Math 235a Homework # 1.

Due in class on April 24. * problems are hard (considered extra credit), and ? problems are open.

For a graph H and a positive integer n, let $c(H;n)$ denote the minimum possible fraction of copies of H in K_n which must be monochromatic in any 2-edge-coloring of K_n.

1. Prove that $c(H) := \lim_{n\to\infty} c(H;n)$ exists and is positive.

2. Prove that if H has m edges, then $c(H) \leq 2^{1-m}$.

3. Prove that the bound in problem 2 is tight if H is a triangle.

4. Prove that the bound in problem 2 is tight if H is a complete bipartite graph $K_{s,t}$.

5. Prove that any sequence of $(n-1)^3+1$ real numbers $a_1, a_2, \ldots, a_{(n-1)^3+1}$ contains a subsequence $a_{i_1}, a_{i_2}, \ldots, a_{i_n}$ of length n (so $i_1 < i_2 < \ldots < i_n$) which is constant ($a_{i_1} = a_{i_2} = \ldots = a_{i_n}$), strictly increasing ($a_{i_1} < a_{i_2} < \ldots < a_{i_n}$), or strictly decreasing ($a_{i_1} > a_{i_2} > \ldots > a_{i_n}$).

6. Prove that for every r there is $N = N(r)$ such that the following holds:
For every r-coloring of the edges of the complete graph on N^2 vertices with vertex set being the grid $[N] \times [N]$, there is a rectangle whose opposite edges have the same color. That is, there are $x \neq x'$ and $y \neq y'$ such that the edges $((x,y),(x,y'))$ and $((x',y),(x',y'))$ have the same color, and the edges $((x,y),(x',y))$ and $((x,y'),(x',y'))$ have the same color.

7. (*) Prove that $N(r)$ in problem 6 satisfies $N(r) = r^{\omega(1)}$. That is, $N(r)$ grows faster than any polynomial in r.

8. (?) Prove $N(r) < (1-c)r^{\binom{r+1}{2}}$ for some absolute constant $c > 0$ and all sufficiently large r.