Math 235a Homework # 3.

Due in class on May 24. * is a hard problem, ? is open.

1. The upper density of an infinite graph is the infimum of all reals x for which the finite subgraphs H of G with $e(H)/\binom{|H|}{2} > x$ have bounded order (number of vertices). Show that this number always takes one of the countably many values $0, 1, 1/2, 2/3, 3/4, 4/5, 5/6, \ldots$.

2. Prove that for each $\epsilon > 0$ there is $\delta > 0$ such that if a K_4-free graph on n vertices has at least $\left(\frac{1}{8} + \epsilon\right) n^2$ edges, then it has independence number at least δn.

3. Let $G = (V, E)$ be a graph. Max-Cut(G) is the maximum number of edges $e(V_1, V_2)$ over all partitions $V = V_1 \cup V_2$ of the vertex set. It is known that it is NP-hard to approximate this parameter with a ratio better than 16/17. Prove that we can approximate Max-Cut for dense graphs. That is, there is a function $f : [0, 1] \to \mathbb{N}$ such that for each $\epsilon > 0$ and graph G on n vertices, there is an algorithm which runs in time $f(\epsilon)n^{O(1)}$ that approximates Max-Cut(G) up to an additive ϵn^2.

4.(?) Let G be a graph on n vertices and at least $n^{1.5}$ edges. The Max-Cut-Ratio of G is defined to be Max-Cut$(G)/e(G)$, which is a rational number between 1/2 and 1. Pick a random induced subgraph G' of G on $n/2$ vertices. Prove that with high probability the Max-Cut-Ratio of G and G' are within $o(1)$ of each other.

5. Fix $t \in \mathbb{N}$. Prove that the number of K_t-free graphs on n vertices is $2^{\left(1-\frac{1}{t-1}+o(1)\right)n^2/2}$.

6.(*) Fix $t \in \mathbb{N}$. Prove that almost all K_t-free graphs on n vertices have chromatic number $t - 1$. (This problem is a strenghthening of the previous problem).