 92]. If the fish remains near the center of its watery abode and if the light rays are considered parallel, then the fish swims in a circle away from the star toward its diametrically opposite image. But if an exact solution is demanded, we need to know the index of refraction and the radius of the hydrosphere. The fish swims toward the image of the star, but the position of this image depends on the eccentric location of the fish, so this is not pursuit of a *given* point.

[E3] **Geodesic Pursuit.** Benjamin F. Finkel, who founded the *American Mathematical Monthly,* proposed [v. 9 (1902), p. 271] the following problem. "A dog at the vertex of a right conical hill pursues a fox at the foot of the hill. How far will the dog run to catch the fox, if the dog runs directly toward the fox at all times, and the fox is continually running around the hill at its foot, the velocity of the dog being 6 feet per second, the velocity of the fox being 5 feet per second, the hill being 100 feet high and 200 feet in diameter at the base?" The next year [v. 10 (1903), p. 104–106] a solution by G. B. M. Zerr was published. A differential equation was derived and the numerical solution $s = 314$ feet was obtained by introducing the approximation that vertex-dog-fox are collinear at all times. But J. E. Sanders objects (p. 205) that "the dog cannot run toward the fox at all times and keep between the fox and the vertex of the hill. If he keeps *between,* Sanders computes $s = 167$ feet from the formula $s = md \arcsin 1/m$ where d is the original distance between dog and fox. R. C. Archibald and H. P. Manning comment: "Professor Finkel made clear the equivalence of his problem" with circular pursuit, conceiving "the surface of the cone to be spread out on a plane." The reader will note that this "equivalence" holds only if we interpret "directly toward" as the shortest path between dog P and fox Q when the curved surface of the cone is developed into a plane. We call this *geodesic* pursuit since it suggests a natural extension to pursuit on a non-developable surface, bringing to mind a spider chasing a fly across a doughnut. But the geodesic path is not the only reasonable interpretation, especially since the dog cannot *see* the fox. An alternate point of view, suggested by motion of a particle constrained to move on a smooth surface, would be to project the vector PQ onto the tangent plane at P.

An earlier version with the hill 200 feet in *radius* appeared in 1888 in *Mathematical Questions from Educational Times*. The proposer obtains a differential equation which is a quarter page long but does not solve it. A numerical solution by B. C. Wallis [v. 5 (1904), p. 30] gives $s =$ 266 feet while the fox runs 222 feet.

This sort of problem is reminiscent of the spider which must follow a path along the walls and ceiling of a room. To obtain a solution the walls must be folded onto the same plane. Since there is more than one way of effecting this projection, the alternative paths must be computed for the room dimensions given, and then the least of these relative minima is the acceptable answer.

And God said, "Let there be light"; and there was light. The Hebrew text uses the same word for the command and its fulfillment. But we can imagine the angelic architect asking for more details: "What path shall light follow in going from P to Q?" And the answer might have been, "Don't bother me with such details. See that it makes the trip in a minimum time." From this *minimal principle* one finds that *for reflection* the angle of incidence should equal the angle of reflection, while *for refraction* at an interface the ratio of the sine of the angle of incidence to the sine of the angle of refraction must equal the ratio of speeds in the two media. And God saw that the light was good.

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