# 1 Introduction

In 1996, Spielman and Teng proved a long-conjectured upper bound on the second eigenvalue of the unnormalized Laplacian for planar graphs: $\lambda_2 = O(1/n)$. This proof had two main consequences. First, it provided a direct proof of the Edge Planar Separator Theorem. Second, it provided a motivation for why spectral partitioning 'works' for planar graphs, in the sense that, when it is used properly, spectral partitioning produces cuts with $O(n)$ vertices in each subset and a cut ratio of $O(\sqrt{n})$ edges to $n$ vertices.

In Section 2, we connect the spectral properties of the normalized Laplacian seen in earlier lectures to those of the unnormalized Laplacian. In Section 3 we present the Vertex Planar Separator Theorem and Edge Planar Separator Theorem. In Section 4, we discuss (without proof) the necessary geometric machinery from the theory of planar graph embeddings needed to prove the conjecture. In Section 5, we prove the conjecture. In Section 6, we prove the Edge PST constructively using spectral partitioning.

# 2 Eigenvalues: normalized vs. unnormalized Laplacian

In the last lecture, we saw a connection between sparse cuts and eigenvectors of the *normalized* Laplacian matrix. However, in some contexts it is easier to work with eigenvalues and eigenvectors of the unnormalized Laplacian, $L_G$. One can generally use eigenvectors of $L_G$ for spectral partitioning, provided one is willing to tolerate weaker bounds for graphs with unbalanced degree sequences. For example, if $y$ is an eigenvector of $L_G$ satisfying $L_G y = \lambda_2 y$ then we can express $Q(y)$ as follows:

$$\nu_2(G) \le Q(y) = d(V)\frac{y^\mathsf{T} L_G y}{\sum_{u<v} d(u)d(v)(y(u)-y(v))^2} = \frac{d(V)\lambda_2\|y\|^2}{\sum_{u<v} d(u)d(v)(y(u)-y(v))^2}.$$

To estimate the denominator, let $d_{\min}$, $d_{\max}$, and $d_{\text{avg}}$ denote the minimum, maximum, and average degree of $G$, respectively. We have

$$\begin{aligned}
\sum_{u<v} d(u)d(v)(y(u)-y(v))^2 &= \frac{1}{2}\sum_u \sum_{v\neq u} d(u)d(v)(y(u)-y(v))^2 \\
&\ge \frac{1}{2} d_{\min}^2 \sum_u \sum_{v\neq u} (y(u)-y(v))^2 \\
&= d_{\min}^2 n \sum_u y(u)^2 = d_{\min}^2 n \|y\|^2.
\end{aligned}$$

Hence

$$Q(y) \leq \frac{d(V)}{n} \frac{\lambda_2 \|y\|^2}{d_{\min}^2 \|y\|^2} = \left( \frac{d_{\text{avg}}}{d_{\min}^2} \right) \lambda_2.$$

Since $d_{\text{avg}} \leq d_{\max}$ (trivially) and $d_{\min} \geq 1$ (follows from connectivity), this gives us the useful relationship

$$\nu_2(G) \leq d_{\max} \lambda_2(G).$$

When combined with Cheeger's inequality ($h(G) \leq \sqrt{2\nu_2(G)}$), this in turn gives us an upper bound on the expansion of a graph $h(G)$ in terms of the spectrum of the unnormalized Laplacian (and the max degree),

$$h(G) \leq \sqrt{2 d_{\max} \lambda_2(G)}.$$

# 3 Planar Separator Theorems

We discuss two types of separations, vertex separators and edge separators.

**Definition 3.1** (Vertex separation). *A vertex separation is a separating vertex set $S \subset V$ of small size, $O(\sqrt{n})$, such that the remaining graph is separated into two comparably sized, $O(n)$, disjoint sets.*

**Definition 3.2** (Edge separation). *An edge separation is a separating edge set $F \subset E$ of small size, $O(\sqrt{n})$, such that the graph is separated into two comparably sized, $O(n)$, disjoint sets. Note that this separation has a cut ratio of $O(1/\sqrt{n})$.*

Notice that an edge separator $F$ gives a vertex separator $S$ (let $S$ be the set of vertices on each end of the edge separator, totalling $2O(\sqrt{n})$ nodes), but not the other way around ($O(\sqrt{n})$ nodes may touch more than $O(\sqrt{n})$ edges).

A seminal result for planar graphs is the Lipton-Tarjan Planar Separator Theorem (PST) for vertex separators, first proved by in 1979. The original proof was given by an algorithmic construction, see Chapter 15 of Kozen, *The Design and Analysis of Algorithms.* Alon, Seymour, and Thomas later gave a non-constructive combinatorial proof. In a proof we will not discuss here, an adaption of the Spielman-Teng results we will discuss can be used to give a geometric proof of this Lipton-Tarjan theorem, but it is important to note that their results do *not* imply this theorem directly.

**Theorem 3.3** (Vertex Planar Separator Theorem, LT79). *For every undirected planar graph $G = (V, E)$, there exists a partition of $V$ into disjoin sets $A$, $B$ and $S$ such that:*

1. $|A|, |B| \leq \frac{2n}{3}$
2. $|S| \leq 4\sqrt{n}$
3. $e(A, B) = \emptyset$ *($S$ is a sperator).*

The Edge Planar Separator Theorem, given below, *does* however follow from their resolution of the conjecture.

**Theorem 3.4** (Edge Planar Separator Theorem, ST96). *For every bounded-degree undirected planar graph $G$, there exists a separation into sets $A$, $B$ such that $|A|$, $|B|$ are $O(n)$ and $h(A,B) = O(1/\sqrt{n})$.*

The goal of this lecture is to prove the Edge PST by proving the conjecture proved by Spielman and Teng ($\lambda_2 = O(1/n)$). Applying the conjecture to Cheeger's inequality,

$$h(G) \leq \sqrt{2d_{\max}\lambda_2} \leq O(1/\sqrt{n}).$$

This guarantees the existence of a cut where $h(A,B) = O(1/\sqrt{n})$, but it doesn't immediately guarantee a partition whereby $|A|$ and $|B|$ are both $O(n)$. We first prove the conjecture, and then we return to this balancing issue, by a simple construction, at the end of the lecture.

What is most remarkable about the Spielman-Teng proof of this theorem is that it is not only constructive, but the construction is explicitly based on spectral partitioning. Thus, their proof killed two birds with one stone: (i) it proved the existence of an edge separator for all bounded-degree planar graphs by proving that (ii) spectral partitioning is such a separator.

The general idea that planar graphs are cuttable is actually rather intuitive. One can reasonably intuit how, as a bounded-degree graph $G$ grows, by virtue of its planarity it is forced to 'spread out' on the plane, and new frontiers of the graph are limited in how attached they can be to the existing interior of the graph. Phrased another way, planarity implies that a graph just can't be made 'all that uncuttable': the expansion, the minimum surface-area-to-volume ratio, will forcibly decrease as $n \to \infty$. Now, some examples:

**Example 3.5.** *Consider the 2-D square lattice of $\sqrt{n} \times \sqrt{n}$ nodes. A cut along an axis of the grid gives $h(G) \leq \frac{\sqrt{n}}{n} = O(1/\sqrt{n})$.*

**Example 3.6.** *Consider a binary tree with $n$ nodes. Cut the central edge. Then $h(G) \leq \frac{1}{\frac{n}{2}-1} = O(1/n)$. So the theorem is not tight for all bounded-degree planar graphs.*

**Example 3.7.** *Consider the star graph on $n$ nodes. $h(G) = 1$, independent of $n$. But the star graph does not have bounded degree, so this is consistent with the theorem.*

In proving the conjecture we will rely on geometric embedding results from the theory of planar graphs to do a lot of heavy lifting. Now for some machinery.

# 4 Geometric embeddings for planar graphs

A range of elegant embedding theorems exist for planar graphs. We state the following theorems in an attempt to provide a somewhat richer context for the ultimate embedding we will use, the Spielman-Teng embedding in Theorem 4.4.

**Theorem 4.1** (Fáry straight line embedding). *$G$ planar $\Rightarrow$ $G$ has a drawing in $\mathbb{R}^2$ where edges are straight lines.*

**Theorem 4.2** (Tutte spring embedding). *$G$ planar, 3-connected $\Rightarrow$ $G$ has a drawing in $\mathbb{R}^2$ where an outside face forms a convex polygon, and all other nodes are at the centroid of their neighbors.*

**Theorem 4.3** (Koebe-Andreev-Thurston circle embedding). *$G = (V, E)$ planar $\Rightarrow$ $G$ has a drawing in $\mathbb{R}^2$ where (i) nodes are circles with disjoint interiors and (ii) $(i, j) \in E$ iff circle $i$ touches circle $j$.*

The Koebe-Andreev-Thurston theorem forms the core idea of the embedding used by Spielman and Teng in their proof that $\lambda_2 = O(1/n)$. Proving the Koebe-Andreev-Thurston theorem is however non-trivial, and we will not prove it here. The cleanest proofs rely on deeper results from algebraic topology of which the circle embedding of planar graphs is an elementary special case. More direct proofs exist but are less easy to follow.

Spielman and Teng transfer this embedding to the sphere via inverse stereographic projection, and then do some extra work to move the centroid of all the points to the origin, for reasons that will become clear. We state their embedding theorem without proof.

**Theorem 4.4** (Spielman-Teng spherical embedding). *$G$ planar $\Rightarrow$ $G$ has an embedding on the unit sphere in $\mathbb{R}^3$ where (i) nodes are caps with disjoint interiors, (ii) $(i, j) \in E$ iff cap $i$ touches cap $j$, and (iii) the centroid of the cap centers are the center of the sphere (the origin).*

# 5 Spielman and Teng's proof that $\lambda_2 = O(1/n)$.

First, recall the following facts about Rayleigh quotients, $Q(x) = \frac{x^T L_G x}{x^T x} = \frac{\sum_{(i,j)\in E}(x_i - x_j)^2}{\sum_i x_i^2}$,

$$\lambda_2 \leq Q(x) \qquad \forall x \in \{x : x \neq 0, \sum_i x_i = 0\}, \tag{1}$$

$$\lambda_2 = Q(x) \qquad \Leftrightarrow \quad L_G x = \lambda_2 x. \tag{2}$$

Thus, each feasible $x = x_1, \ldots, x_n$ gives an upper bound on $\lambda_2$. Each such $x$ can be interpreted as a non-degenerate ($x \neq 0$) embedding of the vertex set $V$ onto $\mathbb{R}$ such that the centroid is at the origin ($\sum_i x_i = 0$).

Our hope is to use to the Spielman-Teng spherical embedding in $\mathbb{R}^3$ (as opposed of $\mathbb{R}$) to place a meaningful upper bound (say, perhaps, $O(1/n)$) on $\lambda_2$. To do this we must prove the following two Lemmas.

**Lemma 5.1.** *Consider a graph $G = (V, E)$ (not necessarily planar). Let $\mathbf{v} = \mathbf{v}_1, \ldots, \mathbf{v}_n$, where each $\mathbf{v}_i \in \mathbb{R}^d$, be an embedding of the vertex set $V$ into $\mathbb{R}^d$. Then the vectorized*

*Rayleigh quotient $Q(\mathbf{v})$ achieves the same minima as the ordinary Rayleigh quotient $Q(x)$, $x \in \mathbb{R}$:*

$$\min_{\substack{\mathbf{v} \neq 0 \\ \sum_i \mathbf{v}_i = \mathbf{0}}} Q(\mathbf{v}) = \min_{\substack{x \neq 0 \\ \sum_i x_i = 0}} Q(x) = \lambda_2,$$

*where $Q(\mathbf{v}) = \frac{\sum_{(i,j)\in E} ||\mathbf{v}_i - \mathbf{v}_j||^2}{\sum_i ||\mathbf{v}_i||^2}$.*

**Lemma 5.2.** *Consider a planar graph $G = (V, E)$ with a Spielman-Teng spherical embedding $\mathbf{v} = \mathbf{v}_1, \dots, \mathbf{v}_n$. Then $Q(\mathbf{v}) = O(1/n)$.*

Lemma 5.1 gives us that $\lambda_2 \leq Q(\mathbf{v}), \forall \mathbf{v} \in \{\mathbf{v} : \mathbf{v} \neq 0, \sum_i \mathbf{v}_i = \mathbf{0}\}$. Because the Spielman-Teng spherical embedding $\mathbf{v}$ belongs to this feasible set by construction (centroid at origin $\Leftrightarrow \sum_i \mathbf{v}_i = 0$), Lemma 5.2 gives us that $\lambda_2 = O(1/n)$.

*Proof of Lemma 5.1.* Define

$$\lambda_2 = \min_{\substack{x \neq 0 \\ \sum_i x_i = 0}} Q(x), \qquad \text{and} \qquad \lambda_2' = \min_{\substack{\mathbf{v} \neq 0 \\ \sum_i \mathbf{v}_i = \mathbf{0}}} Q(\mathbf{v}).$$

We want to show that $\lambda_2' \leq \lambda_2$ and $\lambda_2' \geq \lambda_2$.

First, let us show that $\lambda_2' \leq \lambda_2$. This is easily done by construction. Let $\mathbf{u}_1, \dots, \mathbf{u}_n$ be $\mathbf{u}_1 = (x_1, 0, \dots, 0), \dots, u_n = (x_n, 0, \dots, 0)$. Then $Q(\mathbf{u}) = \lambda_2$, so

$$\lambda_2' = \min_{\mathbf{v}} Q(\mathbf{v}) \leq Q(\mathbf{u}) = \lambda_2.$$

Next, let us show that $\lambda_2' \geq \lambda_2$. Let $\mathbf{v}_1, \dots, \mathbf{v}_n$ be the optimal feasible $\mathbf{v}$ such that $\lambda_2' = Q(\mathbf{v})$. Expanding terms gives us that

$$\lambda_2' = Q(\mathbf{v}) = \frac{\sum_{(i,j)\in E} ||\mathbf{v}_i - \mathbf{v}_j||^2}{\sum_i ||\mathbf{v}_i||^2} = \frac{\sum_{(i,j)\in E} \sum_{\ell=1}^d (v_{i,\ell} - v_{j,\ell})^2}{\sum_i \sum_{\ell=1}^d v_{i,\ell}^2} \tag{3}$$

$$= \frac{\sum_{(i,j)\in E}(v_{i,1} - v_{j,1})^2 + \cdots + \sum_{(i,j)\in E}(v_{i,d} - v_{j,d})^2}{\sum_i v_{i,1}^2 + \cdots + \sum_i v_{i,d}^2}. \tag{4}$$

Identifying terms, let $a_\ell := \sum_{(i,j)\in E}(v_{i,\ell} - v_{j,\ell})^2$ and $b_\ell = \sum_i v_{i,\ell}^2$. We know that for each $\ell$, $\frac{a_\ell}{b_\ell} \geq \lambda_2$, a property of the scalar Rayleigh quotient. We now use the arithmetic fact that

$$\frac{a_\ell}{b_\ell} \geq \lambda_2, \forall i \Rightarrow \frac{\sum_{\ell=1}^d a_\ell}{\sum_{\ell=1}^d b_\ell} \geq \lambda_2,$$

to finish the proof. □

*Proof of Lemma 5.2.* Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be the cap centers of the S-T spherical embedding. Let $r_1, \ldots, r_n$ be the cap radii.

We aim to bound $Q(\mathbf{v}) = \frac{\sum_{(i,j)\in E} ||\mathbf{v}_i - \mathbf{v}_j||^2}{\sum_i ||\mathbf{v}_i||^2}$. Notice that the denominator $\sum_i ||\mathbf{v}_i||^2 = n$, since all the points $\mathbf{v}_i$ are on the unit sphere. Thus, we want to show that the numerator $\sum_{(i,j)\in E} ||\mathbf{v}_i - \mathbf{v}_j||^2 = O(1)$, a constant dependent only on $d_{\max}$.

By the triangle inequality, the distance between each pair of vectors much be at least the sum of their radii, so $||\mathbf{v}_i - \mathbf{v}_j|| \leq r_i + r_j$. Using this,

$$\sum_{(i,j)\in E} ||\mathbf{v}_i - \mathbf{v}_j||^2 \leq \sum_{(i,j)\in E} (r_i + r_j)^2 \leq \sum_{(i,j)\in E} 2r_i^2 + 2r_j^2 = 2\sum_i d_i r_i^2 \leq 2d_{\max} \sum_i r_i^2.$$

Now, some spherical geometry. Recall that the surface area of a sphere of radius $R$ is $SA_S = 4\pi R^2$. Recall that the area of a cylindrical cap with height $d$ is $SA_C = 2\pi d$, which follows from the 'spherical loaf of bread principle'. A simple exercise in geometry (illustration not included here) gives that $d = r^2/2$. Thus, the surface area of a cap can be expressed in terms of its radius, $SA_C = \pi r^2$. Since the embedding guarantees that the caps are surface disjoint, we know that their total area is less than the area of the unit sphere:

$$\sum_{i=1}^{n} \pi r_i^2 \leq 4\pi,$$

which gives us the useful result that $\sum_i r_i^2 \leq 4$, which allows us to finish the proof:

$$\sum_{(i,j)\in E} ||\mathbf{v}_i - \mathbf{v}_j||^2 \leq 8d_{\max}.$$

Thus the numerator is $O(1)$, and so $Q(\mathbf{v}) = O(1/n)$. □

While Lemma 5.1 is valid for all graphs, we rely directly upon planarity for Lemma 5.2, since only planar graphs are guaranteed to have a Spielman-Teng spherical embedding.

**Remark 5.3.** *Notice that the embedding* $\mathbf{v}$ *is in principle completely unrelated to the eigenvectors of* $L_G$*, since it need not be anywhere near the minimizer of* $Q$*.*

# 6 Spectral partitioning works

The proof of Cheeger's inequality given last lecture said that for every feasible $x$, there exists a cut point $t$ (recall: this was by the probabilistic method) such that the cut between $S = \{x : x < t\}$ and $\bar{S} = \{x : x \geq t\}$ has small expansion $h(S, \bar{S}) \leq \sqrt{2dQ(x)}$. Thus, for the eigenvector associated with eigenvalue $\lambda_2$, this bound is minimized, and there exists a cut point $t$ such that $h(S, \bar{S}) \leq \sqrt{2d\lambda_2}$. In a practical context, linear search for the optimal cut point $t$ gives an algorithm guaranteeing a partition with small expansion. But we have not guaranteed anything about the sizes of the two partition sets. We now prove this, completing the proof of the Edge Planar Separator Theorem (Theorem 3.4).

How do we guarantee that spectral partitioning produces cuts with $O(n)$ vertices in each subset and a cut ratio of $O(\sqrt{n})$ edges to $n$ vertices?

**Lemma 6.1.** *Assume that we are given an algorithm that will a find a cut ratio at most $f(k) = c/\sqrt{k}$ for every $k$-node subgraph of $G$, with some constant $c$. Then repeated application of this algorithm finds a* **bisection** *of $G$ with an edge separator of size $O(\sqrt{n})$.*

*Proof.* The following algorithm will find the bisection:

1. Let $D^0 = G$, let $A$, $B$ be empty sets, let $i = 0$.
2. If $D^i$ is empty, return $A$ and $B$, otherwise repeat
   (a) Find a cut of ratio at most $f(|D^i|)$ dividing $D^i$ into $F^i$, $\bar{F^i}$ (WLOG $|F^i| \leq |\bar{F^i}|$).
   (b) IF $|A| \leq |B|$, let $A = A \cup F^i$, otherwise $B = B \cup F^i$.
   (c) Let $D^{i+1} = \bar{F^{i+1}}$, let $i = i + 1$. Return to step (a).

Assume the algorithm terminates after $t$ steps. To show that the algorithm produces a bisection, we need to show that, for all $i$ in the range $0 \leq i < t$, $\min(|A|, |B|) + |F^i| \leq n/2$. Because $|F^i| \leq |\bar{F^i}|$,

$$\min(|A|, |B|) + |F^i| \leq (|A| + |B| + |F^i| + |\bar{F^i}|)/2 = n/2.$$

So now we have a bisection, but we don't yet have a guarantee that this iterated agglomeration has a good cut ratio, even if each step does. Fortunately, it will. The number of edges we cut to separate $F^i$ at each step is at most

$$f(|D^i|)|F^i| = \sum_{j=|D^i|-|F^i|+1}^{|D^i|} f(|D^i|) \leq \sum_{j=|D^i|-|F^i|+1}^{|D^i|} f(j)$$

where the inequality follows from the fact that $f(j) = c/\sqrt{j}$ is monotonically decreasing. The total number of edges cut by the total algorithm is therefore at most

$$\sum_{i=0}^{t-1} f(|D^i|)|F^i| \leq \sum_{i=0}^{t-1} \left( \sum_{j=|D^i|-|F^i|+1}^{|D^i|} f(j) \right) = \sum_{j=1}^{n} f(j) \leq \int_0^n f(x)dx$$

where the last inequality again follows from the fact that $f(j) = c/\sqrt{j}$ is monotonically decreasing. We evaluate the integral to obtain

$$\int_0^n cn^{-1/2}dx = 2c\sqrt{n} = O(\sqrt{n}).$$

□

Since every $k$-node subgraph of a planar graph is itself planar, spectral partitioning is the required inner algorithm of the proof.