Text as Data

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Finding Lower Dimensional Embeddings of Text

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    - Embed our documents in a lower dimensional space

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    - Visualize our documents

Finding Lower Dimensional Embeddings of Text

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    - Inference about similarity

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   a) Labeling exercise: What are the dimensions?

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4) Validation
   a) Labeling exercise: What are the dimensions?
   b) Are the dimensions interesting semantically?

An Introduction to Eigenvectors, Values, and Diagonalization

Definition

Suppose $\boldsymbol{A}$ *is an* $N \times N$ *matrix and* λ *is a scalar.*
If

$$\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x}$$

Then $\boldsymbol{x}$ *is an eigenvector and* λ *is the associated eigenvalue*

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- $\boldsymbol{A}$ stretches the eigenvector $\boldsymbol{x}$

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An Introduction to Eigenvectors, Values, and Diagonalization

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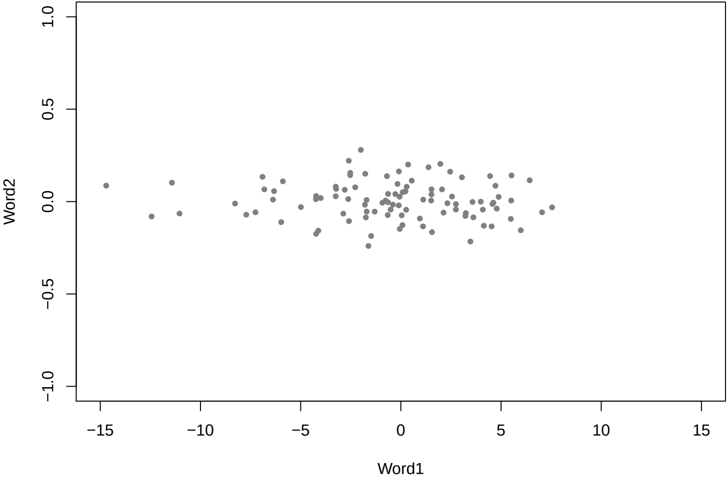
Finding a Lower Dimensional Space (Manifold Learning)



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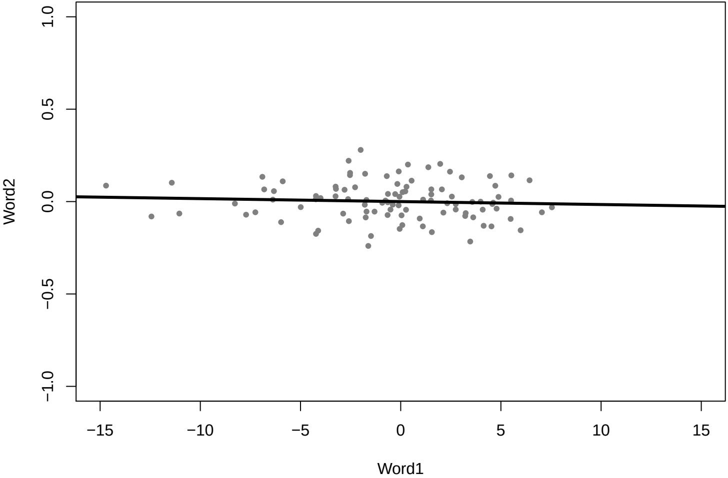

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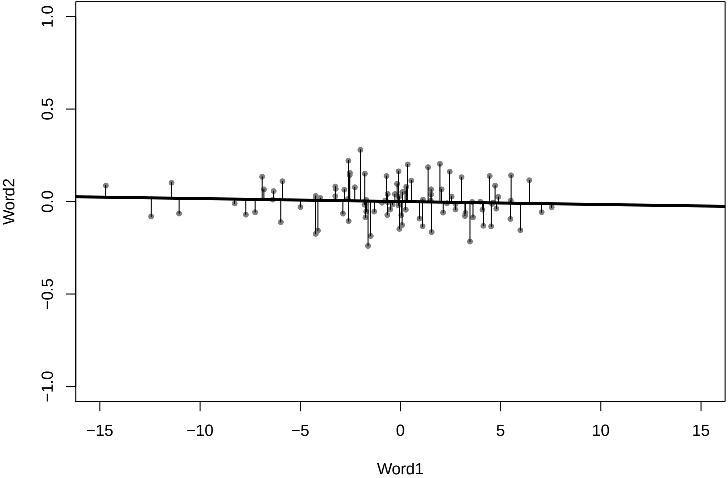

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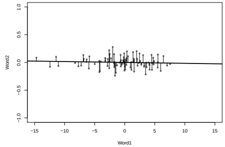
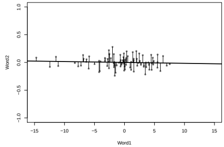


Original data:

Finding a Lower Dimensional Space (Manifold Learning)



Original data:

$$\boldsymbol{x}_i = (x_{i1}, x_{i2})$$

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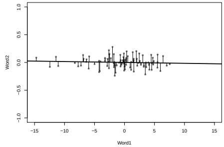

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Which we approximate with

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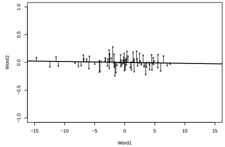


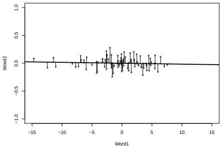
Original data:

$$\boldsymbol{x}_i = (x_{i1}, x_{i2})$$

Which we approximate with

$$\begin{aligned}\tilde{\boldsymbol{x}}_i &= z_i \boldsymbol{w}_1 \\ &= z_i(w_{11}, w_{12})\end{aligned}$$

Finding a Lower Dimensional Space (Manifold Learning)



Original data $\boldsymbol{x}_i \in \Re^J$

$$\boldsymbol{x}_i = (x_{i1}, x_{i2}, \ldots, x_{iJ})$$

Which we approximate with $L(< J)$ weights z_{il} and vectors $\boldsymbol{w}_l \in \Re^J$

$$\tilde{\boldsymbol{x}}_i = z_{i1}\boldsymbol{w}_1 + z_{i2}\boldsymbol{w}_2 + \ldots + z_{iL}\boldsymbol{w}_L$$

Define $\boldsymbol{\theta} = (\underbrace{\boldsymbol{Z}}_{N\times L}, \underbrace{\boldsymbol{W}_L}_{L\times J})$

Principal Component Analysis $\rightsquigarrow$ Objective function

Consider 1-dimensional case ($L = 1$), centered data, and $||\boldsymbol{w}_1|| = 1$.

Principal Component Analysis$\rightsquigarrow$ Objective function

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$$f(\boldsymbol{\theta}, \boldsymbol{X}) = \frac{1}{N}\sum_{i=1}^{N} ||\boldsymbol{x}_i - z_{i1}\boldsymbol{w}_1||^2$$

Principal Component Analysis⇝ Objective function

Consider 1-dimensional case ($L = 1$), centered data, and $||\boldsymbol{w}_1|| = 1$.

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\begin{aligned}
f(\boldsymbol{\theta}, \boldsymbol{X}) &= \frac{1}{N}\sum_{i=1}^{N} ||\boldsymbol{x}_i - z_{i1}\boldsymbol{w}_1||^2 \\
&= \frac{1}{N}\sum_{i=1}^{N} (\boldsymbol{x}_i - z_{i1}\boldsymbol{w}_1)^{'}(\boldsymbol{x}_i - z_{i1}\boldsymbol{w}_1)
\end{aligned}
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&= \frac{1}{N}\sum_{i=1}^{N} \left(\boldsymbol{x}_i^{'}\boldsymbol{x}_i - 2z_{i1}\boldsymbol{w}_1^{'}\boldsymbol{x}_i + z_{i1}^2\right)
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$$

$\boldsymbol{w}_1^{'}\boldsymbol{w}_1 = 1$

Principal Component Analysis$\rightsquigarrow$ Optimization

Optimization:

Principal Component Analysis⇝ Optimization

Optimization:

$$\frac{\partial f(\boldsymbol{\theta}, \boldsymbol{X})}{\partial z_{i1}} = -\frac{2\boldsymbol{w}_1^{'}\boldsymbol{x}_i + 2z_{i1}}{N}$$

Principal Component Analysis $\rightsquigarrow$ Optimization

Optimization:

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\frac{\partial f(\boldsymbol{\theta}, \boldsymbol{X})}{\partial z_{i1}} &= -\frac{2\boldsymbol{w}_1^{'}\boldsymbol{x}_i + 2z_{i1}}{N} \\
0 &= -\frac{2\boldsymbol{w}_1^{'}\boldsymbol{x}_i + 2z_{i1}^{*}}{N}
\end{aligned}
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Principal Component Analysis $\rightsquigarrow$ Optimization

Optimization:

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\frac{\partial f(\boldsymbol{\theta}, \boldsymbol{X})}{\partial z_{i1}} &= -\frac{2\boldsymbol{w}_1^{'}\boldsymbol{x}_i + 2z_{i1}}{N} \\
0 &= -\frac{2\boldsymbol{w}_1^{'}\boldsymbol{x}_i + 2z_{i1}^{*}}{N} \\
z_{i1}^{*} &= \boldsymbol{w}_1^{'}\boldsymbol{x}_i
\end{aligned}
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Principal Component Analysis $\rightsquigarrow$ Optimization

Substituting in z_{i1}^{*}

Principal Component Analysis$\rightsquigarrow$ Optimization

Substituting in z_{i1}^{*}

$$= \frac{1}{N}\sum_{i=1}^{N}(\boldsymbol{x}_i - z_{i1}^{*}\boldsymbol{w}_1)^{'}(\boldsymbol{x}_i - z_{i1}^{*}\boldsymbol{w}_1)$$

Principal Component Analysis⇝ Optimization

Substituting in z_{i1}^*

$$
\begin{aligned}
&= \frac{1}{N}\sum_{i=1}^{N}(\boldsymbol{x}_i - z_{i1}^*\boldsymbol{w}_1)^{'}(\boldsymbol{x}_i - z_{i1}^*\boldsymbol{w}_1) \\
&= \frac{1}{N}\sum_{i=1}^{N}(\underbrace{\boldsymbol{x}_i^{'}\boldsymbol{x}_i}_{\text{Constant}} - 2z_{i1}^*\underbrace{\boldsymbol{w}_1^{'}\boldsymbol{x}_i}_{z_{i1}^*} + (z_{i1}^*)^2\underbrace{\boldsymbol{w}_1^{'}\boldsymbol{w}_1}_{1})
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Principal Component Analysis$\rightsquigarrow$ Optimization

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&= -\frac{1}{N}\sum_{i=1}^{N}(z_{i1}^*)^2 + c
\end{aligned}
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&= -\frac{1}{N}\sum_{i=1}^{N}\boldsymbol{w}_1^{'}\boldsymbol{x}_i\boldsymbol{x}_i^{'}\boldsymbol{w}_1
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Principal Component Analysis$\leadsto$ Optimization

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&= -\frac{1}{N}\sum_{i=1}^{N}\boldsymbol{w}_1^{'}\boldsymbol{x}_i\boldsymbol{x}_i^{'}\boldsymbol{w}_1 \\
&= -\boldsymbol{w}_1^{'}\boldsymbol{\Sigma}\boldsymbol{w}_1
\end{aligned}
$$

Principal Component Analysis$\rightsquigarrow$ Optimization

$$= \; -\sum_{i=1}^{N} \boldsymbol{w}_1^{'} \boldsymbol{\Sigma} \boldsymbol{w}_1$$

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where $\boldsymbol{\Sigma}$ is the :

Principal Component Analysis$\rightsquigarrow$ Optimization

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- Variance of the projected data. Define

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$$\boldsymbol{z}_1 \quad = \quad (\boldsymbol{w}_1\boldsymbol{x}_1, \boldsymbol{w}_1\boldsymbol{x}_2, \ldots, \boldsymbol{w}_1\boldsymbol{x}_N)$$

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$$\begin{aligned} \boldsymbol{z}_1 &= (\boldsymbol{w}_1\boldsymbol{x}_1, \boldsymbol{w}_1\boldsymbol{x}_2, \ldots, \boldsymbol{w}_1\boldsymbol{x}_N) \\ \text{var}(\boldsymbol{z}_1) &= E[\boldsymbol{z}_1^2] - E[\boldsymbol{z}_1]^2 \end{aligned}$$

Principal Component Analysis⇝ Optimization

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\text{var}(\boldsymbol{z}_1) &= E[\boldsymbol{z}_1^2] - E[\boldsymbol{z}_1]^2 \\
&= \frac{1}{N}\sum_{i=1}^{N} z_{i1}^2 - 0
\end{aligned}$$

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\end{aligned}$$

Minimize reconstruction error

Principal Component Analysis$\leadsto$ Optimization

$$= \quad -\sum_{i=1}^{N} \boldsymbol{w}_1^{'} \boldsymbol{\Sigma} \boldsymbol{w}_1$$

where $\boldsymbol{\Sigma}$ is the :

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- Variance of the projected data. Define

$$\begin{aligned}
\boldsymbol{z}_1 &= (\boldsymbol{w}_1 \boldsymbol{x}_1, \boldsymbol{w}_1 \boldsymbol{x}_2, \ldots, \boldsymbol{w}_1 \boldsymbol{x}_N) \\
\text{var}(\boldsymbol{z}_1) &= E[\boldsymbol{z}_1^2] - E[\boldsymbol{z}_1]^2 \\
&= \frac{1}{N}\sum_{i=1}^{N} z_{i1}^2 - 0 \\
&= \frac{1}{N}\sum_{i=1}^{N} \boldsymbol{w}_1^{'} \boldsymbol{x}_i \boldsymbol{x}_i^{'} \boldsymbol{w}_1
\end{aligned}$$

Minimize reconstruction error $\leadsto$ maximize variance of projected data

Principal Component Analysis$\rightsquigarrow$ Optimization

Maximize variance, subject to constraints

Principal Component Analysis $\rightsquigarrow$ Optimization

Maximize variance, subject to constraints

$$g(\boldsymbol{z}^{*}, \boldsymbol{w}_1, \boldsymbol{X}) = \boldsymbol{w}_1^{'} \boldsymbol{\Sigma} \boldsymbol{w}_1 - \lambda_1(\boldsymbol{w}_1^{'} \boldsymbol{w}_1 - 1)$$

Principal Component Analysis $\rightsquigarrow$ Optimization

Maximize variance, subject to constraints

$$\begin{aligned} g(\boldsymbol{z}^*, \boldsymbol{w}_1, \boldsymbol{X}) &= \boldsymbol{w}_1^{'} \boldsymbol{\Sigma} \boldsymbol{w}_1 - \lambda_1(\boldsymbol{w}_1^{'} \boldsymbol{w}_1 - 1) \\ \frac{\partial g(\boldsymbol{z}^*, \boldsymbol{w}_1, \boldsymbol{X})}{\partial \boldsymbol{w}_1} &= 2\boldsymbol{\Sigma} \boldsymbol{w}_1 - 2\lambda_1 \boldsymbol{w}_1 \end{aligned}$$

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$$\boldsymbol{w}_1^{'}\boldsymbol{\Sigma}\boldsymbol{w}_1 = \lambda_1$$

So $\boldsymbol{w}_1$ is eigenvector associated with the largest eigenvalue λ_1

An Introduction to Eigenvectors, Values, and Diagonalization

Theorem

Suppose $\boldsymbol{A}$ is an invertible $N \times N$ matrix. Then $\boldsymbol{A}$ has N distinct eigenvalues and N linearly independent eigenvectors. Further, we can write $\boldsymbol{A}$ as,

$$\boldsymbol{A} = \boldsymbol{W}^{'} \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_N \end{pmatrix} \boldsymbol{W}$$

where $\boldsymbol{W} = (\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_N)$ is an $N \times N$ matrix with the N eigenvectors as column vectors.

An Introduction to Eigenvectors, Values, and Diagonalization

Definition

Suppose A is a covariance matrix. Then, we can write A as

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Where $\lambda_1 > \lambda_2 > \ldots > \lambda_N \geq 0$.
We will call $\boldsymbol{w}_1$ *the first eigenvector,* $\boldsymbol{w}_2$ *the second eigenvector, ...,* $\boldsymbol{w}_j$ *the* j^{th} *eigenvector.*

Back to Principal Components

Theorem

Suppose we want to approximate N observations $\boldsymbol{x}_i \in \Re^J$ with $L < J$ orthogonal-unit length vectors $\boldsymbol{w}_l \in \Re^J$ with associated scores z_{il} to minimize reconstruction error:

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Application of Principal Components in R

Consider press releases from 2005 US Senators

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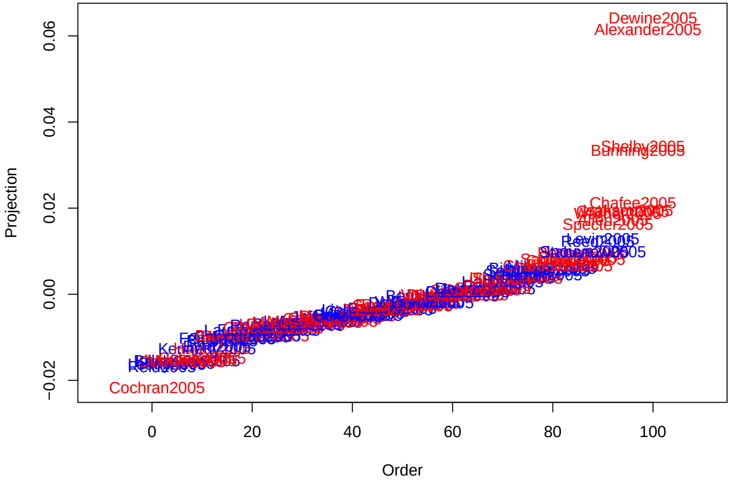
`prcomp(dtm)` applies principal components

Application of Principal Components in R



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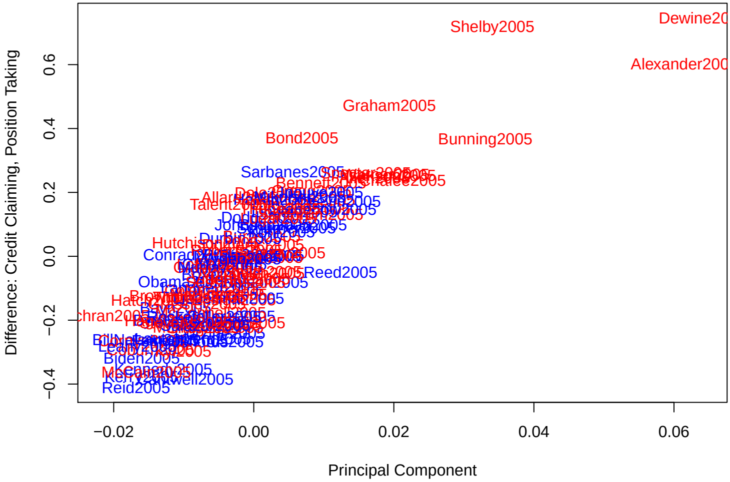

How do we select the number of dimensions L? $\rightsquigarrow$ Model

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Recommendation $\rightsquigarrow$ look for Elbow

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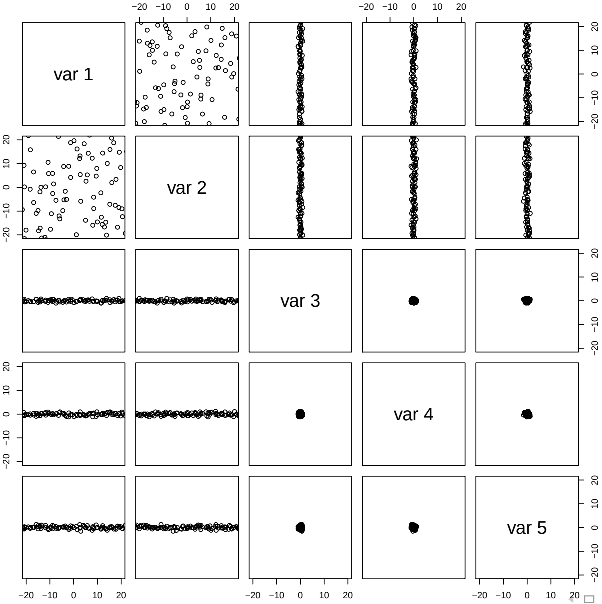

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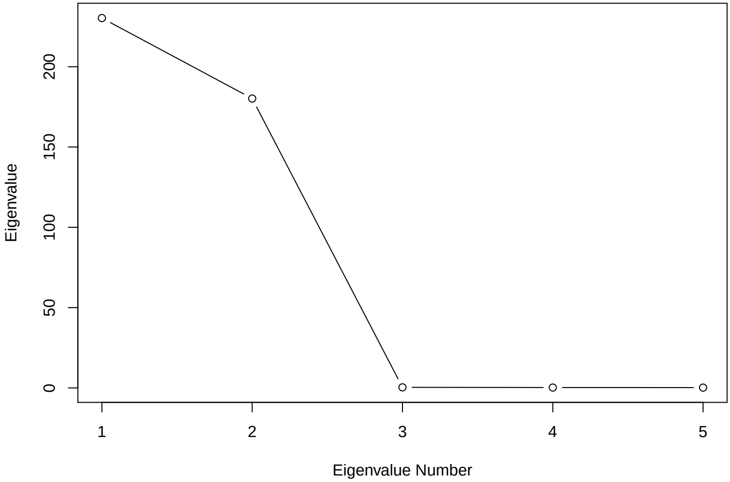

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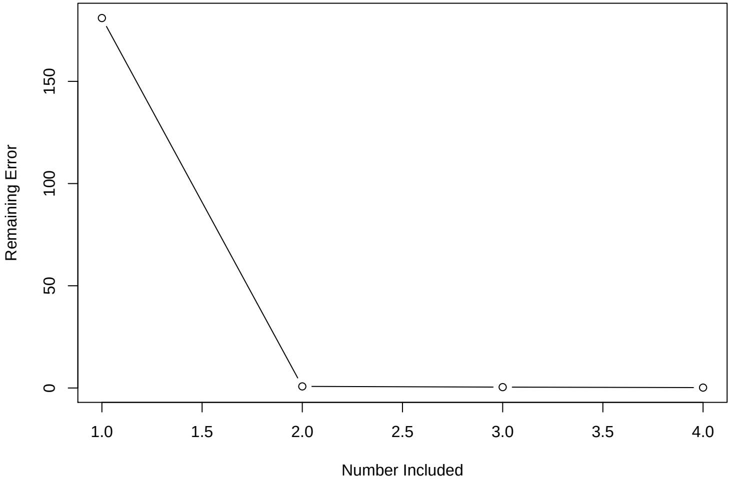

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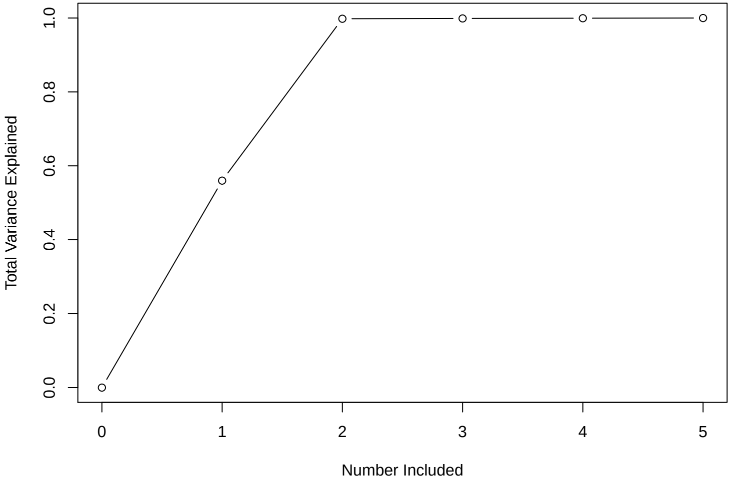

Non-model based evaluations: What's the point?

What is the true underlying dimensionality of $\boldsymbol{X}$?

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Mathematical model $\rightsquigarrow$ insufficient to make modeling decision

Kernel Principal Component Analysis

Recall from Thursday that we define a Kernel ($N \times N$) matrix as:

$$\boldsymbol{K} = \begin{pmatrix} k(\boldsymbol{x}_1, \boldsymbol{x}_1) & k(\boldsymbol{x}_1, \boldsymbol{x}_2) & \ldots & k(\boldsymbol{x}_1, x_N) \\ k(\boldsymbol{x}_2, \boldsymbol{x}_1) & k(\boldsymbol{x}_2, \boldsymbol{x}_2) & \ldots & k(\boldsymbol{x}_2, \boldsymbol{x}_N) \\ \vdots & \vdots & \ddots & \vdots \\ k(\boldsymbol{x}_N, \boldsymbol{x}_1) & k(\boldsymbol{x}_N, \boldsymbol{x}_2) & \ldots & k(\boldsymbol{x}_N, \boldsymbol{x}_N) \end{pmatrix}$$

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Compute PCA of $\boldsymbol{\Phi}$ from $\boldsymbol{\Phi}\boldsymbol{\Phi}^{'}$

Kernel PCA

PCA of $\boldsymbol{X}$

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PCA of $\boldsymbol{X}$ Eigenvectors of $\boldsymbol{X}^{'}\boldsymbol{X}$ ($\frac{1}{N}$ doesn't affect eigenvectors)

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Center $\boldsymbol{K}$?
Use centering matrix $\boldsymbol{H}$

$$\begin{aligned}\boldsymbol{H} &= \boldsymbol{I}_N - \frac{(\mathbf{1}_N\mathbf{1}_N^{'})}{N}\\ \boldsymbol{K}_{\text{center}} &= \boldsymbol{HKH}\end{aligned}$$

Spirling and Indian Treaties

Spirling (2013): model Treaties between US and Native Americans

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Spirling and Indian Treaties

How do we preserve word order and semantic language?

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After stemming, stopping, bag of wording:

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Spirling and Indian Treaties

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After stemming, stopping, bag of wording:

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$\boldsymbol{\phi}(\boldsymbol{x}_i) \approx \binom{32}{5}$ element long count vector

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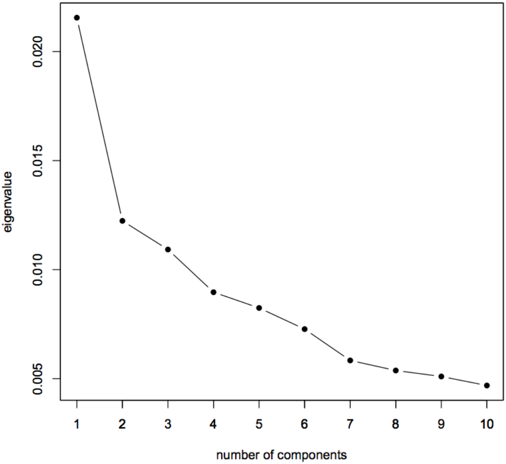

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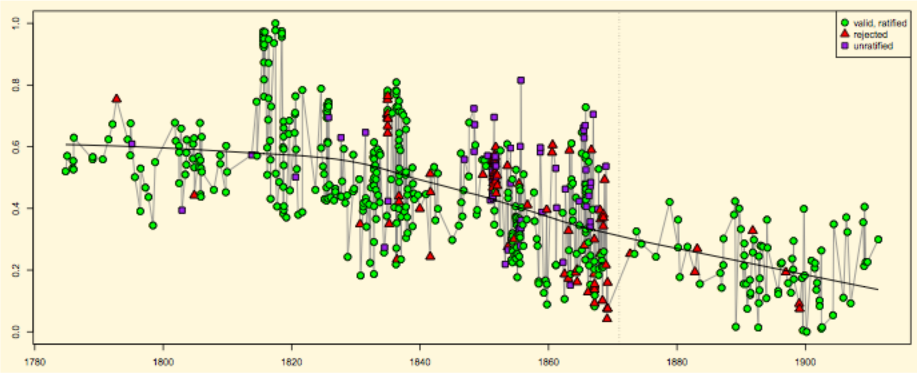

Classic Multidimensional Scaling $\rightsquigarrow$ Objective Function

Suppose we have an $N \times N$ matrix $\boldsymbol{D}$ $\rightsquigarrow$

Classic Multidimensional Scaling $\rightsquigarrow$ Objective Function

Suppose we have an $N \times N$ matrix $\boldsymbol{D} \rightsquigarrow$ matrix of squared Euclidean distances between documents $\boldsymbol{x}_i, \boldsymbol{x}_j \in \Re^J$.

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Recover best low-dimensional representation $\rightsquigarrow$ Eigenvector decomposition of $\boldsymbol{D}$

Classic Multidimensional Scaling $\rightsquigarrow$ Optimization

1) Start with $\boldsymbol{D}$
2) "Center" $\boldsymbol{D}$ to obtain $\boldsymbol{X}\boldsymbol{X}^{'}$
3) Find largest eigenvectors of $\boldsymbol{X}\boldsymbol{X}^{'}$

Classic Multidimensional Scaling $\rightsquigarrow$ Optimization

Center the distance matrix, obtain inner product

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\begin{aligned}
d_{ij}^2 &= \sum_{k=1}^{J} (x_{ik} - x_{jk})^2 \\
&= \sum_{k=1}^{J} (x_{ik}^2 - 2x_{ik}x_{jk} + x_{jk}^2)
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- Subtract off row means

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$$
\boldsymbol{X}\boldsymbol{X}^{'} = \frac{-\boldsymbol{HDH}}{2}
$$

Classic Multidimensional Scaling $\rightsquigarrow$ Optimization

We can write $\boldsymbol{X}\boldsymbol{X}^{'}$ as

$$
\begin{aligned}
\boldsymbol{X}\boldsymbol{X}^{'} &= \boldsymbol{W}^{'} \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_N \end{pmatrix} \boldsymbol{W} \\
&= \underbrace{\boldsymbol{W}^{'} \begin{pmatrix} \sqrt{\lambda_1} & 0 & \ldots & 0 \\ 0 & \sqrt{\lambda_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \sqrt{\lambda_N} \end{pmatrix}}_{\boldsymbol{X}} \underbrace{\begin{pmatrix} \sqrt{\lambda_1} & 0 & \ldots & 0 \\ 0 & \sqrt{\lambda_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \sqrt{\lambda_N} \end{pmatrix} \boldsymbol{W}}_{\boldsymbol{X}^{'}}
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Approximate $\boldsymbol{X}$ with first L eigenvectors

Classic Multidimensional Scaling$\rightsquigarrow$ Optimization

Define $\boldsymbol{z}_i \in \Re^L$ as,

$$\boldsymbol{z}_i^* = \left(\sqrt{\lambda_1}w_{1i}, \sqrt{\lambda_2}w_{2i}, \ldots, \sqrt{\lambda_L}w_{Li}\right)$$

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- Same diagnostics as before: eigenvalues

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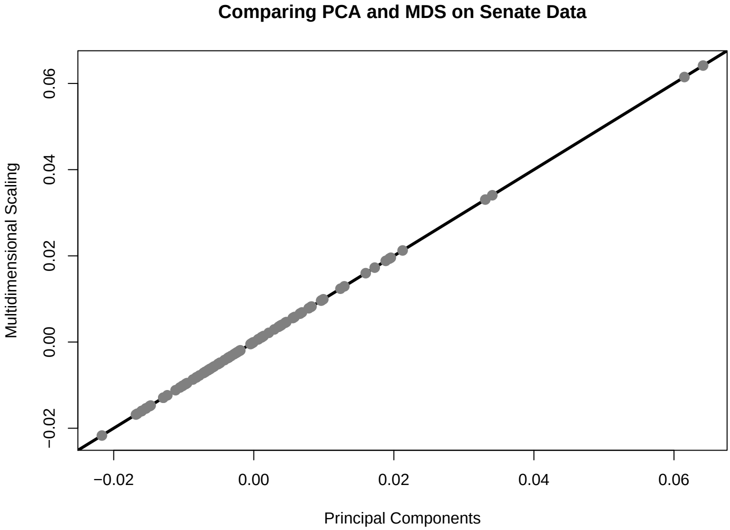
- Same issues in dimension reduction$\leadsto$ what are the dimensions for your task
- Information up to rotation and arbitrary center (same result with any orthogonal matrix $\boldsymbol{M}$)

Comparing MDS and PCA



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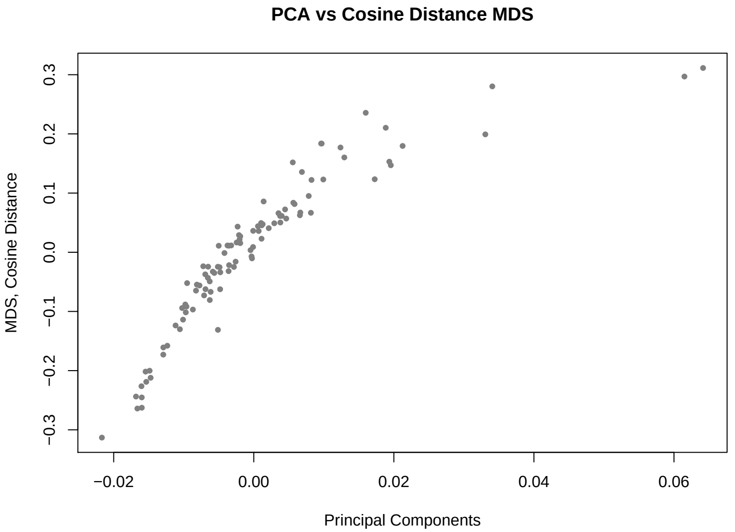

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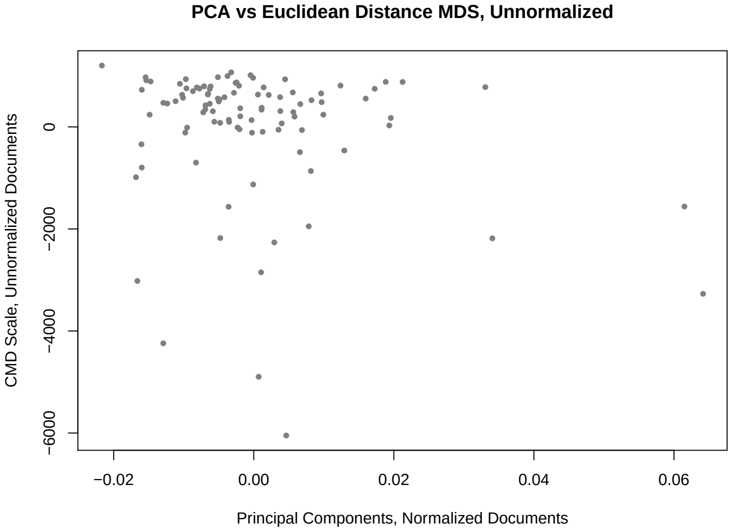

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Common theme$\rightsquigarrow$ Manifold Learning

Low Dimensional Embedding

1) Find lower dimensional space to summarize documents $\rightsquigarrow$ Eigenvectors
2) Thursday: basic language models and the Dirichlet distribution