Logic and Probability
The Quantitative/Qualitative Interface

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Overview of Today

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Today we expand this to prominent kinds of probabilistic reasoning, including inference.

Default reasoning and rules for acceptance of hypotheses in uncertain contexts have been studied in both logical and probabilistic settings. How do these approaches relate?
Some people think it is possible to try to save monotonicity by saying that what was in your mind was not a general rule [...] but a probabilistic rule. So far these people have not worked out any detailed epistemology for this approach, i.e. exactly what probabilistic sentences should be used. Instead AI has moved to directly formalizing nonmonotonic logical reasoning.

—John McCarthy, 1990
Consequence Relations

Preferential Consequence Relations: System \( \mathbf{P} \)

\[ \varphi \sim \varphi \quad (\text{REFL.}) \]
Consequence Relations

Preferential Consequence Relations: System P

\[
\begin{align*}
\varphi & \sim \varphi \\
\models \varphi_1 \leftrightarrow \varphi_2 & \quad \varphi_1 \sim \psi \\
\varphi_2 & \sim \psi
\end{align*}
\]

(REFL.)

(L-E)
Consequence Relations

Preferential Consequence Relations: System $\mathbb{P}$

- $\varphi \not\in\varphi$ (Ref.)
- $\models \varphi_1 \leftrightarrow \varphi_2 \quad \varphi_1 \not\in \psi$ \quad \text{(L-E)}
- $\models \psi_1 \rightarrow \psi_2 \quad \varphi \not\in \psi_1$ \quad \text{(R-W)}
- $\varphi \not\in \psi_2$ \quad \text{(And)}
- $\varphi \not\in \psi \land \psi_2 \quad \varphi \not\in \psi \land \psi_2$ \quad \text{(Or)}
- $\varphi \not\in \psi \land \psi_2 \quad \varphi \not\in \psi \land \psi_2$ \quad \text{(Cautious Monotonicity)}
Preferential Consequence Relations: System $\mathbf{P}$

- **Reflexivity (Ref.)**: $\varphi \not\models \varphi$

- **Left Entailment (L-E)**:
  \[
  \frac{\models \varphi_1 \leftrightarrow \varphi_2 \quad \varphi_1 \not\models \psi}{\varphi_2 \not\models \psi}
  \]

- **Right Weakening (R-W)**:
  \[
  \frac{\models \psi_1 \rightarrow \psi_2 \quad \varphi \not\models \psi_1}{\varphi \not\models \psi_2}
  \]

- **Conjunction (And)**:
  \[
  \frac{\varphi \not\models \psi_1 \quad \varphi \not\models \psi_2}{\varphi \not\models \psi_1 \land \psi_2}
  \]

- **Disjunction (Or)**:
  \[
  \frac{\varphi \not\models \psi_1 \lor \varphi \not\models \psi_2}{\varphi \not\models \psi}
  \]

- **Cautious Monotonicity (And)**:
  \[
  \frac{\varphi \land \psi \not\models \chi}{\varphi \not\models \psi}
  \]
Preferred Consequence Relations: System P

\[ \varphi \models \psi \]

\[ \varphi \models \varphi \quad (\text{REFL.}) \]

\[ \frac{\varphi_1 \Leftrightarrow \varphi_2 \quad \varphi_1 \not\models \psi}{\varphi_2 \not\models \psi} \quad (\text{L-E}) \]

\[ \frac{\varphi \models \psi_1 \quad \varphi \models \psi_2}{\varphi \models \psi_1 \land \psi_2} \quad (\text{AND}) \]

\[ \frac{\psi_1 \rightarrow \psi_2 \quad \varphi \not\models \psi_1}{\varphi \not\models \psi_2} \quad (\text{R-W}) \]

\[ \frac{\varphi \models \psi_1 \quad \varphi \models \psi_2}{\varphi \models \psi_1 \lor \psi_2} \quad (\text{OR}) \]
Preferential Consequence Relations: System $\text{P}$

\begin{align*}
\text{REFL.} & \quad \frac{}{\varphi \sim \varphi} \\
\text{L-E} & \quad \frac{\varphi_1 \leftrightarrow \varphi_2 \quad \varphi_1 \sim \psi}{\varphi_2 \sim \psi} \\
\text{AND} & \quad \frac{\varphi \sim \psi_1 \quad \varphi \sim \psi_2}{\varphi \sim \psi_1 \land \psi_2} \\
\text{R-W} & \quad \frac{\varphi \sim \psi_1}{\varphi \sim \psi_2} \\
\text{OR} & \quad \frac{\varphi_1 \sim \psi \quad \varphi_2 \sim \psi}{\varphi_1 \lor \varphi_2 \sim \psi} \\
\text{Cautious Monotonicity} & \quad \frac{\varphi \sim \psi \quad \varphi \sim \chi}{\varphi \land \psi \sim \chi}
\end{align*}
Semantics for System $\textbf{P}$: Preferential Models

A preferential model is a triple $(W, \preceq, V)$ of a set $W$ of states, a preorder $\preceq$, and a propositional valuation $V : \text{Prop} \rightarrow \wp(W)$. 

Let $M = (W, \preceq, V)$ be a model, then:

$\llbracket \phi \rrbracket_M = \{ s \in W | M, s \models \phi \}$

$\min \preceq (X) = \{ s \in X | \text{there is no } t \in X, \text{ such that } t \prec s \}$

We can then extend a propositional logical language to include a conditional operator $\phi; \psi$:

$M | = \phi; \psi$ iff $\min \preceq (\llbracket \phi \rrbracket_M) \subseteq \llbracket \psi \rrbracket_M$. 

Semantics for System \( \textbf{P} \): Preferential Models

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Given a model \(\mathcal{M} = (W, \preceq, V)\), define:

\[
\mathcal{M}[\varphi] = \{ s \in W \mid \mathcal{M}, s \models \varphi \}
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\text{min}_{\preceq}(X) = \{ s \in X \mid \text{there is no } t \in X, \text{ such that } t \prec s \}
\]

We can then extend a propositional logical language to include a conditional operator \(\varphi \leadsto \psi\):

\[
\mathcal{M} \models \varphi \leadsto \psi \quad \text{iff} \quad \text{min}_{\preceq}(\llbracket \varphi \rrbracket_\mathcal{M}) \subseteq \llbracket \psi \rrbracket_\mathcal{M}.
\]
**Theorem (Veltman 1986; Kraus et al. 1990)**

For any consequence relation $\vdash$ closed under the rules of $\mathbb{P}$, there is a model $\mathcal{M}$, such that:

$$
\phi \vdash \psi \quad \text{iff} \quad \mathcal{M} \models \phi \sim \psi .
$$
Theorem (Veltman 1986; Kraus et al. 1990)

For any consequence relation $\models \sim$ closed under the rules of $P$, there is a model $M$, such that:

$$\varphi \models \sim \psi \quad \text{iff} \quad M \models \varphi \sim \psi.$$ 

Conversely, given any model $M$, there is a $P$-consequence relation $\models \sim$ satisfying the above equivalence.
Some have proposed a stronger rule than Cautious Monotonicity:

\[
\frac{\varphi \not\sim \psi_1 \quad \varphi \not\sim \neg \psi_2}{\varphi \land \psi_2 \not\sim \psi_1}
\]

(Rational Monotonicity)

The system that results from adding this rule to \( \mathbf{P} \) is called \( \mathbf{R} \).
Consequence Relations

Semantics for System $\mathbf{R}$

The models for $\mathbf{R}$ are just like those for $\mathbf{P}$, namely triples $(W, \preceq, V)$, except $\preceq$ is now assumed to be a total preorder:

$$\text{for all } s, t \in W, \text{ either } s \preceq t \text{ or } t \preceq s \text{ (or both).}$$
The models for $R$ are just like those for $P$, namely triples $(W, \preceq, V)$, except $\preceq$ is now assumed to be a total preorder:

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Semantics for System $\mathbf{R}$

The models for $\mathbf{R}$ are just like those for $\mathbf{P}$, namely triples $(W, \preceq, V)$, except $\preceq$ is now assumed to be a total preorder:

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Theorem (Lehmann & Magidor)

$\mathbf{R}$ consequence relations correspond to total models.
Another Angle: Belief Revision

- Imagine some agent with current knowledge $\mathcal{K}$, say, given by a set of formulas.
Another Angle: Belief Revision

- Imagine some agent with current knowledge $\mathcal{K}$, say, given by a set of formulas.

- Suppose this agent receives (veridical) information $\varphi$. How should $\mathcal{K}$ be revised in light of this fact?
Belief Revision

Alchourrón-Gärdenfors-Makinson (AGM) Belief Revision

1. $K^*$ is closed under (classical) consequence.
2. $\phi \in K^*$.
3. $K^* \phi \subseteq \text{Cl} (K \cup \{\phi\})$.
4. If $\neg \phi \not\in K$, then $\text{Cl} (K \cup \{\phi\}) \subseteq K^* \phi$.
5. $K^* \phi = \text{Cl} (\{\bot\})$, if and only if $\models \neg \phi$.
6. If $\models \phi_1 \leftrightarrow \phi_2$, then $K^* \phi_1 = K^* \phi_2$.
7. $K^* (\phi \land \psi) \subseteq \text{Cl} (K^* \phi \cup \{\psi\})$.
8. If $\neg \psi \not\in K^* \phi$, then $\text{Cl} (K^* \phi \cup \{\psi\}) \subseteq K^* (\phi \land \psi)$.
Alchourrón-Gärdenfors-Makinson (AGM) Belief Revision

1 $K^* \varphi$ is closed under (classical) consequence;
Alchourrón-Gärdenfors-Makinson (AGM) Belief Revision

1. \( \mathcal{K} \ast \varphi \) is closed under (classical) consequence;
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4. If \( \neg \varphi \not\in \mathcal{K} \), then \( \text{Cl}(\mathcal{K} \cup \{\varphi\}) \subseteq \mathcal{K} \ast \varphi \);

5. \( \mathcal{K} \ast \varphi = \text{Cl}(\{\perp\}) \), if and only if \( \models \neg \varphi \).
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5. $\mathcal{K} \ast \varphi = \text{Cl}(\{\bot\})$, if and only if $\vDash \neg \varphi$;

6. If $\vDash \varphi_1 \leftrightarrow \varphi_2$, then $\mathcal{K} \ast \varphi_1 = \mathcal{K} \ast \varphi_2$;

7. $\mathcal{K} \ast (\varphi \land \psi) \subseteq \text{Cl}(\mathcal{K} \ast \varphi \cup \{\psi\})$;
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Belief Revision

Theorem (Folklore)

1 Suppose we are given a knowledge base \( \mathcal{K} \) and a revision operation \(*\). Define a consequence relation \( \models \) so that

\[
\phi \models \psi \quad \text{iff} \quad \psi \in \mathcal{K} * \phi.
\]

Then \( \models \) satisfies all the rules of \( R \).
Belief Revision

Theorem (Folklore)

1. Suppose we are given a knowledge base $\mathcal{K}$ and a revision operation $\ast$. Define a consequence relation $\triangleright$ so that

$$\phi \triangleright \psi \quad \text{iff} \quad \psi \in \mathcal{K} \ast \phi.$$ 

Then $\triangleright$ satisfies all the rules of $\mathcal{R}$.

2. Suppose $\triangleright$ is a consequence relation satisfying $\mathcal{R}$, and such that $\phi \triangleright \bot$ only if $\models \neg \phi$. If we define

$$\mathcal{K} \triangleq \{ \psi \mid \top \triangleright \psi \} \quad \text{and} \quad \mathcal{K} \ast \phi \triangleq \{ \psi : \phi \triangleright \psi \},$$

then this gives us an AGM belief revision operation.
How do these systems relate to probability?
Probabilistic Semantics for System P

\[ \phi \models \psi \]

**REFL.**

\[ \phi \models \phi \]

**L-E**

\[ \phi_1 \leftrightarrow \phi_2 \models \phi_1 \models \psi \]

\[ \phi_2 \models \psi \]

**R-W**

\[ \psi_1 \rightarrow \psi_2 \models \phi \models \psi_1 \]

\[ \phi \models \psi_2 \]

**AND**

\[ \phi \models \psi_1 \]

\[ \phi \models \psi_2 \]

\[ \phi \models \psi_1 \land \psi_2 \]

**OR**

\[ \phi_1 \models \psi \]

\[ \phi_2 \models \psi \]

\[ \phi_1 \lor \phi_2 \models \psi \]

**Cautious Monotonicity**

\[ \phi \models \psi \]

\[ \phi \models \chi \]

\[ \phi \land \psi \models \chi \]
Let \( \Delta \) be a finite set \( \{ \alpha_1 \sim \beta_1, \ldots, \alpha_n \sim \beta_n \} \).
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We write

$$\Delta \models_A \varphi \sim \psi,$$
Probabilistic Semantics

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We write

$$\Delta \Vdash_A \varphi \sim \psi,$$

if $\forall \epsilon > 0, \exists \delta > 0$, such that for all $P : \mathcal{L} \rightarrow [0, 1]$:

$$\text{if } P(\beta_i | \alpha_i) > 1 - \delta \text{ for all } i \leq n, \text{ then } P(\psi | \varphi) > 1 - \epsilon.$$
Theorem (Adams 1966)

A default statement $\varphi \vdash \psi$ is derivable from a set $\Delta$ of default statements using the rules of $\mathbf{P}$, if and only if $\Delta \models_A \varphi \vdash \psi$. 
Definition
Define a conditional probability function to be any function $P : \mathcal{L} \times \mathcal{L} \rightarrow [0, 1]$ satisfying the following:

1. For some $\phi, \psi$, we have $P(\phi | \psi) \neq 1$.
2. $P(\phi | \phi) = 1$.
3. $P(\neg \phi | \psi) = 1 - P(\phi | \psi)$.
4. $P(\phi \land \psi | \chi) = P(\phi | \psi \land \chi) P(\psi | \chi)$. 
Probabilistic Semantics for System R

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Probabilistic Semantics for System $\mathcal{R}$

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2. $\mathbb{P}(\varphi|\varphi) = 1$.
3. $\mathbb{P}(\neg \varphi|\psi) = 1 - \mathbb{P}(\varphi|\psi)$.
4. $\mathbb{P}(\varphi \land \psi|\chi) = \mathbb{P}(\varphi|\psi \land \chi)\mathbb{P}(\psi|\chi)$.
Probabilistic Semantics for System \( R \)

Theorem (Harper 1975; Hawthorne 1998)

Given a conditional probability function \( P \), define a consequence relation \( \varphi \models \psi \), which holds just in case \( P(\psi \mid \varphi) = 1 \). Then \( \models \) is an \( R \) relation.
Probabilistic Semantics for System $\mathbf{R}$

**Theorem (Harper 1975; Hawthorne 1998)**

1. Given a conditional probability function $P$, define a consequence relation $\varphi \mid\sim \psi$, which holds just in case $P(\psi \mid \varphi) = 1$. Then $\mid\sim$ is an $\mathbf{R}$ relation.

2. For any $\mathbf{R}$ relation $\mid\sim$, there is a conditional probability function $P$ such that:

   - $P(\psi \mid \varphi) = 1$, iff $\varphi \mid\sim \psi$;
   - $P(\psi \mid \varphi) = 0$, iff $\varphi \mid\sim \neg \psi$;
   - $0 < P(\psi \mid \varphi) < 1$, iff $\varphi \mid\not\sim \psi$ and $\varphi \mid\sim \neg \psi$. 
Instead of starting with qualitative intuitions and trying to find quantitative representations, we can go the other direction.
Probability Threshold Semantics

• Instead of starting with qualitative intuitions and trying to find quantitative representations, we can go the other direction.

• Natural thought: $\varphi \sim \psi$ holds just in case $P(\psi | \varphi) > t$. 

Rational Monotonicity fails: Define $P$ so that:

$P(A \land \neg B \land \neg C) = P(\neg A \land B \land \neg C) = P(\neg A \land \neg B \land C) = \frac{1}{3}$

and let $t = 0.5$. Then $\top \sim A \lor B$ and $\top \not\sim \neg (A \lor C)$, but $A \lor C \not\sim A \lor B$. 

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- What is the logic of this?
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&= P(\neg A \land \neg B \land C) = 1/3
\end{align*}
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and let $t = 0.5$. Then $\top \sim A \lor B$
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\]

and let $t = 0.5$. Then $\top \sim A \lor B$ and $\top \not\sim \neg(A \lor C)$. 
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\end{align*}
\]

and let $t = 0.5$. Then $\top \sim A \lor B$ and $\top \not\sim \neg(A \lor C)$, but $A \lor C \sim A \lor B$. 
System $\mathcal{P}$

- **RefL.**
  \[ \varphi \models \varphi \]

- **L-E**
  \[ \frac{\varphi_1 \leftrightarrow \varphi_2 \quad \varphi_1 \models \psi}{\varphi_2 \models \psi} \]

- **L-E**
  \[ \frac{\varphi \models \psi_1 \quad \varphi \models \psi_2}{\varphi \models \psi_1 \land \psi_2} \]

- **R-W**
  \[ \frac{\psi_1 \rightarrow \psi_2 \quad \varphi \models \psi_1}{\varphi \models \psi_2} \]

- **R-W**
  \[ \frac{\varphi_1 \models \psi \quad \varphi_2 \models \psi}{\varphi_1 \lor \varphi_2 \models \psi} \]

- **And**
  \[ \frac{\varphi \models \psi_1 \quad \varphi \models \psi_2}{\varphi \models \psi_1 \land \psi_2} \]

- **Very Cautious Monotonicity**
  \[ \frac{\varphi \models \psi \land \chi}{\varphi \land \psi \models \chi} \]
System O

Probabilistic Semantics

\[ \phi \vdash \phi \] (Refl.)

\[ \vdash \phi_1 \leftrightarrow \phi_2 \quad \phi_1 \vdash \psi \quad \phi_2 \vdash \psi \] (L-E)

\[ \vdash \psi_1 \rightarrow \psi_2 \quad \phi \vdash \psi_1 \] (R-W)

\[ \phi \vdash \psi_1 \quad \phi \wedge \neg \psi_2 \vdash \psi_2 \] (W-A)

\[ \phi \wedge \chi \vdash \psi \quad \phi \wedge \neg \chi \vdash \psi \] (W-O)

\[ \phi \vdash \psi \wedge \chi \] (Very Cautious Monotonicity)
Another rule, called **Negation Rationality**: 

\[
\frac{\phi \pesim \psi}{\phi \land \chi \pesim \psi \quad \text{OR} \quad \phi \land \neg \chi \pesim \psi}
\]
Another rule, called Negation Rationality:

\[
\frac{\varphi \sim \psi}{\varphi \land \chi \sim \psi} \quad \text{OR} \quad \frac{\varphi \land \neg \chi \sim \psi}
\]

Theorem (Paris & Simmonds 2009)
Threshold semantics cannot be axiomatized by any finite set of rules of this form. In particular system \( \mathbf{O} \) is not complete.
A direct approach: acceptance rules

\[ \varphi \ | \sim \psi \text{ if and only if } \mathbb{P}(\psi \mid \varphi) > t \]

From a more general perspective, these can be seen as encoding a logic of **acceptance**:

*Given prior \( \mathbb{P} \), after learning \( \varphi \), accept \( \psi \).*

Implicit **acceptance rule**:
A direct approach: acceptance rules

\[ \varphi \sim_P \psi \text{ if and only if } P(\psi | \varphi) > t \]

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Implicit *acceptance rule*: “Accept \( \varphi \) if and only if \( P(\varphi) > t \)”

(*Lockean rule*)
A direct approach: acceptance rules

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*Given prior \( P \), after learning \( \varphi \), accept \( \psi \).*

Implicit acceptance rule: “Accept \( \varphi \) if and only if \( \mathbb{P}(\varphi) > t \)”  

*(Lockean rule)*

Compare this to

\[ \varphi \models \neg \psi \text{ iff } \psi \in K \ast \varphi . \]

Question: how do we get \( K \) and \( \ast \) from \( P \)?
Acceptance rules aim at answering the following question:

**Question**

*Suppose our information/degrees of belief are described by a probability space* $(W, \mathcal{E}, \mu)$. *Which propositions/hypotheses should we (tentatively) accept simpliciter?*

- Bridge between probabilistic credences and qualitative beliefs (collection of accepted propositions).
- Relating probability spaces to nonmonotonic consequence relations.
- Relating dynamics of probabilistic update to belief revision operators.
Acceptance rules and the logic of uncertain reasoning

Acceptance rule: translation between credences and all-or-nothing beliefs.

What are well-behaved acceptance rules?

Desiderata:
- Non-trivial: do not rule out the acceptance of uncertain hypotheses.
- Consistent: never accept contradictions.
- Conjunctive: accepting $A$ and accepting $B$ entails accepting $A \cap B$.
- High-probability: only accept high-probability propositions (above some threshold $t > 0.5$)
A direct approach: acceptance rules

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⚠️ The Lockean rule does not meet these criteria! (Lottery Paradox)
Acceptance rules on finite spaces

Let $(W, \mathcal{E}, \mu)$ be a finite probability space, $X, Y, \ldots \in \mathcal{E}$ and $\Delta_{\mathcal{E}}$ the set of all probability measures on $\mathcal{E}$. An acceptance rule is a map $\alpha : \Delta_{\mathcal{E}} \rightarrow \mathcal{E}$. The agent accepts $X \in \mathcal{E}$ if and only if $\alpha(\mu) \subseteq X$; i.e., $\alpha(\mu)$ is the strongest accepted proposition.
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The tracking problem

Revision & the Reverend: Bayes & Boole

$\mu$
Revision & the Reverend: Bayes & Boole

\[ \mu \xrightarrow{\text{Bayesian conditioning}} \mu(\cdot | X) \]
The tracking problem

Revision & the Reverend: Bayes & Boole

\[ \mu \xrightarrow{\mathcal{K}} \mu(\cdot | X) \]

Bayesian conditioning
The tracking problem

Revision & the Reverend: Bayes & Boole

\[ \mu \xrightarrow{\mu(\cdot \mid X)} \mu(\cdot \mid X) \]

Bayesian conditioning

\[ \mu \]

\[ K \]
The tracking problem

Revision & the Reverend: Bayes & Boole

\[ \mu \mid_X \rightarrow \mu(\cdot \mid X) \]

Bayesian conditioning

\[ K \star_X \rightarrow K^*X \]

revision operator
The tracking problem

Revision & the Reverend: Bayes & Boole

\[ \mu \xrightarrow{\mid X} \mu(\cdot \mid X) \]

Bayesian conditioning

\[ K \xrightarrow{\ast_X} K^*X \]

revision operator

\[ \text{acceptance} \]

\[ \text{acceptance} \]
The tracking problem

Dynamic compatibility: tracking

**Tracking.** A qualitative revision policy maps each $\mu \in \Delta \mathcal{E}$ to a belief set $\alpha(\mu)$ and a revision operator $\ast$ applicable to that belief set. It *tracks* Bayesian conditioning if, for any measure $\mu$ and proposition $X \in \mathcal{E}$ with $\mu(X) > 0$,

$$\alpha(\mu) \ast X = \alpha(\mu(\cdot \mid X)).$$

\[\begin{array}{c}
\mu \xrightarrow{\mid X} \mu(\cdot \mid X) \\
\text{Bayesian conditioning}
\end{array}\]

\[\begin{array}{c}
\alpha(\mu) \xrightarrow{\ast X} \alpha(\mu(\cdot \mid X))
\end{array}\]
Stability (Skyrms, Arló-Costa, Leitgeb)

Let \((\mathcal{W}, \mathcal{E}, \mu)\) a probability space and \(t \in (0.5, 1]\). A hypothesis \(H \in \mathcal{E}\) is \((\mu, t)\)-stable if and only if \(\forall Y \in \mathcal{E}\) such that \(H \cap Y \neq \emptyset\) and \(\mu(Y) > 0\), \(\mu(H | Y) \geq t\).

\(H\) is stable if it has resiliently high probability under new information.

\(\therefore\) On finite spaces, the collection of stable sets, written \(\mathcal{S}^t(\mu)\), is well-ordered by logical strength. (why?)
Stability (Skyrms, Arló-Costa, Leitgeb)

Let $S = (\mathcal{W}, \mathcal{E}, \mu)$ a probability space and $t \in (0.5, 1)$. We define the collection $L(\mu)$ of accepted hypotheses as follows:

$$L(\mu) := \{X \in \mathcal{E} \mid \exists A \in \mathcal{G}^t(\mu), A \subseteq X\}$$

$$= \{X \in \mathcal{E} \mid \tau(\mu) \subseteq X\}$$

where $\tau(\mu) = \min \mathcal{G}^t(\mu)$
The stability rule gives an AGM revision policy. Does this AGM policy commute with conditioning?

\[ \mu \xrightarrow{\text{inclusion}} \mu(\cdot | X) \]

\[ \xrightarrow{\text{compatibility}} \]

\[ L(\mu) \xrightarrow{\text{compatibility}} L(\mu) \times X = \mu \]

The revision \( L(\mu) \rightarrow L(\mu(\cdot | X)) \) is not in general AGM, as it can fail Inclusion.

\[ \text{Tracking fails} \]

for the stability rule.
Theorem (No-Go Theorem (Lin & Kelly, 2012))

Let $\alpha : \Delta_\mathcal{E} \to \mathcal{E}$ be any sensible acceptance rule. No AGM revision policy based on $\alpha$ tracks Bayesian conditioning.
Belief revision as ‘lossy’ Bayesian reasoning

In spite of the No-Go Theorem, is there still a way to bridge AGM revision and Bayesian conditioning? How can a Bayesian agent make sense of AGM revision?

Observation: AGM revision can be represented as Bayesian conditioning on a distinguished probabilistic representative of the agent’s qualitative belief state.

Consider an agent with a belief set $K$. Credences compatible with $K$: all $\mu$ such that $L(\mu) = K$. 
Max Entropy Principle

Entropy: \( \mathcal{H}(\mu) = \sum_{\omega \in \mathcal{W}} -\mu(\omega) \log \mu(\omega) \)

- Entropy as a measure of uncertainty.

**Maximum Entropy Principle (MEP):**

*Select a distribution with maximal entropy consistent with the constraints imposed by the available information (if such distributions exist).*

- Max Entropy distribution thought of as the least biased representation of the agent’s belief state, given the constraints.
Max Entropy Principle

In making inferences on the basis of partial information we must use that probability distribution which has maximum entropy subject to whatever is known. This is the only unbiased assignment we can make; to use any other would amount to arbitrary assumption of information which by hypothesis we do not have. [...] The maximum entropy distribution may be asserted for the positive reason that it is uniquely determined as the one which is maximally noncommittal with regard to missing information.

AGM from Bayes via Max Entropy

For finite probability spaces:

Let $K$ a consistent belief set on $\mathcal{E}$, and fix $t \in (0.5, 1)$. Then there is a unique maximal entropy distribution $\mu \in \Delta \mathcal{E}$ such that $L(\mu) = K$. Moreover, for any positive probability $X \in \mathcal{E}$, we have $L(\mu(\cdot | X)) = K^* X$, where $*$ is the AGM revision operator generated by $\mathcal{S}(\mu)$. 
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Belief revision as ‘lossy’ Bayesian reasoning

The same obtains even for a finer representation of belief: a plausibility ranking.

\[ \mu \rightarrow \mu(\cdot \mid X) \]

Max Entropy

\[ \mathcal{G} \rightarrow \mathcal{G} \mid X \]

Restriction to \( X \)

\[ L(\mu) \rightarrow L(\mu(\cdot \mid X)) \]

AGM

(For the revision to be autonomous—fully determined by the qualitative structure alone—the agent must retain the plausibility ranking \( \mathcal{G} \).)
AGM revision = stable belief + MaxEnt + Bayes

• AGM reasoners behave as if they were lossy Bayesian reasoners; they revise their beliefs as if they were relying on the maximum entropy representative of their belief state.

• Dynamic compatibility can be (approximately) recovered, and AGM represented as ‘coarse-grained’ Bayesian reasoning.

• A grain of salt: representation vs. rationalisation.

• A grain of truth? Resource-rational approaches to cognition [Lieder & Griffiths 2020].

Other approaches

Bridges: a potpourri

\[ \mu \xrightarrow{\text{Bayes}} \mu(\cdot | X) \]

acceptance rule \[ \alpha \]

\[ \alpha(\mu) \xrightarrow{*X} \alpha(\mu)^*X = \alpha(\mu(\cdot | X)) \]

revision operator

General models for revision policies: \((W, E, \sigma)\) with \(\sigma : E \rightarrow E\) a selection function: here, \(\sigma(A)\) is the revised state after learning \(A\).

A selection function specifies a belief set \(K = \sigma(W)\) together with a qualitative revision policy \(K^*A := \sigma(A)\).

\[ \varphi \sim \psi \text{ if and only if } \sigma(\llbracket \varphi \rrbracket) \subseteq \llbracket \psi \rrbracket \]
Other approaches

Bridges: a potpourri

\[ \mu \xrightarrow{x} \mu(\cdot \mid X) \]

\[ \alpha(\mu) \xrightarrow{\cdot \mid X} \alpha(\mu)X = \alpha(\mu(\cdot \mid X)) \]

**General models for revision policies:** \((W, E, \sigma)\) with \(\sigma : E \rightarrow E\) a **selection function:** here, \(\sigma(A)\) is the revised state after learning \(A\).

A selection function specifies a belief set \(K = \sigma(W)\) together with a qualitative revision policy \(K^*A := \sigma(A)\).

\[ \varphi \sim \psi \text{ if and only if } \sigma([\varphi]) \subseteq [\psi] \]
Other approaches

Bridges: a potpourri

\[
\begin{align*}
\mu & \xrightarrow{\mu(X)} \mu(\cdot | X) \\
\alpha(\mu) & \xrightarrow{\alpha(\mu)^X} \alpha(\mu)(\cdot | X)
\end{align*}
\]

Bayes acceptance rule? \(\alpha\)

AGM acceptance rule? \(\kappa\)
Other approaches

**AGM = Stability + MaxEnt + Bayes**

\[ \mu \xrightarrow{\text{Bayes}} \mu(\cdot|X) \]

\[ \text{Max Ent} \quad L \quad \mu \]

\[ L(\mu) \xrightarrow{\ast} L(\mu)^*X = L(\mu(\cdot|X)) \]

Nonmonotonic consequence relations closed under **system R**.

\[ \sigma(X) := \min_{\preceq}(X) \quad \text{for } \preceq \text{ a total preorder on } W. \]

\[ \omega_i \preceq \omega_j \quad \text{if and only if for any } (\mu, t)\text{-stable } H, \text{ if } \omega_i \in H \text{ then } \omega_j \in H \]
Other approaches

**System \( \mathbf{P} = \text{Odds Ratio} + \text{Bayes} \)** *(Lin and Kelly, 2012)*

\[
\begin{align*}
\mu & \xrightarrow{x} \mu(\cdot \mid X) \\
\kappa & \downarrow \\
\kappa(\mu) & \xrightarrow{*x} \kappa(\mu)^*X = \kappa(\mu(\cdot \mid X))
\end{align*}
\]

- **Bayes**
- **odds-ratio rule**
- **Shoham revision**

Nonmonotonic consequence relations closed under system \( \mathbf{P} \).

\[
\sigma(X) := \min_{<}(X) \text{ for } < \text{ a } \text{partial order} \text{ on } W.
\]

\[
\omega_i < \omega_j \text{ if and only if } \mu(\omega_i) > q \cdot \mu(\omega_j) \quad (q \geq 1)
\]
Other approaches

Probabilistically stable revision

\[ \mu \xrightarrow{x} \mu(\cdot | X) \]

\[ L(\mu) \xrightarrow{\text{Stability}} L(\mu(\cdot | X)) \]

\[ \mu \xrightarrow{\text{Bayes}} \mu(\cdot | X) \]

\[ L(\mu)^*X := L(\mu(\cdot | X)) \]
Probabilistically stable revision

\[ \mu \xrightarrow{\text{Bayes}} \mu(\cdot | X) \]

\[ L(\mu) \xrightarrow{\ast_{\times}} L(\mu)^{*}X := L(\mu(\cdot | X)) \]

Stability

Revision operator?
Other approaches

Probabilistically stable revision

\[ \mu \xrightarrow{\text{Bayes}} \mu(\cdot | X) \]

\[ L(\mu) \xrightarrow{s.s.s \text{ operator } \sigma_\mu} L(\mu)^* X := L(\mu(\cdot | X)) \]

Characterise strongest stable set operators \( \sigma_\mu : X \mapsto \tau(\mu(\cdot | X)) \).

Representation result: these operators \( \sigma \) encode the behaviour of a (special class of) comparative probability orders.

Probabilistically stable revision is not trackable using order minimisation.
Probabilistically stable revision

Theorem (M.)

Let \((W, \mathcal{E}, \sigma)\) be a selection structure. Then \(\sigma\) is a strongest stable set operator \(\sigma_\mu : X \mapsto \tau(\mu(\cdot | X))\) for some (regular) measure \(\mu\) on \(\mathcal{E}\) if and only if the following hold:

1. \((S1)\) \(\sigma(X) = \emptyset\) only if \(X = \emptyset\)
2. \((S2)\) \(\sigma(X) \subseteq X\)
3. \((S3)\) If \(\sigma(A) \cap B \neq \emptyset\), then \(\sigma(A \cap B) \subseteq \sigma(A) \cap B\)
4. \((S4)\) If \(X_i \setminus X_j \subseteq \sigma(X_i)\) for all \(i \neq j \leq n\), then \(\sigma(\bigcup_{i \leq n} X_i) \subseteq \bigcup_{i \leq n} \sigma(X_i)\)
5. \((SC)\) If \((A_i)_{i \leq n} = 0 (B_i)_{i \leq n}\) and \(\forall i \leq n, A_i \succ^* \sigma B_i\), then \(\forall i \leq n, A_i \preccurlyeq^* B_i\).

Say \(A \gg B\) iff \(\sigma(A \cup B) \subseteq A \setminus B\). We define

\[A \succ^*_\sigma B \iff \text{either } [A \gg B] \text{ or } [B \text{ is a } \mathcal{E}\text{-atom } \& B \gg A]\]

\(\leadsto (SC)\): Finite Cancellation-style axiom for comparative probability.
Other approaches

Probabilistically stable revision

\[
\begin{align*}
\phi \not\models \phi & \quad \text{(REFL.)} \\
\vdash \phi_1 \iff \phi_2 & \quad \phi_1 \not\models \psi \\
\phi_2 \not\models \psi & \quad \text{(L-E)} \\
\phi \not\models \psi_1 & \quad \phi \not\models \psi_2 \\
\phi \not\models \psi_1 \land \psi_2 & \quad \text{(AND)} \\
\phi \not\models \neg \psi & \quad \phi \not\models \chi \\
\phi \land \psi \not\models \chi & \quad \text{(RATIONAL MONOTONICITY)}
\end{align*}
\]
Probabilistically stable revision

\( \varphi \models \varphi \) \quad (\text{Refl.})

\( \models \varphi_1 \leftrightarrow \varphi_2 \quad \varphi_1 \models \psi \quad \varphi_2 \models \psi \) \quad (\text{L-E}) \quad \models \varphi_1 \rightarrow \varphi_2 \quad \varphi \models \psi_1 \quad \varphi \models \psi_2 \) \quad (\text{R-W})

\( \varphi \models \psi_1 \quad \varphi \models \psi_2 \) \quad \varphi \models \psi_1 \land \psi_2 \) \quad (\text{AND})

\( \varphi_1 \models \psi \quad \varphi_2 \models \psi \) \quad \varphi_1 \lor \varphi_2 \models \psi \) \quad (\text{OR})

\( \varphi \models \neg \psi \quad \varphi \models \chi \) \quad (\text{Rational Monotonicity})
Other approaches

Probabilistically stable revision

\[ \varphi \models \varphi \tag{REFL.} \checkmark \]

\[ \vdash \varphi_1 \leftrightarrow \varphi_2 \quad \varphi_1 \models \psi \quad \varphi_2 \models \psi \tag{L-E} \checkmark \]

\[ \vdash \varphi_1 \rightarrow \varphi_2 \quad \varphi \models \psi_1 \quad \varphi \models \psi_2 \tag{R-W} \checkmark \]

\[ \varphi \models \psi_1 \quad \varphi \models \psi_2 \quad \varphi \models \psi \wedge \psi \tag{AND} \checkmark \]

\[ \varphi \models \psi \quad \varphi \models \psi \wedge \psi \quad \varphi \models \psi \rightarrow \psi \tag{Rational Monotonicity} \checkmark \]

\[ \varphi \models \psi \quad \varphi \models \neg \psi \quad \varphi \models \psi \wedge \psi \models \psi \tag{OR} \times \]

\[ \varphi \models \psi \quad \varphi \models \psi \wedge \psi \models \psi \]

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Thomas Icard & Krzysztof Mierzewski: Default Reasoning, Acceptance Rules, The Quantitative/Qualitative Interface 41
We saw:

- Common systems for non-monotonic reasoning, their standard ‘qualitative’ and probabilistic semantics, and belief revision;
- The tracking problem: in what sense can logical belief revision be compatible, or emerge from, probabilistic reasoning and Bayesian updating in particular?
- Acceptance rules as bridge principles between quantitative and all-or-nothing beliefs, and the non-monotonic/defeasible reasoning principles they validate.
Next time:

- What is the relationship between measure-theoretic probability and probability functions on formal languages?
- When logic and probability shed light on each other: random structures and the 0-1 law.