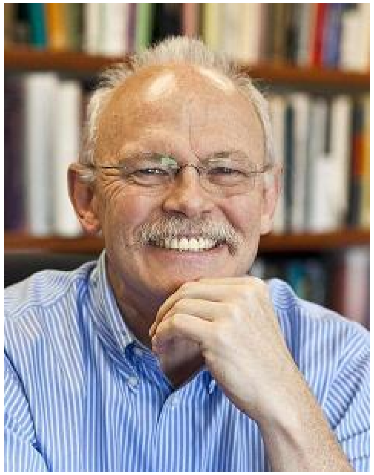
A PDP Approach to Mathematical Cognition

Heineken Prize Lecture

Cognitive Science Society Meeting, 2015



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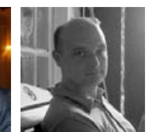
Thanks to …


- Charlene de Carvalho-Heineken
- Bob Glushko & Pam Samuelson
- David Rumelhart, Geoff Hinton & past PDP Collaborators









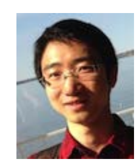


- The lab:
  - Kevin Mickey, Cam McKenzie, Will Zou, Andrew Saxe
  - Steven Hansen
  - Frank Kanayet, Arianna Yuan
  - Qihong Lu, Harry Blayan








Motivating Questions

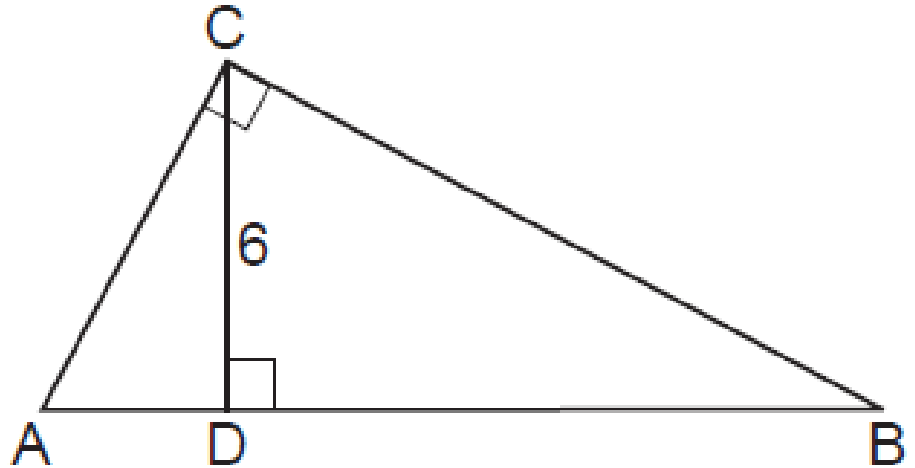
- Neural networks are back: But can they *think?*
- Can neural networks help us understand how math learning occurs, and why math is hard to learn?
- How must neural networks be extended, to capture mathematical cognition?

A Test

Could a neural network learn mathematics well enough to pass the New York State Regents Exam in Geometry?

Inspired by the Allen Institute's AI Fellowship Competition, 2014

An Example Problem



In right triangle ABC, CD is the altitude to hypotenuse AB. If CD = 6 and the ratio of AD to AB is 1:5, determine and state the length of BD.

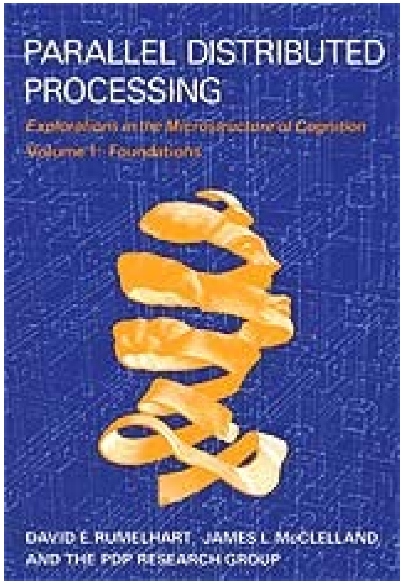
[Only algebraic solutions receive full credit]

The Acid Test

Could the network learn, act and explain its own actions in a human-like way from human-like input?

Talk Plan

- A perspective on
  - The nature of mathematics and the basis of mathematical intuition
  - The nature of knowledge and learning
- Empirical findings and models in mathematical cognition consistent with this perspective
- Toward taking the acid test
  - Key tenets, issues, and a 10-year research plan
- Discussion



What is Mathematics?
One View

- “All mathematics is symbolic logic” – Russell (1903)

- Fodor & Pylyshyn (1988) characterize systematic cognition as *the manipulation of symbolic expressions according to structure-sensitive rules*

- Since the 80’s computer programs based on these ideas have been able to process any well-formed integro-differential equation, such as the one below

$$(p_i - p_0)\left[\int_{t_{j-1}}^{t_j} R(t)dt + \left(p_0 T_j - C_r\right)R\left(t_{j-1}\right) + C_r R(t_j)\right] \Big/ R\left(t_{j-1}\right) - C_i$$

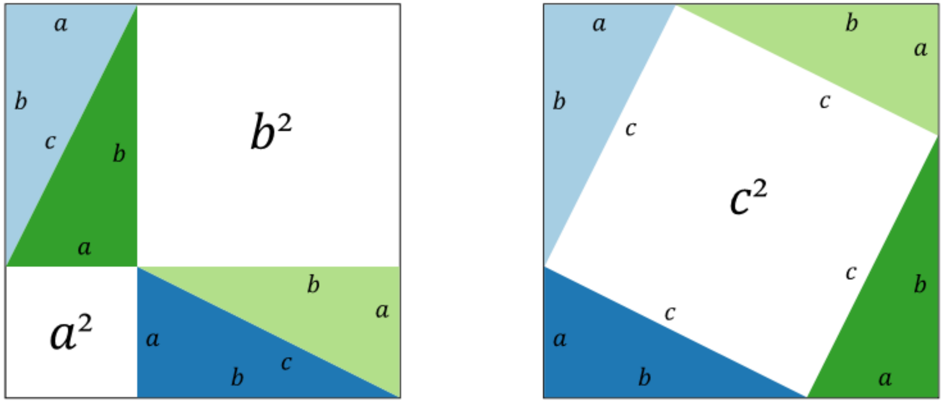
An Alternative Vision

- Manipulating mathematical symbols without seeing what they mean is like writing music without ever hearing a note.
  – T. Needham, *Visual complex analysis*

- Mental transformation of visualized objects of thought can *convince us of the validity* of mathematical and physical laws
  – R. Shepard, *The efficacy of thought experiments*

A proof of the Pythagorean theorem

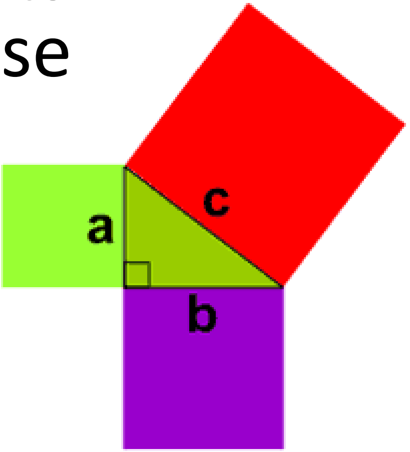
- "The sum of the areas of the squares on the perpendicular sides of a right triangle is equal to the area of a square on the hypotenuse" – Euclid, 4th Century BC



As presented in Shepard's *Rumelhart Prize Lecture*

Two Key Tenets (Thompson, 1992)

- Mathematics is a set of culturally constructed systems for characterizing and reasoning about quantifiable properties of real or imagined objects.
  - The *area* of a *square.*
  - The *cardinality* of a *set.*
- Mathematical expressions are statements about the measurable properties of these objects and the relationships between them.



$$a^2+b^2=c^2$$

Transformational Geometry[1]

- Congruence is established by showing that objects with given properties can be brought into correspondence using transformations that preserve shape: translation, rotation, and reflection

- Intuitions about the shape-preserving properties of these transformations provide a basis for proof that can be made rigorous

[1]From Camenga, K. A. (2013). *Understanding Congruence with Reflections, Rotations, and Translations.* NCTM.

Where do these intuitions come from?

- Shepard and others emphasize innate constraints inherent in the human mind
  - Dehaene appeals to the Kantian idea of *pure intuitions* that are *a priori* and *universal*
- I emphasize the roles of experience and education with culturally-constructed mathematical systems (Hirsh, 1997)
- The resulting *habits of mind* affect how we see both objects and mathematical expressions (Margolis, 1987)
- These habits may become so ingrained that we forget we acquired them making them *seem* self-evident or innate

The nature of knowledge and learning

What allows us to behave in accordance with a rule or principle?

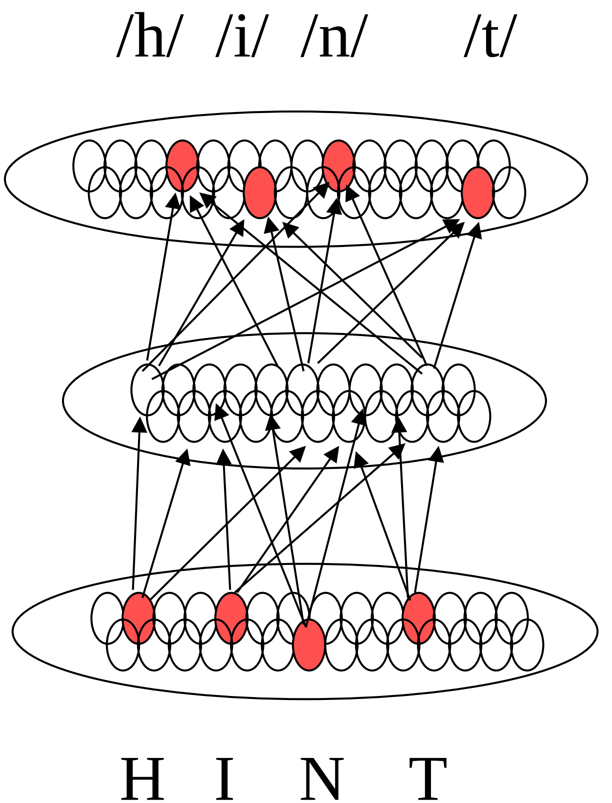
The classical answer

- We use the rule as such, following it by appealing to it as we think.
- If the rule or principle is not given *a priori,* then there must be a 'Eureka moment' (Pinker, 1999) when it is discovered or induced.
- For example, Izard, Pica, Spelke & Dehaene (2008) wrote:

  The exact nature of the insight that children experience when they reach the state of CP-knower is unclear. In order to understand it better, we need to determine what children know just before this insight, what triggers it, and what they finally derive from their newly acquired numeric competences.

The Emergentist / PDP Alternative (Rumelhart & McClelland, 1986)

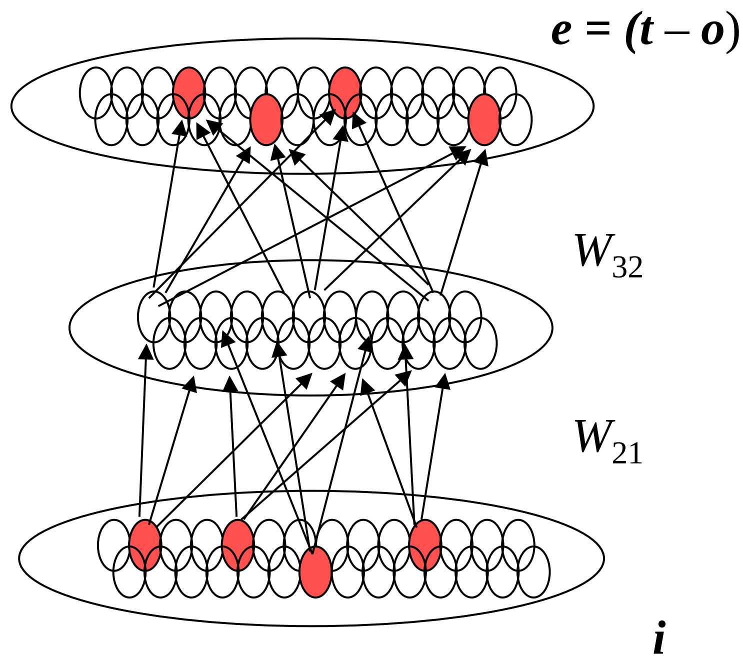
- We gradually acquire implicit encoding and responding abilities that *can be described* by rules
- … guided by an environment that encourages these tendencies
- Deep neural network / PDP models capture these characteristics



Network model of Plaut *et al.* (1996)

A key characteristic of multi-layer neural networks (Saxe *et al*, 2013)

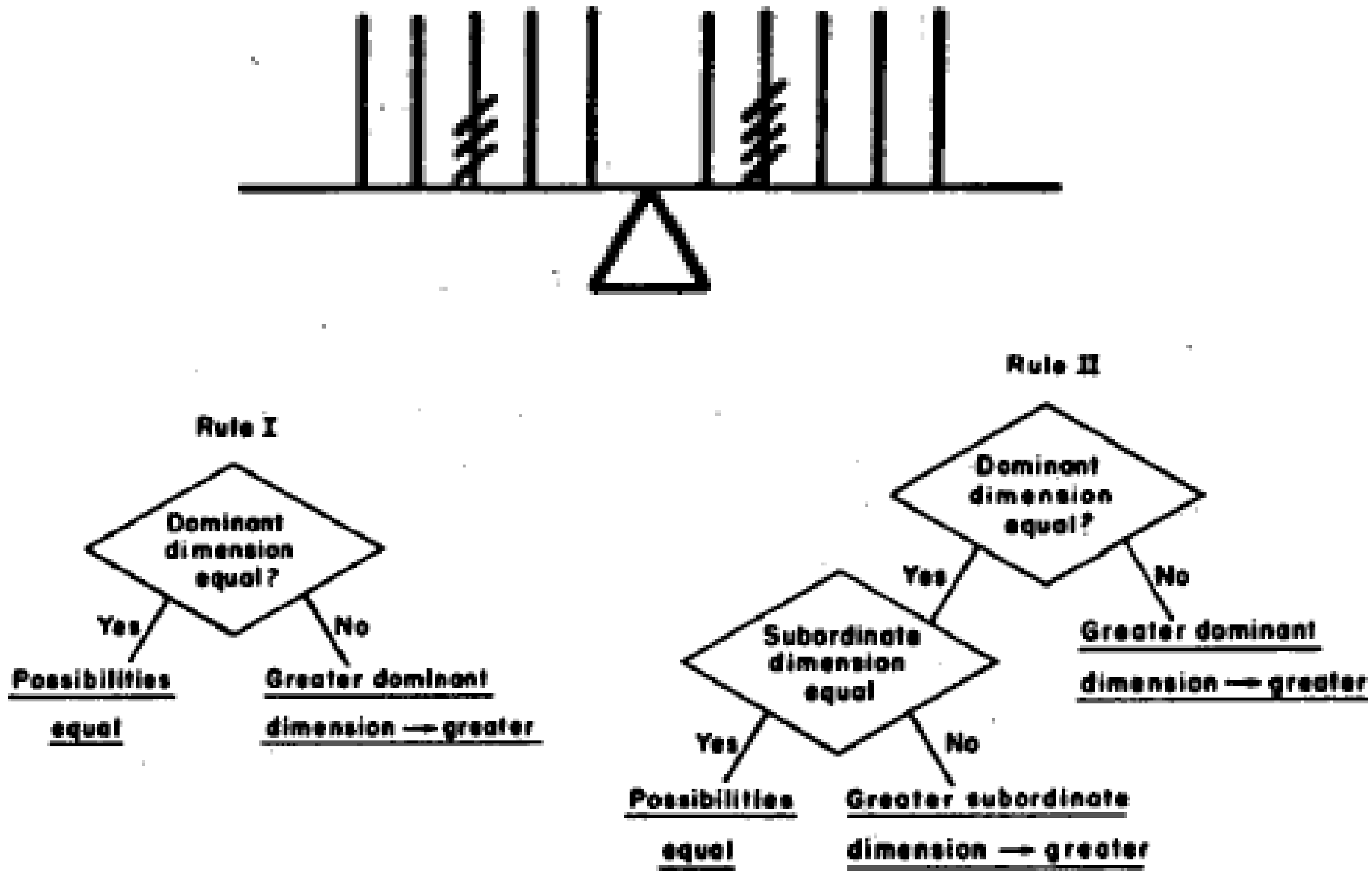
- They exhibit a non-linear dependency between the strength of existing knowledge and how rapidly they learn
- This can help us understand
  - Why learning can be slow at first
  - How stage transitions occur
  - Changes in readiness to learn



$$\Delta W_{21} = \lambda[eW_{32}]' i$$

$$\Delta W_{32} = \lambda e' \, [W_{12} i']'$$

Case Study: The Balance Scale (Siegler, 1977; McClelland, 1995)



Rules or Connections?

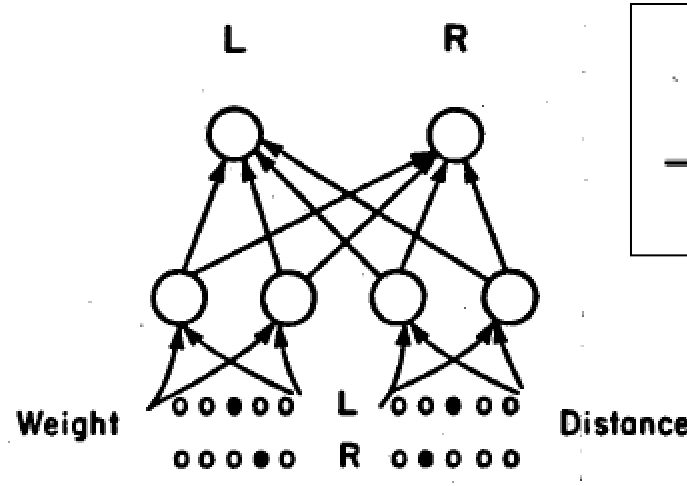
- Children's performance often scores in accordance with simple rules, but with subtle discrepancies

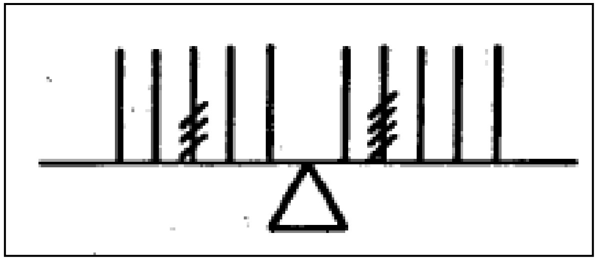
- Children are sensitive to *amount* of difference in distance, to a degree than increases with age

- Children's explanations are often consistent with their choices, but not always

- Within children said to use Rule 1:
  - older children progress from training with conflict problems
  - younger children do not

Balance Scale Model





Input units for number of weights and distance on each side of a balance

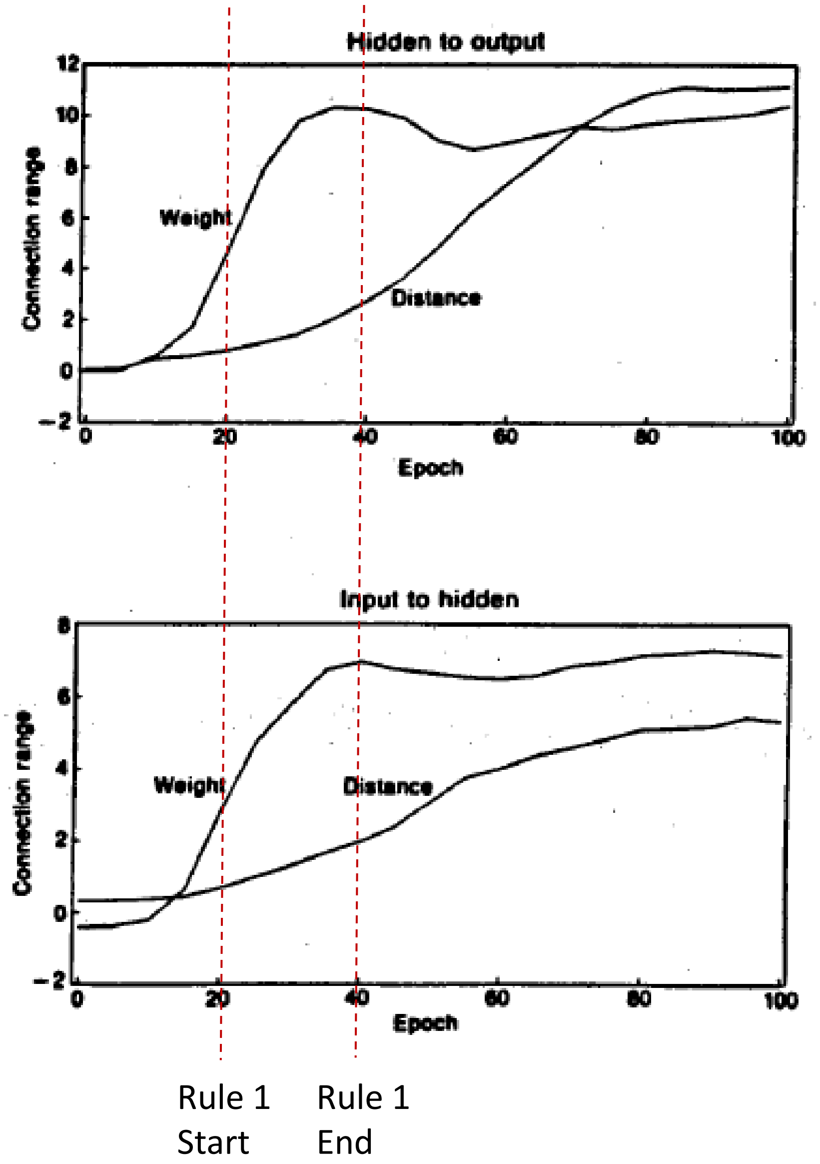
Task: Activate the output unit for the side that should go down (set both to .5 if balance)

Initialization: Connections start with small random values

Training environment: Weight differs between the two sides more often than distance

Results

- Learning is continuous, showing graded sensitivity to cues
- Performance matches Rule 1 from epoch 20 to 40, then Rule 2
- Train on conflict problems at epoch 20:
  - Regression toward guessing
- Same training at epoch 40:
  - Onward progress to Rule 2



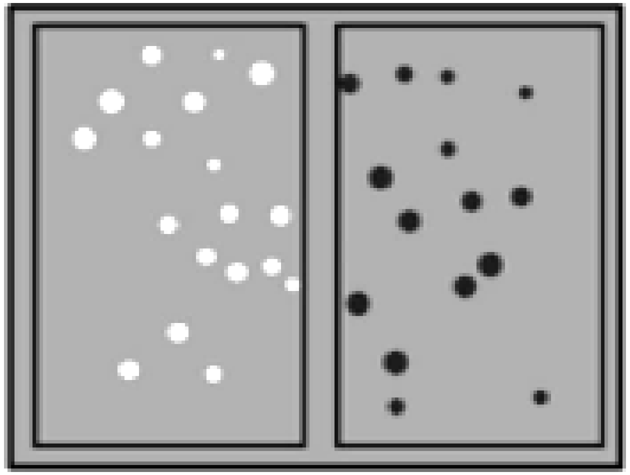
Interim Summary

- The ability to act in accordance with principles or rules can emerge through a sub-symbolic learning process in a neural network
- These models exhibit properties that help explain
  - Gradual, sometimes punctuated development
  - Graded sensitivity to quantitative variables
  - Changes in readiness to learn with experience

Findings from research on mathematical cognition

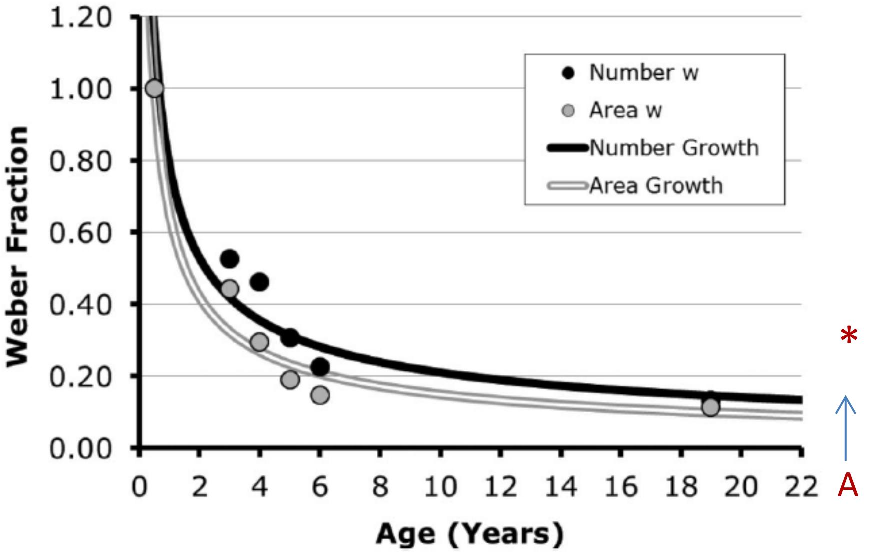
Interpreted and modeled from a PDP perspective

Are Approximate and Exact Number Concepts Innate?



- Many hold that humans and animals are endowed with an innate, approximate ‘number sense’ (Dehaene, 1997)
- Signature characteristic: ratio, not difference dependence
- But acuity of this sense is now thought to be age- and culture/schooling-dependent

From Odic *et al* (2013)



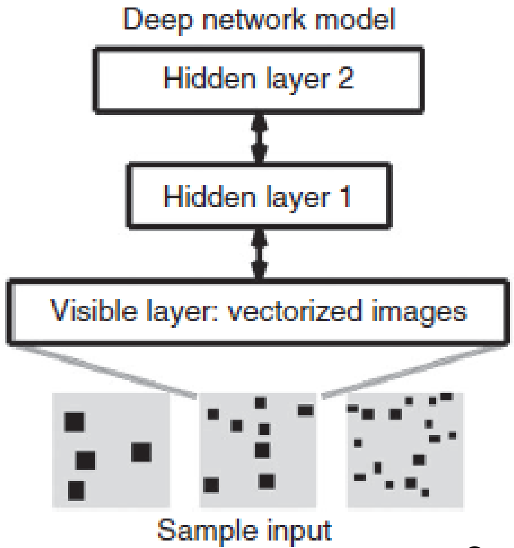
A = Uneducated Mundurucu Adults (Pica *et al,* 2013)

Approximate Number in a Neural Network

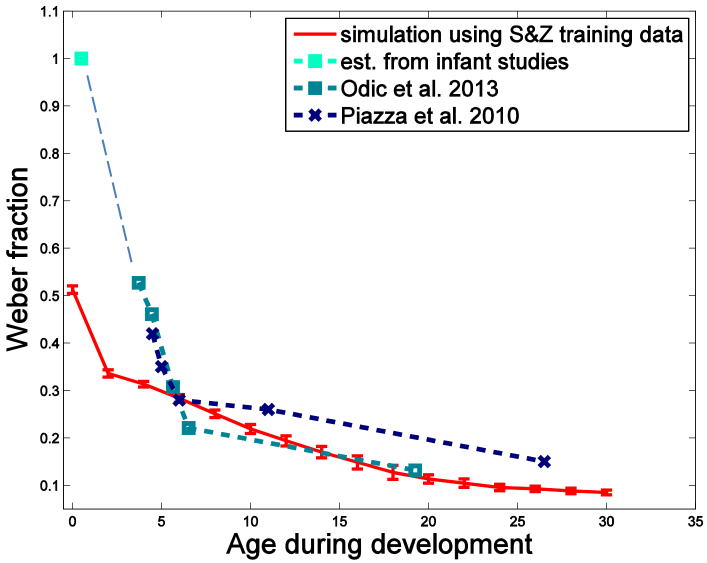
- Representations formed in generic neural networks support signature characteristics of the ANS *without explicit training to represent number* (Stoianov & Zorzi, 2012)

- Extending these simulations, we find (Zou, Testolin & McClelland, submitted)
  - They can perform as well as 3 year olds using random initial weights
  - They can gradually refine their acuity through incremental unsupervised learning.

- Perhaps generic, rather than number specific mechanisms, can support this important cognitive ability.

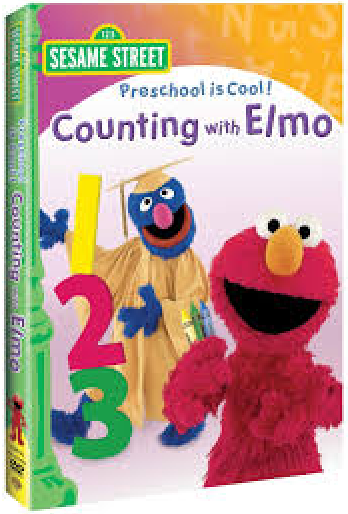


Stoianov & Zorzi, 2012



What about Exact Number?

- Some once thought that an understanding of counting and natural numbers is innate
  - But individuals from cultures lacking counting systems fail basic tests of counting principles (Gordon, 2004)
  - Thus many now suppose that an understanding of exact number is also culturally transmitted



Does Learning to Count Involve a Semantic Induction?

- Sarnecka & Carey (2008) argued that children make a ‘semantic induction’ of the ‘cardinality principle’ at around the age of 4, indexed by success at the *give-N* task.

- Systems possessing symbolic computational primitives can account for this inductive leap (Piantadosi, Tenenbaum & Goodman, 2012).

- But it is not clear if counting involves such an induction

Gradual Development of an Understanding of Exact Number

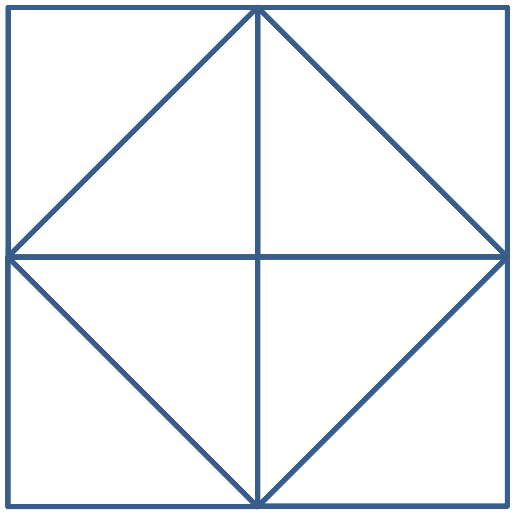
- Davidson, Eng & Barner (2012) looked at children who can correctly 'give-N' for numbers in the 6-8 range. All could count at least to 10.
- They then tested children on the 'Unit' test of CP-knowledge from S&C

  "How many do I have if I add 1 more to a set of 5?
  Do I have 6 or do I have 7"?

- Children who could count to 10 but not much higher performed at chance
- Success spread to larger numbers as the count list grew, but did not generally extend to all numbers within a child's count list
- The authors proposed that children only achieve a general understanding of exact number after considerable experience
- Should we see this as resulting from an 'induction' or from the gradual emergence of a cognitive ability?

Some Observations in Geometry

- Geometrical understanding develops gradually with age, through a series of ‘levels’ (van Heile, 1973; Burger & Shaughnessy, 1986)
  - A year’s course in Geometry has no special impact on student’s ‘level’
  - Many good high-school geometry students have not reached ‘Level IV’, associated with proof and systematic mathematical reasoning
- ‘Failure of a didactic exercise’ (Goldin *et al.* 2011):
  - Students are walked through the Socratic Dialog (Plato, ~350 BC) on how to create a square with twice the area of a given square.
  - In spite of professing to understand afterwards, students with many prior misconceptions could not demonstrate the solution on a new square.
- We see a similar pattern after a brief lesson in trigonometry (Mickey & McClelland, *in preparation)*



Interim Conclusion

- The PDP perspective seems consistent with findings across many domains of mathematics
- It may be worthwhile to see if we can develop the approach further

Preparing for the Test

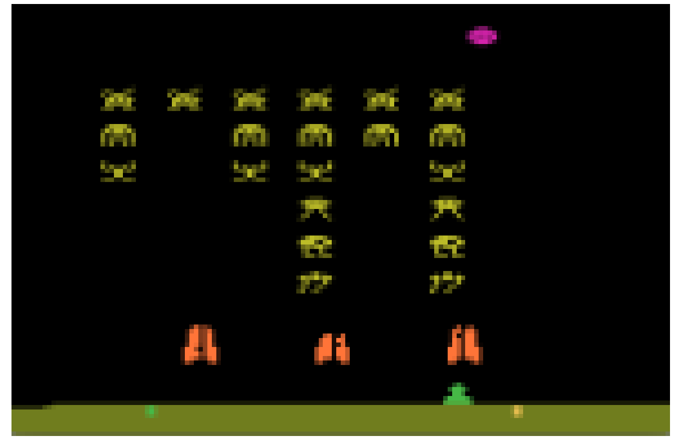
Could a neural network learn mathematics well enough to pass the Regents Exam in Geometry, in a human-like way from human-like input?

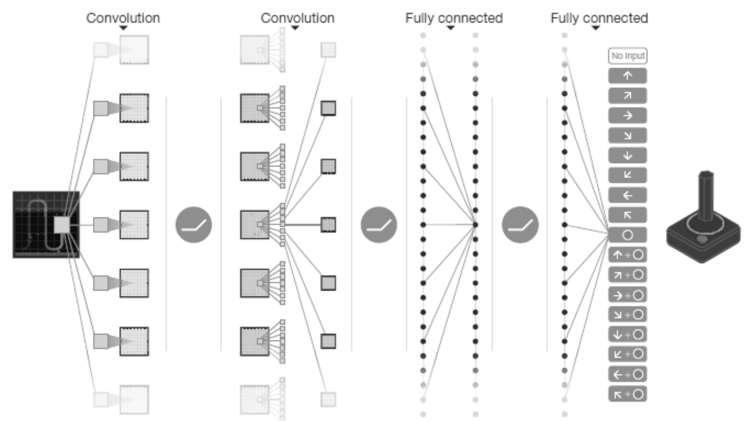
Tenets of the Approach

- Mathematics offers **culturally constructed model systems** that support reasoning in number, geometry, and many other domains
- **Some characteristics** of such systems are **implicit in everyday experience** interacting with objects in the physical world
- Others arise from **structured interactions with physical instantiations of these models**, or with symbols that are grounded in such models, **guided by peers, caregivers, and teachers**
- The guidance is generally **sensitive to the capabilities of the learner** and **combines demonstration, explanation, practice, encouragement, and testing with feedback**
- **Neural networks have many of the right characteristics** to capture how humans acquire an understanding of such models, **but need to be extended** to succeed.

The Project

- Create an artificial world and a simulated learning agent
  - Inspired by the DeepMind Atari Project (Mnih *et al,* 2015)
- Let it explore its world and learn through structured experiences guided by supportive teachers
- Allow it to acquire intuitions and skills related to number, measurement, algebra and geometry
- Gradually build toward multi-step reasoning and problem solving under continuing structured guidance

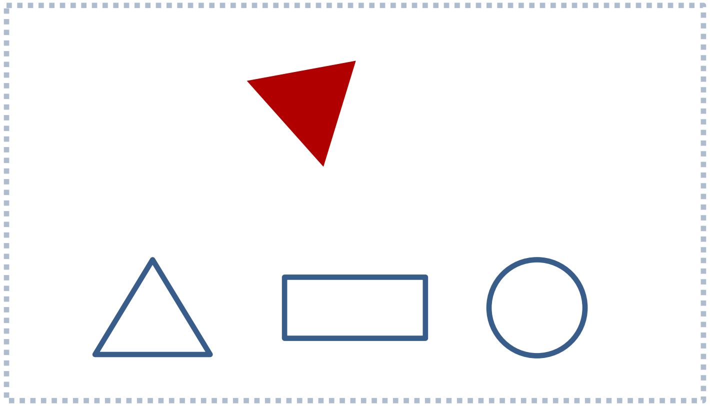




Deep network trained to predict which action will produce the greatest expected reward (Mnih *et al,* 2015)

The Artificial World

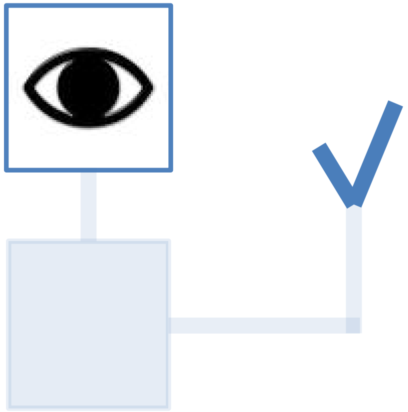
- A large two-dimensional planar surface
  - Where flat objects can be placed
  - On which drawings can be made
    - by the agent or its teachers
- A symbolic communication channel
  - Like a TTY (c.f. *Turing test*)
  - Teacher can give instructions and ask questions in words or mathematical expressions
  - Agent can reply with words and expressions



```
T: Can you touch
the red triangle?
```

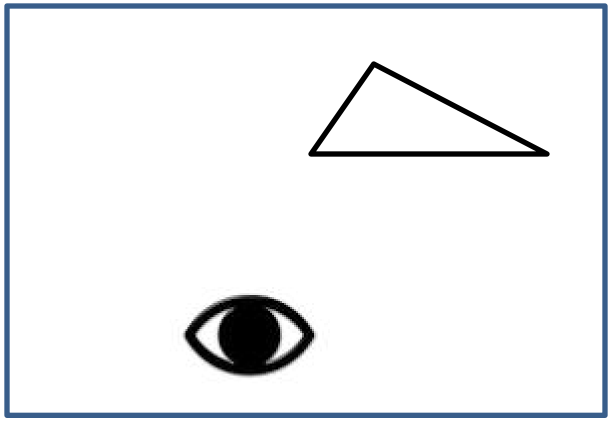
The Agent

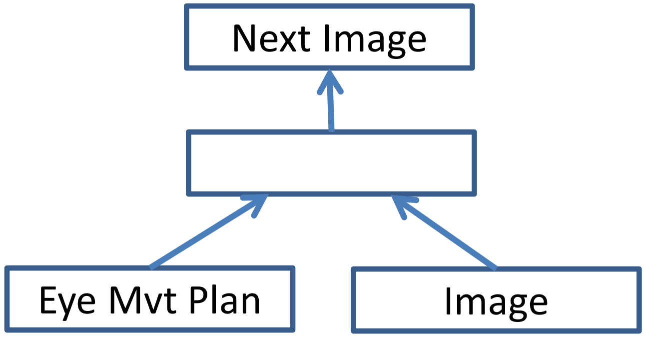
- The agent will have a head with a fixed eye and a simplified hand
  - It will be able to move its head to fixate anywhere on the plane
  - It will be able to move its hand to touch, drag, rotate, or flip objects
  - And to manipulate tools so it can draw, copy and measure objects
- The agent will also read and write linguistic and symbolic expressions, treated as character strings on the TTY



Initial thoughts about learning

- Learning will be exploration, reinforcement, correction, and imitation-driven
- E.g., Learning invariant shape representations through exploration
  - Hinton's transforming auto-encoder as a mechanism for learning an invariant representation of shape
  - Can learn separate representations of objects and transformations
  - Can be extended to capture effects of translation and rotation carried out by the eye and hand.
- Curriculum-based learning
  - Teacher sets tasks at the edge of the competence of the learner, demonstrating, observing, correcting, and reinforcing
  - Tasks are set using linguistic or symbolic inputs along with virtual objects and diagrams
  - Explanation and justification is part of what has to be learned



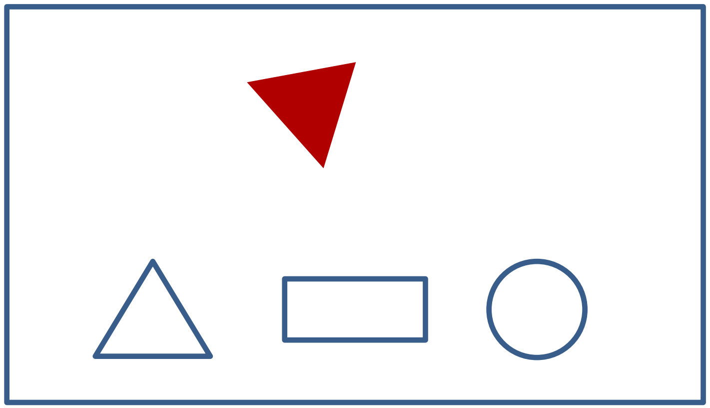


Early Curriculum for Teaching the Agent

Early sensory-motor interaction with shapes

- Move/zoom viewpoint to achieve translational and scale invariance
- Grasp and move objects to center of field of view
- Rotate, translate and flip objects to fit ‘shape sorter’ cutouts in the plane

- Learn to recognize and identify through Agent-teacher interactions



```
T: What kind of shape
is the red one?
A: Triangle?
T: That’s right, A!
```

Two key issues for the agent

- Structured sequential behavior
  - Neural networks work well for one-step computations – can they learn to produce a structured sequenced of operations?
- Generalization & abstraction
  - Neural networks rely on similarity-based generalization – will this allow them to acquire skills they can apply generally, without exhaustive experience?

Counting objects and positions

- A could watch T count objects, touching each one sequentially from left to right, learning to anticipate the next item T will touch and the next word T will produce.
- A could then attempt to replicate T's behavior, receiving a combination of kinds of feedback from T
- If the agent starts at the left, and moves eye and hand to each successive object, T could observe if A is following the correct procedure. This could guide feedback, external control, and demonstration (Alibali & Di Russo, 1999).
- Successive fixation provides a natural basis for generalization of action
  - Current object is always at fixation
  - Count next object if there is one
  - Absence of additional objects provides the stopping condition

Where does structured action come from?

- Structure emerges from culturally defined components that can be assembled.
  - Addition as two Give-N's and a How-Many

T: How many is 2 plus 3?

Where does structured action come from?

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```
T: How many is 2 plus 3?
A: 1, 2
T: Um Hum
```

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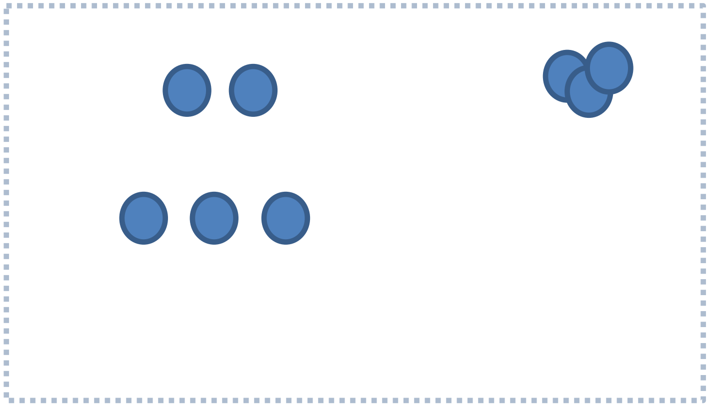
T: How many is 2 plus 3?
A: 1,2
T: Um Hum
A: 1,2,3
T: OK
```



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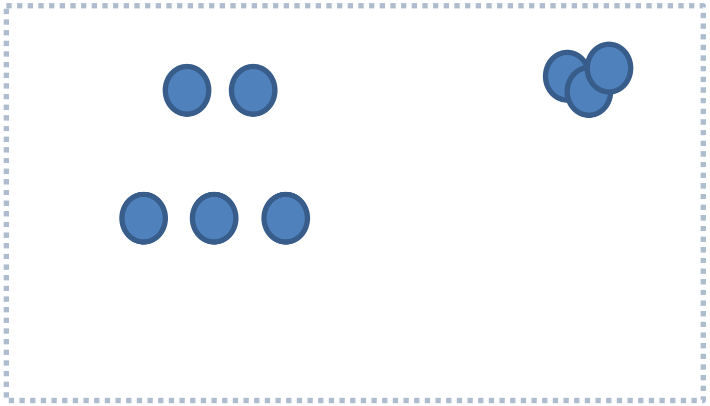
```
T: How many is 2 plus 3?
A: 1,2
T: Um Hum
A: 1,2,3
T: OK
A: 1,2,3,4,5. 5!
T: Great job, A!
```



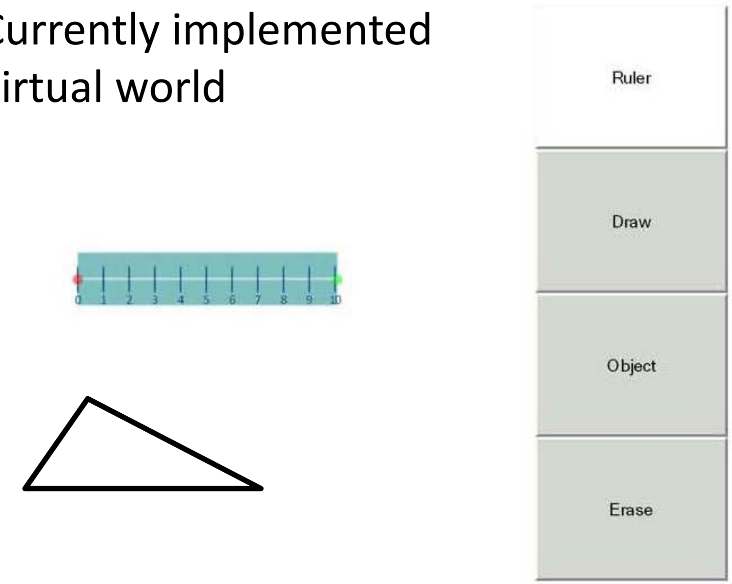
Scaling up to take on Geometry

Too hard for a neural network?

Construction, Measurement, Copying, Scaling, and Establishing Relationships Among Objects

- Provide the agent with a ruler object that can be
  - Translated, rotated and scaled
  - Used to draw lines and arcs
- Teach the agent to
  - Copy a given figure
  - Draw a figure to specification
  - Reproduce a figure at a different scale
  - Measure lengths and angles, assess parallelism, similarity, and congruence
- Teach agent to use the Cartesian coordinate system when given axes, a unit, and a origin

Currently implemented virtual world



Advanced Topics in Geometry

Carry out Euclidean constructions, e.g.,

- Construct perpendicular to segment
- Construct Δ's and other shapes from alternative specifications

Determine perimeter and area of polygons and circles in given units

Establish correspondence between figures, given information about shared properties

Solve geometry problems requiring intermediate inferences and computations

- Lay out intermediate results algebraically
- Justify steps based on learned justifications
- Assess provability

Is it possible?

- Although there are clearly many gaps to fill, some of what I've proposed seems doable
  - The first stages are already being pursued, by us and others (e.g., DeepMind)
- Two big gaps as this stands:
  - Learning to manipulate mathematical expressions, rather than just objects, while representing what they mean
  - Learning to explain, justify, and formulate a proof
- Since proof and justification are major challenges for humans, it makes sense to leave this as the final challenge for A as well.

We'll learn a lot along the way

- New ways of understanding where structure comes from in mathematical reasoning
- New ways of teaching neural networks
- A deeper understanding of how teachers shape the behavior of learners
- New ways of thinking about what it means to be able to think mathematically

Wish me luck, and thanks for listening!