Logic and Probability

Probabilities on rich languages, random structures and 0-1 laws

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August 10, 2022
Probability spaces in the measure theoretic sense are structures $(\Omega, \mathcal{E}, \mu)$ with

- $\Omega$ is an arbitrary set
- $\mathcal{E}$ is a $\sigma$-algebra over $\Omega$, i.e., a subset of $\mathcal{P}(\Omega)$ closed under complement and countable unions.
- $\mu : \mathcal{E} \to [0, 1]$ a countably additive measure, i.e.
  - $\mu(\Omega) = 1$
  - $\mu(\bigcup_{i \in \mathbb{N}} E_i) = \sum_{n=0}^{\infty} \mu(E_i)$, when $E_i \cap E_j = \emptyset$ for $i \neq j$.

How do these relate to probabilities defined directly on logical languages?
Everyone says “consider the probability that $X \geq 0$,” where $X$ is a random variable, and only the pedant insists on replacing this phrase by “consider the measure of the set $\{\omega \in \Omega : X(\omega) \geq 0\}$.” Indeed, when a process is specified, only the distribution is of interest, not a particular underlying sample space. In other words, practice shows that it is more natural in many situations to assign probabilities to statements rather than sets.

—Scott & Krauss 1966
Suppose we have a countable propositional language $L$:

$$
\varphi \ ::= \ A_1 \mid A_2 \mid \ldots \mid \varphi \land \varphi \mid \neg \varphi
$$

We can define a probability $\mathbb{P} : L \to [0, 1]$ directly on $L$:

- $\mathbb{P}(\varphi) = 1$, for any tautology $\varphi$;
- $\mathbb{P}(\varphi \lor \psi) = \mathbb{P}(\varphi) + \mathbb{P}(\psi)$, whenever $\models \neg (\varphi \land \psi)$.

Equivalent set of requirements:

- $\mathbb{P}(\varphi) = 1$, for any tautology $\varphi$;
- $\mathbb{P}(\varphi) \leq \mathbb{P}(\psi)$ whenever $\models \varphi \rightarrow \psi$;
- $\mathbb{P}(\varphi) = \mathbb{P}(\varphi \land \psi) + \mathbb{P}(\varphi \land \neg \psi)$.
Some measure-theoretic notions

A family of subsets $\mathcal{R} \subseteq \wp(\Omega)$ forms a **ring** if

- $\emptyset \in \mathcal{R}$
- If $A, B \in \mathcal{R}$ then $A \cup B \in \mathcal{R}$ and $A \setminus B \in \mathcal{R}$

A measure $\mu$ is **finite** if $\mu(\Omega)$ is finite.

Given a family of subsets $\mathcal{F} \subseteq \wp(\Omega)$, let $\sigma(\mathcal{F})$ the smallest $\sigma$-algebra containing $\mathcal{F}$.

**Theorem (Carathéodory’s Extension Theorem)**

*Let $\mu$ be a measure on a ring $(\Omega, \mathcal{R})$. If $\mu$ is a finite measure that is $\sigma$-additive on $\mathcal{R}$, then there is a unique $\sigma$-additive measure $\mu'$ on $\sigma(\mathcal{R})$ that extends $\mu$.***
From Probabilities on Languages to Spaces

- Let $\mathcal{V}$ be the set of all valuations in language $\mathcal{L}$.

- Let $\mathcal{O} \equiv \{ \llbracket \varphi \rrbracket : \varphi \in \mathcal{L} \}$, where $\llbracket \varphi \rrbracket = \{ v : v \models \varphi \}$. Then $\mathcal{O}$ forms a Boolean algebra, hence also a ring. Moreover, any probability measure $\mathbb{P}$ generates a measure that is $\sigma$-additive on $\mathcal{O}$. By the Carathéodory Extension Theorem, it uniquely extends to a $\sigma$-additive measure on the smallest $\sigma$-algebra extending $\mathcal{O}$ [this uses Compactness!].

- In fact, $\mathcal{O}$ forms a clopen basis of a topology on $\mathcal{V}$, which is homeomorphic to standard Cantor space (coin-tossing space: space of infinite binary sequences with clopen basis of cylinder sets). The $\sigma$-algebra generated by $\mathcal{O}$ is the standard Borel $\sigma$-algebra on Cantor space.

- In this way we can show that all functions $\mathbb{P} : \mathcal{L} \to [0, 1]$ can define all the usual probability measures (Borel measures).
Probabilities on propositional calculi are general, but not particularly expressive.
Let $\mathcal{L}$ be a first-order logical language, given by:

- a set $\mathcal{V}$ of individual variables;
- a set $\mathcal{C}$ of individual constants;
- a set $\mathcal{P}$ of predicate variables.

Terms and formulas of $\mathcal{L}$ are defined as usual:

$$\varphi ::= R(t_1, \ldots, t_n) \mid \varphi \land \varphi \mid \neg \varphi \mid \exists x \varphi \mid \forall x \varphi$$
Define $S_L$ to be the set of sentences of $L$, i.e., formulas with no free variables, and $S^0_L$ to be the set of quantifier-free sentences of $L$.

A probability on $\mathcal{L}' \subseteq S_L$ is a function $\mathbb{P} : \mathcal{L}' \to [0, 1]$, with

- $\mathbb{P}(\varphi) = 1$, for any first-order validity $\varphi$;

- $\mathbb{P}(\varphi \lor \psi) = \mathbb{P}(\varphi) + \mathbb{P}(\psi)$, whenever $\vDash \neg(\varphi \land \psi)$.

Question: Given a probability $\mathbb{P} : S^0_L \to [0, 1]$, is there a natural extension of $\mathbb{P}$ to all of $S_L$?
Question: Given a probability $\mathbb{P}: S_\mathcal{L}^0 \rightarrow [0, 1]$, is there a natural extension of $\mathbb{P}$ to all of $S_\mathcal{L}$?

If there are only finitely many constants $c$ such that $\mathbb{P}(R(c)) > 0$, then:

$$\mathbb{P}(\exists x R(x)) = \mathbb{P}(\bigvee_{c \in \mathcal{C}} R(c))$$

What about in the case where the size of $\mathcal{C}$ is infinite?
Example

Consider a simple first-order arithmetical language $\mathcal{L}$, with a constant $n$ for each $n \in \mathbb{Z}^+ = \{1, 2, 3 \ldots \}$. Let $R(x)$ be a one-place predicate. Define a probability function $\mathbb{P} : \mathcal{S}_\mathcal{L}^0 \rightarrow [0, 1]$ on the quantifier-free sentences so that:

- $\mathbb{P}(R(n)) = 2^{-(n+1)}$, for all $n \in \mathbb{N}$;
- $\mathbb{P}(\bigwedge_{i \leq k} R(n_i)) = \prod_{i \leq k} \mathbb{P}(R(n_i))$.

In this case we should expect:

$$
\mathbb{P}(\exists x R(x)) = \sum_{n=2}^{\infty} \frac{1}{2^n} = \frac{1}{2}.
$$
Let us assume in what follows that we have a countably infinite set of constant symbols.
Definition (Gaifman’s Condition)

A probability \( P : S^0_\mathcal{L} \to [0, 1] \) satisfies the Gaifman condition if for all formulas with one free variable \( \varphi(x) \):

\[
P(\exists x \varphi(x)) = \sup \left\{ P \left( \bigvee_{i=1}^n \varphi(c_i) \right) \mid c_1, \ldots, c_n \in C \right\},
\]

or equivalently,

\[
P(\forall x \varphi(x)) = \inf \left\{ P \left( \bigwedge_{i=1}^n \varphi(c_i) \right) \mid c_1, \ldots, c_n \in C \right\}.
\]

Theorem (Gaifman 1964)

Given \( P' : S^0_\mathcal{L} \to [0, 1] \), there is exactly one extension \( P \) of \( P' \) to all of \( S_\mathcal{L} \) that satisfies the Gaifman condition.
Theorem (Gaifman 1964)

Given $\mathbb{P}' : S^0_L \to [0, 1]$, there is exactly one extension $\mathbb{P}$ of $\mathbb{P}'$ to all of $S_L$ that satisfies the Gaifman condition.

Proof of Uniqueness.

Suppose we have $\mathbb{P}_1$ and $\mathbb{P}_2$ that agree on all of $S^0_L$. We show by induction on quantifier complexity that they agree on all $\varphi \in S_L$. Suppose the $\mathbb{P}_i$’s agree all $\Pi_n$ sentences. Let $\varphi$ a $\Sigma_{n+1}$ sentence. We have $\varphi = \exists \vec{x} \psi(\vec{x})$ where $\psi(\vec{x})$ is $\Pi_n$. Now, since both satisfy the Gaifman condition, we have

$$\mathbb{P}_i(\varphi) = \lim_{n \to \infty} \mathbb{P}_i\left( \bigvee_{k_1, \ldots, k_m < n} \psi(c_{k_1}, \ldots, c_{k_m}) \right).$$

Each $\psi(c_{k_1}, \ldots, c_{k_m})$ is a $\Pi_n$ sentence. Since $\Pi_n$ sentences are closed under disjunctions, each such $\bigvee_{k_1, \ldots, k_m < n} \psi(c_{k_1}, \ldots, c_{k_m})$ is also a $\Pi_n$ sentence, and by inductive hypothesis $\mathbb{P}_1$ and $\mathbb{P}_2$ must agree on it. This uniquely determines the limit above, and so the $\mathbb{P}_i$’s must agree on $\varphi$. The same argument works for $\Pi_{n+1}$ sentences, using the closure of $\Sigma_n$ sentences under conjunctions. \qed
Theorem (Gaifman 1964)
Given $\mathbb{P}' : S^0_L \to [0, 1]$, there is exactly one extension $\mathbb{P}$ of $\mathbb{P}'$ to all of $S^0_L$ that satisfies the Gaifman condition.

Proof Sketch of Existence.
Consider the space $\text{Mod}_\omega$ of all countable models with a fixed countable domain (take as domain set of constants $C$). As in propositional case, let $\llbracket \phi \rrbracket \trianglerighteq \{ \mathcal{M} = (C, \mathcal{I}) : \mathcal{M} \models \phi \}$ for each $\phi \in S^0_L$. This defines a Boolean algebra $B_0$ (hence a ring) in the obvious way, and we can define a measure $\mu(\llbracket \phi \rrbracket) = \mathbb{P}(\phi)$, which can be canonically uniquely extended (by Carathéodory again) to a (countably additive) measure $\mu^*$ on the full $\sigma$-algebra $\sigma(B_0)$ (NB. we use compactness!). Lastly, $\llbracket \exists x \phi(x) \rrbracket = \bigcup_{c \in C} \llbracket \phi(c) \rrbracket$, so all sets of this form are in the $\sigma$-algebra. If we define $\mathbb{P}^*(\exists x \phi(x)) \trianglerighteq \mu^*(\llbracket \exists x \phi(x) \rrbracket)$, then countable additivity guarantees the Gaifman condition. $\square$
The space of models

We have built a measure $\mu$ on the space of countable models.

$\text{Mod}_\omega$ is the space of countable structures $\{\mathcal{M} \text{ an } \mathcal{L}\text{-model} \mid \text{dom}(\mathcal{M}) = \omega\}$ with the topology generated by opens

$$\llbracket \pm R(\bar{a}) \rrbracket := \{\mathcal{M} \in \text{Mod}_\omega \mid \mathcal{M} \models \pm R(\bar{a})\} \text{ with } \bar{a} \in \omega^{<\omega}$$

This is a Polish space: it is homeomorphic to the Cantor space $(2^\omega, \mathcal{O})$ with $\mathcal{O}$ generated by cylinder sets.
The same is true in the propositional case. If we take the space \((\mathcal{V}, \mathcal{O})\) with \(\mathcal{O}\) the topology generated by \(\left[\bigwedge_{i \leq n} \pm p_i\right] = \{v \in \mathcal{V} \mid v \vDash \bigwedge_{i \leq n} \pm p_i\}\).

In both cases, we can treat probability functions on our language \(\mathcal{L}\) as probability measures on the standard Borel space \(\text{Mod}_\omega\).

In this sense we can get all the standard Borel measures: and we already have this with measures on propositional languages with countably many atomic propositions.
From Probabilities on Languages to Spaces

Given a probability measure $\mathbb{P}$ on $\mathcal{L}$, we can see it as

- A measure on the Lindenbaum-Tarski algebra $\mathcal{L}/\equiv$ (the algebra of equivalence classes of formulas modulo logical equivalence), where we let

  $$\mathbb{P}^*([\varphi]) := \mathbb{P}(\varphi)$$

- The induced countably additive measure $\mu$ on the space of models (/valuations), which satisfies:

  $$\mu(\{v \in V \mid v \models \varphi\}) = \mathbb{P}(\varphi)$$

One should be careful about treating these as the same thing!
One important difference:

• Consider a probability measure $\mathbb{P}$ on an infinite (countable) propositional language. The measure $\mu$ induced by $\mathbb{P}$ on $\text{Mod}(\mathcal{L})$ is countably additive.

• ...but the measure $\mathbb{P}^*$ on the Lindenbaum-Tarski algebra always fails to be countably additive [Amer, 1985] and even badly so [Seidenfeld].
Takeway:

We can translate between the logical and measure-theoretic perspective without losing anything essential. (There are however some subtle points to take into consideration, such as the issue of $\sigma$-additivity.)

Now: when can logic and probability genuinely illuminate one another?

From logic to probability and back: the case of random structures.
Probabilities on First Order Formulas

Asymptotic probability of graph properties

What is a typical property of a graph?

• Let $G_n$ the set of all (labelled) graphs on $n$ vertices.

• For a well-defined graph property $F$, define

$$p_n(F) := \frac{|\{G \in G_n \mid G \text{ has } F\}|}{|G_n|}$$

• When does $P(F) = \lim_{n \to \infty} p_n(F)$ exist? What proportion of finite graphs has property $P$ (asymptotically)?
Asymptotic probability

Consider various properties for $F$:

- $G$ has a complete subgraph of size $m$: $\lim_{n \to \infty} p_n(F) = 1$.
- $G$ is planar: $\lim_{n \to \infty} p_n(F) = 0$.
- $G$ has an odd number of vertices: no asymptotic probability.

Which properties have a limiting probability? Which ones are typical, in the sense of occurring almost surely?
0-1 law.

Let \( \varphi \) a FOL sentence. Define

\[
p_n(\varphi) := \frac{\left| \left\{ G \in G_n \mid G \models \varphi \right\} \right|}{|G_n|}
\]

0-1 law. Let \( \varphi \) a FOL sentence. Then \( \lim_{n \to \infty} p_n(\varphi) \) always exists, and takes a value in \( \{0, 1\} \).

All first-order properties (1) have a limiting probability and (2) are either typical or atypical!
Alice’s Restaurant Property

You can get anything you want at Alice’s Restaurant.

\[ \forall x_1, \ldots, x_k, y_1, \ldots, y_m \]
\[
\left( \bigwedge_{i \leq k, j \leq m} x_i \neq y_j \rightarrow \exists z \left( \bigwedge_{i \leq k} z \neq x_i \wedge R(z, x_i) \wedge \bigwedge_{i \leq m} z \neq y_i \wedge \neg R(z, y_i) \right) \right)
\]

Given \( X = \{x_1, \ldots, x_k\} \) and \( Y = \{y_1, \ldots, y_m\} \) we say \( z \) as above is a witness for \( X \) and \( Y \): we write \( W(z, X, Y) \).

There is a unique (up to isomorphism) countably infinite graph with the ARP.

Uniqueness: the AFP gives a winning strategy for Duplicator in \( \text{EF}_\omega \).
Existence?
Probabilistic construction

Take $\mathbb{N}$ as vertex set, and for each $(n, m) \in \mathbb{N}^2$ with $n \neq m$, toss a fair coin to decide if $R(n, m)$. This random process generates a countable random structure $(\mathbb{N}, R)$. Now:

**The Random graph.** The procedure above almost surely generates a graph satisfying the Alice’s Restaurant Property.

So by drawing edges independently at random with probability $1/2$, we almost-surely generate the *unique* countable graph satisfying ARP. This is the **Random/Rado graph** $\mathbb{R}$. 
Probabilistic construction

Proof.

Fix \( A = \{a_1, ..., a_k\} \) and \( B = \{b_1, ..., b_m\} \) two disjoint sets of vertices. List all vertices \( \langle v_n \rangle_{n \in \omega} \) not belonging to either set. For any such \( v_n \),
\[
P(W(A, B, v)) = \frac{1}{2^{k+m}}.
\]
The probability that no other vertex is a witness is
\[
P(\bigcap_n \neg W(v_n, A, B)) = \lim_{n \to \infty} P(\bigcap_{i \leq n} \neg W(v_n, A, B))
\]
\[
= \lim_{n \to \infty} (1 - \frac{1}{2^{k+m}})^n = 0
\]
(edges are drawn independently). Now \( P(\neg ARP) \) is at most
\[
P\left( \bigcup_{A, B \in S} \left( \bigcap_n \neg W(v_n, A, B) \right) \right)
\]
where \( S \) ranges over disjoints pairs of finite sets of vertices. This is a countable union of probability 0 events, so it has probability 0.
Asymptotic probabilities and random structures

Now for the 0-1 law. Let $\alpha_{k,m}$ denote the sentence

$$\forall x_1, \ldots, x_k, y_1, \ldots, y_m \left( \bigwedge_{i \leq k, j \leq m} x_i \neq y_j \rightarrow \exists z \ W(z, x_1, \ldots, x_k, y_1, \ldots, y_m) \right)$$

Let $T_R := \{ \alpha_{n,m} \mid n, m < \omega \}$.

**Thm** (Glebskii et al. [1969], Fagin [1976]). Let $\varphi$ a first-order sentence. The following are equivalent:

- $\lim_{n \to \infty} p_n(\varphi) = 1$
- $\varphi$ holds on the random graph;
- $T_R \vdash \varphi$.

A sentence $\varphi$ holds almost surely—in almost all finite graphs—if and only if it holds on the random graph.

$$Cn(T_R) = Th(\mathcal{R}) = \{ \varphi \in \text{Sent} \mid \lim_{n \to \infty} p_n(\varphi) = 1 \}$$
The 0-1 law

Proof.

By our back-and-forth argument, $T_R$ is $\omega$-categorical, and has no finite models: so it is complete. It has $\mathcal{R}$ as a model, and so $T_R \vdash \varphi$ is equivalent to $\varphi$ holding on the random graph. Next, we show that $T_R \vdash \varphi$ entails $\lim_{n \to \infty} p_n(\varphi) = 1$. $T_R \vdash \varphi$ means that there is a finite set $\Gamma$ of extension axioms $\alpha_{k,m}$ such that $\Gamma \vdash \varphi$. It is enough to show that each $\alpha_{k,m}$ holds (asymptotically) almost surely.

As before, for a finite graph $G$ of size $n$ and two disjoint subsets $A, B \subseteq G$ of respective sizes $k$ and $m$, the probability that no $v \in G \setminus (A \cup B)$ is a witness is $(1 - 1/2^{k+m})^{n-k-m}$. 

\[\square\]
The 0-1 law

Proof.

For sufficiently large $n$, $\alpha_{k,m}$ fails with probability at most

$\binom{n}{k} \binom{n-k}{m} (1 - 1/2^{k+m})^{n-k-m}$

and indeed an cruder upper bound for $\lim_{n \to \infty} p_n(\neg \alpha_{k,m})$ is

$\lim_{n \to \infty} n^{k+m} (1 - 1/2^{k+m})^{n-k-m} = 0$,

Now the expression is of the form $n^\alpha \times \beta^{n-\alpha}$ with $\alpha, \beta$ constants and $0 < \beta < 1$: the term $\beta^{n-\alpha}$ going to 0 exponentially, while $n^\alpha$ has only polynomial growth. So it goes to 0, and so we conclude $\lim_{n \to \infty} p_n(\neg \alpha_{k,m}) = 0$. $\square$
The 0-1 law

Proof.
Lastly, we show that \( \lim_{n \to \infty} p_n(\varphi) = 1 \) entails \( T_R \vdash \varphi \). Suppose \( T_R \not\vdash \varphi \). By completeness of \( T \), we have \( T_R \vdash \neg \varphi \). By the previous argument, this means that \( \lim_{n \to \infty} p_n(\neg \varphi) = 1 \), and so \( \varphi \) cannot hold in almost all finite graphs. \( \square \)
Consequences

**Thm** Let $\varphi$ a first-order sentence. The following are equivalent

- $\lim_{n \to \infty} p_n(\varphi) = 1$
- $\varphi$ holds on the random graph;
- $T_R \vdash \varphi$.

Trakhtenbrot:

Sure properties over finite structures are **undecidable**

The theory $T_R := \{\alpha_{n,m} \mid n, m < \omega\}$ is $\omega$-categorical and so it is complete. The axiomatisation is also recursive. Consequence:

Almost sure properties over finite graphs are **decidable**! (in fact, PSPACE)
We built the random graph by randomly (i.i.d) deciding on each potential edge 
\((a, b) \in \mathbb{N}^2\). But the infinite random graph is easy to get.

The brute-force construction: starting from the empty graph, build an infinite 
increasing sequence of graphs \(G_0 \subseteq \ldots G_n \subseteq G_{n+1} \subseteq \ldots\) as follows:

Given \(G_n = (V_n, E_n)\), let \(G_{n+1} = (V_{n+1}, E_{n+1})\) where

- \(V_{n+1} := V_n \cup \{v_A \mid A \subseteq V_n\}\),
- \(E_{n+1} \cap V_n^2 = E_n\),
- for all \(A \subseteq V_n\), we let \(E_{n+1}(v_A, x) \iff x \in A\)

At each stage, for each subset of vertices, we add a vertex that has precisely 
this subset as neighbours.

By design, \(G_\omega := \bigcup_{n \in \mathbb{N}} G_n\) is an infinite countable graph satisfying ARP.
Set-theoretic construction

- Take \((M, \in)\), a countable model of ZFC.
- For \(a, b \in M\), define \(R(a, b)\) if and only if \(a \in b\) or \(b \in a\).
- Then \((M, R)\) is isomorphic to the Rado graph.

Why? Foundation! Let \(a_1, \ldots, a_n, b_1, \ldots, b_m \in M\) with the \(a\)'s and \(b\)'s pairwise distinct. Consider the set

\[ z := \{a_1, \ldots, a_n, \{b_1, \ldots, b_m\}\} \]

Note that \(R(z, b_i)\) would mean that there are \(\in\)-cycles in \(M\).

\((M, R)\) is thus a countable graph satisfying ARP, and so \((M, \in) \cong \aleph_1\).

*(what if we take non well-founded set theory, e.g. ZFA?)*
Number-theoretic construction (Payley)

Let \( V := \{ p \in \mathbb{P} \mid p \equiv 1 \pmod{4} \} \), and let \( R(p, q) \) if and only if \( \exists x \in \{0, \ldots, q\}, p \equiv x^2 \pmod{p} \). Then \((V, R) \cong \mathbb{N}\).

Let \( \{u_1, \ldots, u_k\} \) and \( \{v_1, \ldots, v_m\} \) disjoint sets in \( V \). Pick some \( b_i \)'s st.
\[-\exists x, x^2 \equiv b_i \pmod{v_i} \]  

By the Chinese Remainder Theorem, there is an \( x \in \mathbb{N} \) such that
\[
x \equiv 1 \pmod{4} \\
x \equiv 1 \pmod{u_i} \quad \text{for } i \leq k \\
x \equiv b_i \pmod{v_i} \quad \text{for } i \leq m
\]

and any number in the progression \( \langle x + nd \rangle_n \) \((d = 4u_1 \ldots u_k v_1 \ldots v_n)\) is also a solution to the above congruences. By Dirichlet's Theorem on arithmetic progressions, there exists a prime \( p' \) of this form, so that \( p' = x \) satisfies the above. Then \( p' \) is a witness for \( \{u_1, \ldots, u_k\} \) and \( \{v_1, \ldots, v_m\} \), as desired.
Properties of the random graph

What is special about the random graph?

- Uniqueness (back-and-forth)
- Almost-sure theory
- Universality
- Symmetry (ultra-homogeneous)
- Its relation to the class of finite graphs: a kind of limit, encoding probabilistic information.
This construction (and the 0-1 law) generalises to finite relational signatures: we can carry over the same general model-theoretic construction for the class of all finite models (via Fraïssé limits).
Random structures offer fertile ground for exploring different notions of typicality:

- Asymptotic over finite structures
- Measure theoretic
  - Probability space \((\text{Mod}_{\omega}, \mathcal{F}, \mu)\) with \(\mathcal{F}\) the Borel algebra of the underlying topology. The Lebesgue measure concentrates on the isomorphism class of the random graph (assigns it measure one). [Symmetric probabilistic constructions: \(\mu\) a \(S_\infty\)-invariant measure, i.e. for every Borel set \(A\) and permutation \(g \in S_\infty\), \(\mu(A) = \mu(gA)\)].
- Topological:
  - Seeing \(\text{Mod}_{\omega}\) as a topological space, the isomorphism class of the random graph forms a co-meagre set (topologically large).

But these notions of typicality need not always agree with one another. How to they relate? By virtue of which property of a theory or class of structures?
Conclusion

Random structures lie at the cusp of probability and logic, bridging together model theory and combinatorics. They can be put to use to:

- establish asymptotic 0-1 laws for logics over classes of finite structures
- display infinitary structures ‘approximating’ finite ones
- investigate the connection between symmetries of a structure and probabilistic models
- explore the relationship between topological and measure-theoretic notions of typicality.
Tomorrow:

Probabilistic grammars and probabilistic programs