Laboratory Games and Quantum Behaviour: The Normal Form with a Separable State Space

Peter J. Hammond*

September 12, 2011

Abstract

The subjective expected utility (SEU) criterion is formulated for a particular four-person "laboratory game" that a Bayesian rational decision maker plays with Nature, Chance, and an Experimenter who influences what quantum behaviour is observable by choosing an orthonormal basis in a separable complex Hilbert space of latent variables. Nature chooses a state in this basis, along with an observed data series governing Chance's random choice of consequence. When Gleason's theorem holds, imposing quantum equivalence implies that the expected likelihood of any data series w.r.t. prior beliefs equals the trace of the product of appropriate subjective density and likelihood operators.

1 Introduction and Outline

1.1 Subjective Expected Utility (SEU) under Quantum Uncertainty

Heisenberg's uncertainty principle is a prominent example in quantum mechanics where observing one physical variable, such as a particular particle's position vector, may preclude observing simultaneously some other variable, such as the same particle's momentum. Physical laws force a choice between mutually exclusive experiments that can observe either the particle's position, or its momentum, but not both simultaneously. Similarly, in the two-slit laboratory experiment famously discussed by Feynman (1951), any apparatus that detects which slit each particle in a stream passes through will destroy whatever wave-like interference effects would otherwise have been observed.

The difficulties posed by such "weird" quantum behaviour have not deterred Deutsch (1999), Pitowsky (2003) and various successors from seeking decision-theoretic foundations for quantum probability. These include "quantum Bayesians" who have sought to extend Bayesian statistics to a quantum setting.[1] Much of this literature, however, limits itself to: (i) a finite-dimensional Hilbert space of possible quantum states, which obviously excludes many important physical phenomena; (ii) what economists call a "risk-neutral" agent, whose decisions maximize the subjective expectation of *wealth* rather than that of the *utility of wealth*. The latter SEU criterion, of course, is typical in decision theory, and allows the agent's preferences to be risk averse or even risk seeking.

There is a more serious obstacle in SEU theory, however, especially in what many of us regard as a superior version due to Anscombe and Aumann (1963). First, the outcomes of what they call

*Department of Economics, University of Warwick, Coventry CV4 7AL, UK; e-mail: `p.j.hammond@warwick.ac.uk`. This is a significant revision of the paper presented to the "Quantum Physics meets TARK" workshop held at the University of Groningen on 15th July 2011. The European Commission provided research support from 2007 to 2010 for a Marie Curie Chair under contract number MEXC-CT-2006-041121. Without implicating him in any errors, many thanks also to Patrick Suppes whose stimulating discussions at Stanford, along with foundational work such as Suppes (1961, 1976), did so much to provoke and inform my interest in quantum probability. And to Ariane Lambert-Mogiliansky, as well as the workshop organizer Sonja Smets, for more encouragement than I deserved.

[1] See in particular Barnum *et al.* (2000), Schack, Brun and Caves (2001), and Caves, Fuchs and Schack (2002).

"roulette lotteries" have hypothetical "objective" probabilities attached to their outcomes, as in von Neumann and Morgenstern (1953). By contrast, the outcomes of "horse lotteries" are given subjective probabilities that depend on tastes, with ratios equal to the agent's (constant) willingness to exchange, contingent on different relevant events, shifts in objective probability away from any one consequence toward a better alternative — see Hammond (1998b). Yet, like the hidden parameter vector of any likelihood function in Bayesian statistics, quantum states are inherently latent variables which can only be observed indirectly, at best. And it would be remarkably futile, of course, to place bets on a horse race which is run in a fog so thick that no physical apparatus could possibly establish its result.

This paper, accordingly, develops a Bayesian decision model that, when combined with Bayesian statistics, may be able to meet the quantum Bayesians' principal aims. To do so, it applies the SEU hypothesis to a "laboratory" game in normal form that faces a decision maker with uncertainty about: (i) quantum states belonging to a complex Hilbert space; (ii) observations in a space of classical physical configurations.[2] This framework is rich enough to accommodate many of the stochastic quantum phenomena that physicists have been able to predict so precisely in experimental settings. For now, we limit our analysis to a separable Hilbert space where operators have matrix representations;[3] non-separable Hilbert spaces rich enough to allow general "likelihood operators", whose spectral decompositions may require integrals instead of only sums, are left for later work.

1.2 Outline of Paper

Section 2 sets out a four-person "laboratory game".[4] Its first three players are the Decision Maker ($\mathbf{D}$), Nature ($\mathbf{N}$), and Chance ($\mathbf{C}$), who all have essential roles in any non-trivial game that is consistent with the Anscombe–Aumann approach to subjective probability. But here they are joined by a fourth player — an Experimenter ($\mathbf{E}$) whose strategy, possibly in the form of a choice of physical apparatus, determines (perhaps inadvertently) what data generating process (DGP) is postulated to explain the observed result x of the experiment. In this setting, SEU theory requires player $\mathbf{D}$ to have a subjective probability measure defined on an appropriate σ-algebra over the space of triples (e, s, x) that combine: (i) player $\mathbf{E}$'s choice of experiment e; (ii) player $\mathbf{N}$'s choice, consistent with e, of both a latent parameter vector s and data observation x. Though in principle player $\mathbf{D}$'s subjective probability of any observable event could depend on which experiment e is used to detect its presence or absence, we follow Vorob′ev (1962) in postulating that it does not.[5]

Following the discussion in section 2 of general experiments, the next section 3 specifies the laboratory game in a quantum setting. Each state s chosen by Nature is identified with a "pure state" or "wave function" which is represented by a *latent stochastic parameter* vector $\mathfrak{h}$ in the unit sphere $\mathcal{S}$ of a separable Hilbert space $\mathcal{H}$ over the complex field $\mathbb{C}$. We assume that $\mathbf{E}$'s choice of e, as interpreted by Nature, determines an orthonormal basis $K_e \subset \mathcal{H}$, along with a compatible equivalence relation $\sim_e$ of physical instinguishability defined on the multi-dimensional space of all possible observations x. Quantum theory, however, treats all the orthonormal subsets of $\mathcal{H}$ that have the same closed linear span as one equivalent event with the same probability. Then the probabilities of different observations x can usually be calculated by applying the standard trace formula extension of the Born squared modulus rule to the product of appropriate density and likelihood operators on $\mathcal{H}$.

The final Section 4 contains concluding remarks and an agenda for further research.

[2]Danilov and Lambert-Mogiliansky (2008, 2010) consider partially ordered sets in a more general framework that admits as a special case orthonormal subsets of (finite-dimensional) Hilbert space. La Mura (2009) limits himself to "projective" expected utility on finite-dimensional Euclidean space.

[3]Recall that a topological space is *separable* if there is a countable subset whose closure is the whole space.

[4]There are key differences from the games considered by Shafer and Vovk (2001, pp. 189–191) or Pitowsky (2003).

[5]Hess and Philipp (2005) appear to have been first to realize how relevant Vorob′ev's insufficiently appreciated work is to quantum theory, though Pitowsky (1994) does note Boole's (1862) related ideas from exactly 100 years earlier. See also Khrennikov (2008). Somewhat similar ideas appear in work by Slavnov (2001) and by Janssens (2004), who nevertheless seem unaware of Vorob′ev's contribution.

2 A Four-Person Laboratory Game

2.1 Players, Strategies, and Payoffs

Anscombe and Aumann (1963) limited themselves to decision problems with a finite set S of possible states s. Our laboratory game allows: (i) a more general *Polish* (i.e., complete and separable metric) **data space** X, with its Borel σ-algebra $\mathcal{B}$, whose typical element x is a time series that describes in full the observed random result of the laboratory experiment; (ii) a second measurable **state space** $(S, \mathcal{S})$, whose typical element s is a **latent variable** parameter vector of the stochastic process that, for each possible choice of laboratory experiment, generates the observed data outcome $x \in X$. As usual in Bayesian statistics, the latent parameter vector s cannot be observed directly; at best it can be inferred from x using Bayes' rule to determine an updated posterior probability distribution over S.

To avoid inessential measurability issues, we assume that the game involves a non-empty finite set E of possible experiments, as well as a finite **consequence domain** Y. To avoid trivialities, we assume that Y has at least two elements, one of which is definitely superior to the other.

We now consider a game with the following four players and their respective strategy sets:

Player D is a Decision Maker, who chooses a decision strategy or action a from a non-empty set A.

Player E is an Experimenter, who chooses an experiment $e \in E$ that then determines a foursome $(\sim_e, \mathcal{X}_e, S_e, \mathcal{S}_e)$ consisting of: (i) an equivalence relation $\sim_e$ of **observational indistinguishability** on X, which for each $x \in X$ induces the **information set** $[x]_e \subseteq X$ as the unique $\sim_e$-equivalence class that includes x; (ii) a corresponding sub-σ-algebra $\mathcal{X}_e$ of $\mathcal{B}$ on X, whose members take the form $\cup_{x \in B} [x]_e$ for some $B \in \mathcal{B}$ — that is, if $x \in B \in \mathcal{X}_e$ and $x' \sim_e x$, then $x' \in B$; (iii) an $\mathcal{S}$-measurable subset $S_e \subset S$ of states relevant to the experiment $e \in E$; (iv) a σ-algebra $\mathcal{S}_e \subseteq \mathcal{S}$ of measurable subsets of the relevant state space S_e.

Player N is Nature, whose choice for each possible $e \in E$ is modelled as the outcome $(s_e, [x]_e)$ of a "horse lottery" which combines a latent state $s_e \in S_e$ with an equivalence class or information set $[x]_e \subset X$ of observationally indistinguishable data series $x \in X$.

Player C is Chance, who, given the chosen experiment $e \in E$: (i) is informed of the pair $(a, [x])$ consisting of player $\mathbf{D}$'s action $a \in A$, along with player $\mathbf{N}$'s choice of equivalence class $[x] = [x]_e \subset X$; (ii) remains otherwise entirely uninformed about either the state $s \in S$ or the experiment $e \in E$; (iii) as a function of (a, x), sets up a "roulette lottery" $Y \ni y \mapsto \lambda(y; a, x) \in [0, 1]$ such that, for each fixed $a \in A$ and $y \in Y$, there is an induced $\mathcal{B}$-measurable map $X \ni x \mapsto \lambda(y; a, x)$, which also satisfies $\lambda(y; a, x') = \lambda(y; a, x)$ for all $x' \in [x]$.

Note that this formulation allows Heisenberg's uncertainty principle, for example, to be respected by having the variable equivalence relation $\sim_e$ treat either the position or the momentum vector of a specific particle as unobservable. Furthermore, player $\mathbf{D}$'s choice of action $a \in A$ determines, in effect, a bet that has to settled using only the *observed* outcome of any experiment.

To complete the description of the game, we must specify the four players' payoff functions. In fact players $\mathbf{E}$, $\mathbf{N}$ and $\mathbf{C}$ will be treated as passive, meaning that their payoffs are arbitrary constants, entirely independent of the game's outcome. Player $\mathbf{D}$'s preferences, on the other hand, are assumed to concern only the random consequences $y \in Y$ governed by the known probability law $(a, x, y) \mapsto \lambda(y; a, x)$; in particular, they do not depend on the latent state s.

2.2 Subjective Expected Utility (SEU) under Experimental Uncertainty

In the context of this laboratory game, standard SEU theory requires player $\mathbf{D}$, our decision maker, to have a unique triple consisting of:

1. a Bayesian prior subjective probability measure[6] $\mathcal{F} \ni F \mapsto \pi(F) \in [0,1]$ over the sample space $\Omega := \{(e,s,x) \in E \times S \times X \mid s \in S_e\}$ of triples describing player **E** and **N**'s combined strategy choices, where $\mathcal{F}$ denotes the σ-algebra of all sets $F \subseteq \Omega$ whose respective sections

$$F_e := \{(s,x) \in S_e \times X \mid (e,s,x) \in F\} \tag{1}$$

are $\mathcal{S}_e \otimes \mathcal{X}_e$-measurable, for all $e \in E$;

2. a cardinal equivalence class of von Neumann–Morgenstern utility functions (NMUFs) $Y \ni y \mapsto u(y) \in \mathbb{R}$;[7]

3. a preference ordering[8] over A represented by the SEU function

$$A \ni a \mapsto U(a) := \int_\Omega \left[\sum\nolimits_{y\in Y} \lambda(y;a,x)\, u(y)\right] d\pi \tag{2}$$

for each NMUF u in the cardinal equivalence class.

Define $E \ni e \mapsto p_e := \pi(\{e\} \times S_e \times X) \in [0,1]$ as the marginal probability density function applying to player **E**'s choice of experiment $e \in E$. Then, given any $e \in E$ with $p_e > 0$, define

$$\mathcal{S}_e \otimes \mathcal{X}_e \ni J \mapsto q_e(J) := \pi(\{e\} \times J)/p_e \quad \text{for all } J \in \mathcal{S}_e \otimes \mathcal{X}_e, \tag{3}$$

thus making q_e the conditional probability measure on $S_e \times X$.

Let $\Delta(X, \mathcal{X}_e)$ denote the set of probability measures on $(X, \mathcal{X}_e)$. Because $(X, \mathcal{X}_e)$ is Polish, the prior probability measure $\mathcal{F} \ni F \mapsto \pi(F) \in [0,1]$ has values given by the composition

$$\pi(F) = \sum\nolimits_{e\in E} p_e\, q_e(F_e) = \sum\nolimits_{e\in E} p_e \int_{S_e} \left[\int_X 1_F(e,s,x)\, \ell_e(dx|s)\right] P_e(ds) \tag{4}$$

where, given any $e \in E$ with $p_e > 0$, Nature randomly chooses a **Bayesian probability system** that combines:

1. a state $s \in S_e$ according to a **prior** probability measure $P_e(ds)$ on $(S_e, \mathcal{S}_e)$ satisfying $P_e(G) = \pi(\{e\} \times G \times X)/p_e$ for each $G \in \mathcal{S}_e$.

2. conditional on each $s \in S_e$, an experimental outcome $x \in X$ according to the **data generating process** (DGP) or **likelihood function** $S_e \ni s \mapsto \ell_e(dx|s) \in \Delta(X, \mathcal{X}_e)$ that is $\mathcal{S}_e$-measurable and satisfies

$$\int_{S_e} \left[\int_X 1_J(s,x)\, \ell_e(dx|s)\right] P_e(ds) = q_e(J) \quad \text{for all } J \in \mathcal{S}_e \otimes \mathcal{X}_e. \tag{5}$$

Then, for each fixed $e \in E$ with $p_e > 0$, the marginal ξ_e on $(X, \mathcal{X}_e)$ of q_e is the probability measure

$$\mathcal{X}_e \ni B \mapsto \xi_e(B) := q_e(S_e \times B) = \int_{S_e} \ell_e(B|s)\, P_e(ds) \tag{6}$$

whose value is the expected likelihood $\ell_e(B|s)$ of $B \in \mathcal{X}_e$, given the prior $P_e(ds)$ on $(S_e, \mathcal{S}_e)$. Finally, each of player **D**'s possible actions $a \in A$ generates the unconditional roulette lottery

$$Y \ni y \mapsto \eta_e(y;a) := \int_X \lambda(y;a,x)\, \xi_e(dx) = \int_{S_e} \left[\int_X \lambda(y;a,x)\ell_e(dx|s)\right] P_e(ds) \tag{7}$$

[6] Note that π is a countably additive probability *measure*, as in Arrow (1971) and Fishburn (1982). By contrast, Savage (1954) derived only finitely additive probability, as do Gyntelberg and Hansen (2009) in the quantum context.

[7] Two NMUFs $u, \tilde{u} : Y \to \mathbb{R}$ are *cardinally equivalent* iff there exist both an additive constant $\alpha \in \mathbb{R}$ and a positive multiplicative constant $\gamma \in \mathbb{R}$ such that $\tilde{u}(y) \equiv \alpha + \gamma u(y)$ on Y.

[8] A *preference ordering* on A is a binary relation $\succsim$ on A that is complete and transitive. It is *represented* by the *utility function* $U : A \to \mathbb{R}$ just in case $U(a) \geq U(a')$ iff $a \succsim a'$, for all $a, a' \in A$.

on Y whose conditional expected utility to player $\mathbf{D}$, given $e \in E$, is specified by the function

$$A \ni a \mapsto V_e(a) := \sum\nolimits_{y \in Y} \eta_e(y; a)\, u(y). \tag{8}$$

This decomposition allows the SEU formula (2) to be rewritten as $U(a) = \sum_{e \in E} p_e V_e(a)$.

2.3 A Vorob′ev Consistent Probability System

As the experiment $e \in E$ varies, the Bayesian structure set up in Section 2.2 places no restrictions at all on inudced changes in: either (i) the subjective prior probability measure $\mathcal{S}_e \ni G \mapsto P_e(G)$ over S_e; or (ii) the $\mathcal{S}_e$-measurable DGP $S_e \ni s \mapsto \ell_e(dx|s)$ on $(X, \mathcal{X}_e)$. Yet presumably good experiments should perturb both of these as little as possible. So, as e varies over E, only the domain $(S_e, \mathcal{S}_e)$ and the σ-algebra $\mathcal{X}_e$ will be allowed to change. By contrast, the values of the probability measure $\mathcal{S}_e \ni G \mapsto P_e(G)$ and of the DGP $S_e \ni s \mapsto \ell_e(dx|s) \in \Delta(X, \mathcal{X}_e)$ must both remain constant.

Formally, therefore, there must exist a **probabilistic set function** $P : \mathcal{S}^* \to [0, 1]$ defined on the domain $\mathcal{S}^* := \cup_{e \in E} \mathcal{S}_e$, along with a **likelihood law** $(s, B) \mapsto \ell(dx|s) \in [0, 1]$ defined on $\cup_{e \in E}(S_e \times \mathcal{X}_e)$, both independent of e, such that for each $e \in E$, $G \in \mathcal{S}_e$, $B \in \mathcal{X}_e$ and $s \in \mathcal{S}_e$, the pair (P_e, ℓ_e) satisfy the **consistency conditions**:

$$\text{(i) } P_e(G) = P(G); \text{ (ii) } \ell_e(B|s) = \ell(B|s); \text{ (iii) } S_e \ni s \mapsto \ell(B|s) \text{ must be } \mathcal{S}_e\text{-measurable.} \tag{9}$$

These conditions require in particular that the various random data series $x \in X$ which can be observed in different experiments $e \in E$ must all belong to overlapping probability spaces $(X, \mathcal{X}_e, \xi)$, where only the σ-algebra $\mathcal{X}_e$ depends on e. That is, the collection

$$(X, \{\mathcal{X}_e\}_{e \in E}, \xi) \tag{10}$$

must meet Vorob′ev's (1962) definition of a **consistent probability system**.[9]

Note that the usual additivity condition $\xi(B \cup B') = \xi(B) + \xi(B')$ needs to hold *only if* the two sets B and B' are not only disjoint Borel subsets of X, but also belong to the same σ-algebra $\mathcal{X}_e$ for at least one common experiment $e \in E$.[10] Furthermore, given any countable[11] collection $\{B_n\}_{n \in N}$ of pairwise disjoint Borel subsets of X, the usual countable additivity condition

$$\xi\left(\bigcup\nolimits_{n \in N} B_n\right) = \sum\nolimits_{n \in N} \xi(B_n) \tag{11}$$

must hold *if* there exists at least one experiment $e \in E$ such that $B_n \in \mathcal{X}_e$ for all $n \in N$.

Finally, it is instructive to compare Vorob′ev consistency with a stronger condition for the existence of a Kolmogorov extension. The latter requires ξ to be defined throughout $\sigma(\mathcal{B}^*)$, the smallest σ-algebra containing all the sets in $\mathcal{B}^*$; this is typically much larger than the union $\mathcal{B}^*$. Moreover, (11) must hold for any collection $\{B_n\}_{n \in N}$ of pairwise disjoint subsets in $\sigma(\mathcal{B}^*)$.

3 Application to Quantum Experiments

3.1 Quantum States

In the quantum version of the game set out in Section 2, players $\mathbf{D}$, $\mathbf{E}$ and $\mathbf{C}$ have exactly the same strategy spaces. Moreover, there is still an arbitrary Polish space $(X, \mathcal{B})$ of possible experimental observations, with an indistinguishability relation $\sim_e$ on X and a corresponding σ-algebra $\mathcal{X}_e \subset \mathcal{B}$

[9]Discussing the obvious relationship to the quantum theoretic idea of context independence is deferred till later work.
[10]This is an obvious extension to σ-algebras of Griffiths' (2003) "single-framework rule" for Boolean algebras.
[11]We regard any finite set as countable, as well as any countably infinite set.

that both depend on the experiment $e \in E$. Furthermore, the random consequence $y \in Y$ is still the result of player **C**'s chosen roulette lottery, governed by the probability law $(a, x, y) \mapsto \lambda(y; a, x)$.

Now, however, there is a **quantum state space** $(S, \mathcal{S})$ where: (i) S becomes the unit sphere $\mathbb{S}$ of a particular physically relevant separable Hilbert space $\mathcal{H}$ over the complex field $\mathbb{C}$;[12] (ii) $\mathcal{S}$ is the Borel σ-algebra on $\mathbb{S}$ generated by those of its subsets that are relatively open w.r.t. the norm topology of $\mathcal{H}$. Also, for each experiment $e \in E$, the measurable space $(S_e, \mathcal{S}_e)$ has its own special structure that we shall now explain.

First, recall that a set $G \subset \mathcal{H}$ is *orthonormal* just in case the inner products of all its pairs $\mathfrak{h}, \mathfrak{h}' \in G$ satisfy $\langle \mathfrak{h}, \mathfrak{h}' \rangle = \delta_{\mathfrak{h}\mathfrak{h}'}$, where $G \times G \ni (\mathfrak{h}, \mathfrak{h}') \mapsto \delta_{\mathfrak{h}\mathfrak{h}'} \in \{0, 1\}$ is the Kronecker delta function. Obviously, any orthonormal set G is linearly independent and a subset of $\mathbb{S}$. When $\mathcal{H}$ has finite dimension d, then $\#G \leq d$; in any case, separability of $\mathcal{H}$ implies that G is countable.[13]

Given any subset $G \subseteq \mathcal{H}$ (not necessarily orthonormal), let $\overline{\mathrm{sp}}\, G$ denote the closure in $\mathcal{H}$ of the linear subspace spanned by G; it is also a linear subspace of $\mathcal{H}$. An orthonormal set $G \subset \mathcal{H}$ is *complete*, or an *orthonormal basis* of $\mathcal{H}$, if $\overline{\mathrm{sp}}\, G = \mathcal{H}$ — i.e., for any $\mathfrak{h} \in \mathcal{H}$, there exist matching countable sets of basis elements $\{\mathfrak{e}_n\}_{n \in N} \subset G$ and of scalars $\{c_n\}_{n \in N} \subset \mathbb{C}$ such that $\mathfrak{h} = \sum_{n \in N} c_n \mathfrak{e}_n$, where the sum converges in Hilbert norm if N is infinite. Let $\mathcal{G}$ and $\mathcal{K}$ denote the families of all orthonormal sets G and all orthonormal bases K, respectively, of the space $\mathcal{H}$.

In our laboratory game, we follow Section 2.1 in supposing that player **E**'s choice of experiment $e \in E$ determines a foursome $(\sim_e, \mathcal{X}_e, K_e, 2^{K_e})$, but with the general set $S_e \subset S$ replaced by an orthonormal basis $K_e \in \mathcal{K}$, and with the σ-algebra $\mathcal{S}_e$ replaced by the power set 2^{K_e}. We derive results, however, not just for a finite collection of foursomes $(\sim_e, \mathcal{X}_e, K_e, 2^{K_e})$ $(e \in E)$, but for arbitrary foursomes $(\sim, \mathcal{X}, K, 2^K)$, where $\sim$ is any equivalence relation partitioning X into Borel measurable subsets that generate the σ-algebra $\mathcal{X}$, and $K \in \mathcal{K}$ is any orthonormal basis of $\mathcal{H}$ with 2^K as its power set.[14] Reality constrains which of these foursomes are physically compatible, but that is not really relevant for the mathematical structure of our purely decision-theoretic results.

3.2 Equivalent Quantum Events

The usual physical formulation of quantum mechanics (QM) represents uncertain latent events, not as subsets whose members are latent vectors belonging to the unit sphere $\mathbb{S}$ of $\mathcal{H}$, but as members of the set $\mathcal{L}$ of all closed linear subspaces of $\mathcal{H}$ — or, equivalently, of the set $\mathcal{P}$ of all orthogonal projections of $\mathcal{H}$ onto such subspaces. In our framework, this implies that QM imposes a **quantum equivalence relation** $\overset{Q}{\sim}$ defined on the set $\mathcal{G}$ of all orthonormal subsets of $\mathcal{H}$ by

$$G \overset{Q}{\sim} G' \iff \overline{\mathrm{sp}}\, G = \overline{\mathrm{sp}}\, G'. \tag{12}$$

Thus, each orthonormal set $G \subset \mathbb{S}$ belongs to one equivalence class $[G]$ of events, which corresponds to the unique closed linear subspace $\overline{\mathrm{sp}}\, G \in \mathcal{L}$, as well as to the associated orthogonal projector $\Pi_{[G]} \in \mathcal{P}$ mapping $\mathcal{H}$ onto $\overline{\mathrm{sp}}\, G$. In fact there are obvious bijections

$$[G] \leftrightarrow \overline{\mathrm{sp}}[G] \leftrightarrow \Pi_{[G]} \tag{13}$$

between the three spaces: (i) $[\mathcal{G}] := \mathcal{G}/\overset{Q}{\sim}$ of equivalence classes of orthonormal subsets $G \in \mathcal{G}$; (ii) $\mathcal{L}$ of closed linear subspaces of $\mathcal{H}$; and (iii) $\mathcal{P}$ of orthogonal projections onto closed subspaces in $\mathcal{L}$. A well known result of quantum logic is that $\mathcal{L}$, or equivalently $\mathcal{P}$, can be given an orthomodular lattice structure. Then, for each fixed orthonormal basis $K \in \mathcal{K}$, the family $\mathcal{L}_K := \{\overline{\mathrm{sp}}[G] \mid G \subseteq K\}$, or its image $\mathcal{P}_K$ under the bijection between $\mathcal{L}$ and $\mathcal{P}$, is an orthocomplemented Boolean sublattice.

[12]The space $\mathcal{H}$ should be the domain of the relevant Hamiltonian energy operator that appears in the Schrödinger wave equation. Using the complex field $\mathbb{C}$, rather than the more intuitively appealing real field $\mathbb{R}$, allows any solution of this wave equation to be conveniently represented by means of unitary operators mapping $\mathbb{S}$ isometrically into itself.

[13]This is because any orthonormal basis is countable — see Friedman (1982, Lemma 6.4.7) for a concise proof.

[14]Equivalently, each K is a complete set of mutually orthogonal one-dimensional projectors, as in Caves, Fuchs and Schack (2002) for the case when $\mathcal{H}$ is finite-dimensional.

3.3 Quantum Prior Probabilities

Section 2.2 introduced, for each experiment $e \in E$, a Bayesian prior P_e on $(S_e, \mathcal{S}_e)$. In our quantum context this becomes, for each orthonormal basis $K \in \mathcal{K}$, a probability measure P_K on $(K, 2^K)$. Then, because the union $\cup_{K \in \mathcal{K}} 2^K$ obviously equals the entire family $\mathcal{G}$ of all orthonormal subsets of $\mathcal{S}$, the consistency condition (i) of (9) implies that there must exist a probabilistic set function $P \colon \mathcal{G} \to [0,1]$ such that $P_K(G) = P(G)$ whenever $G \subseteq K \in \mathcal{K}$.

Beside this first form of consistency, an obvious extra condition is **quantum consistency**, which requires all quantum equivalent events in $\mathcal{S}$ to have equal probability, independent of which orthonormal bases include them as subsets. Specifically, whenever two orthonormal sets G and G' satisfy $[G] = [G']$ because $\overline{\mathrm{sp}}\, G = \overline{\mathrm{sp}}\, G'$, we insist that $P(G) = P(G')$. Then the bijections (13) allow us to define the two set functions $\mathcal{L} \ni L \mapsto \nu(L) \in [0,1]$ and $\mathcal{P} \ni \Pi \mapsto \mu(\Pi) \in [0,1]$ to satisfy $P(G) = \nu(\overline{\mathrm{sp}}\, G) = \Pi_{[G]}$ whenever $G \in [G] \in [\mathcal{G}]$.

Suppose that $\{\Pi_n\}_{n \in N}$ is any countable collection of **pairwise orthogonal** projectors — i.e., they satisfy $\Pi_i \Pi_j = 0$ whenever $i, j \in N$ with $i \neq j$. This is equivalent to their respective ranges $\{L_n\}_{n \in N}$ being pairwise orthogonal closed subspaces of $\mathcal{H}$. For each $n \in N$, let G_n be an orthonormal basis of the subspace L_n. Then the sets $\{G_n\}_{n \in N}$ are pairwise disjoint, and their union $G := \cup_{n \in N} G_n$ is also an orthonormal set. Note also that the orthogonal projection onto $L := \overline{\mathrm{sp}}\, G$ satisfies $\Pi_{[G]} = \Pi_L = \sum_{n \in N} \Pi_n$. Hence, the mapping $\mathcal{P} \ni \Pi \mapsto \mu(\Pi) \in [0,1]$ satisfies the countable additivity condition

$$\mu\left(\sum\nolimits_{n \in N} \Pi_n\right) = \mu(\Pi_L) = P(G) = \sum\nolimits_{n \in N} P(G_n) = \sum\nolimits_{n \in N} \mu(\Pi_n) \tag{14}$$

whenever the projectors $\{\Pi_n\}_{n \in N}$ are pairwise orthogonal. In particular, the triple $(\mathcal{S}, \mathcal{P}, \mu)$ succinctly summarizes any prior probability system $\{P_K\}_{K \in \mathcal{K}}$ that satisfies quantum consistency, as well as consistency condition (i) of (9).

3.4 Quantum Likelihood Operators

Also in the quantum context, for each foursome $(\sim, \mathcal{X}, K, 2^K)$ specified in Section 3.1, we will impose the consistency condition (ii) of (9) on the likelihood function of Section 2.2. The result is a **quantum likelihood law** $\mathcal{S} \ni \mathfrak{h} \mapsto \ell(dx|\mathfrak{h}) \in \Delta(X, \mathcal{X})$, which specifies how the conditional DGP $\mathcal{X} \ni B \mapsto \ell(B|\mathfrak{h})$ changes as $\mathfrak{h}$ varies over the whole of $\mathcal{S}$.

Next, for each foursome $(\sim, \mathcal{X}, K, 2^K)$, consider the operator-valued set function

$$\mathcal{X} \ni B \mapsto L_{K\mathcal{X}}(B) := \sum\nolimits_{\mathfrak{e} \in K} \ell(B|\mathfrak{e})\, \Pi_{[\{\mathfrak{e}\}]} \tag{15}$$

whose value, for each $B \in \mathcal{X}$, is a self-adjoint or Hermitean (linear) **quantum likelihood operator** $\mathfrak{h} \mapsto L_{K\mathcal{X}}(B)\, \mathfrak{h}$ on $\mathcal{H}$. Note that $L_{K\mathcal{X}}(B)$ is a non-negatively weighted average of the orthogonal family $\Pi_{[\{\mathfrak{e}\}]}$ ($\mathfrak{e} \in K$) of one-dimensional projectors on $\mathcal{H}$, which implies that it is a *positive* operator — i.e., $\langle \mathfrak{h}, L_{K\mathcal{X}}(B)\, \mathfrak{h} \rangle \geq 0$ for all $\mathfrak{h} \in \mathcal{H}$. Finally, the countable range $\Lambda_{K\mathcal{X}}(B) := \cup_{\mathfrak{e} \in K} \{\ell(B|\mathfrak{e})\} \subseteq [0,1]$ of possible likelihood numbers must constitute the (pure point or discrete) spectrum of eigenvalues for the operator $L_{K\mathcal{X}}(B)$. Indeed, each (real) eigenvalue $r \in \Lambda_{K\mathcal{X}}(B)$ has its own eigenspace $\overline{\mathrm{sp}}\, \Gamma^r_{K\mathcal{X}}(B)$, where $\Gamma^r_{K\mathcal{X}}(B) := \{\mathfrak{e} \in K \mid \ell(B|\mathfrak{e}) = r\}$. Of course, eigenspaces associated with distinct eigenvalues must be orthogonal.

For each foursome $(\sim, \mathcal{X}, K, 2^K)$ specified in Section 3.1, both the joint distribution q_e defined by (3) and its marginal ξ_e defined by (6) also have their quantum counterparts. For q_e, it is the joint probability measure $q_{K\mathcal{X}}$ on the product $(K, 2^K) \times (X, \mathcal{X})$ given by

$$2^K \otimes \mathcal{X} \ni J \mapsto q_{K\mathcal{X}}(J) = \sum\nolimits_{\mathfrak{e} \in K} \left[\int_X 1_J(\mathfrak{e}, x)\, \ell(dx|\mathfrak{e})\right] P(\{\mathfrak{e}\}) = \sum\nolimits_{\mathfrak{e} \in K} \ell(J_\mathfrak{e}|\mathfrak{e}) P(\{\mathfrak{e}\}), \tag{16}$$

where $J_{\mathfrak{e}} := \{x \in X \mid (\mathfrak{e}, x) \in J\}$ denotes the appropriate section of the set J, for each $\mathfrak{e} \in K$. Following the pattern of (15), its operator equivalent is the set function defined by

$$2^K \otimes \mathcal{X}_e \ni J \mapsto Q_{K\mathcal{X}}(J) := \sum\nolimits_{\mathfrak{e}\in K} \ell(J_{\mathfrak{e}}|\mathfrak{e})\, \Pi_{[\{\mathfrak{e}\}]}. \tag{17}$$

The counterpart of ξ_e is the probability measure $\xi_{K\mathcal{X}}$ on $(X, \mathcal{X})$ which, because of (16), is given by

$$\mathcal{X} \ni B \mapsto \xi_{K\mathcal{X}}(B) = q_{K\mathcal{X}}(K \times B) = \sum\nolimits_{\mathfrak{e}\in K} \ell(B|\mathfrak{e}) P(\{\mathfrak{e}\}), \tag{18}$$

whose operator equivalent, which again follows (15) because of (17), is given by

$$\mathcal{X} \ni B \mapsto \Xi_{K\mathcal{X}}(B) := Q_{K\mathcal{X}}(K \times B) = \sum\nolimits_{\mathfrak{e}\in K} \ell(B|\mathfrak{e})\, \Pi_{[\{\mathfrak{e}\}]}. \tag{19}$$

3.5 Gleason's Theorem and the Trace Formula

Let $\{\mathfrak{e}_n\}_{n\in N}$ be any orthonormal basis of $\mathcal{H}$. Then the *trace* of any positive self-adjoint operator ρ is defined by $\operatorname{tr} \rho := \sum_{n\in N} \langle \mathfrak{e}_n, \rho \mathfrak{e}_n\rangle$ even if this sum of non-negative terms diverges to $+\infty$; its value is preserved by applying the same unitary transformation to all the vectors in $\mathcal{H}$, which is equivalent to changing its orthonormal basis. A *density operator* on $\mathcal{H}$ is any positive operator satisfying $\operatorname{tr} \rho = 1$.

Suppose the separable space $\mathcal{H}$ has dimension $d \geq 3$. By a corollary of Gleason's (1957) theorem due to Parthasarathy (1992, Theorem 9.18), the countable additivity condition (14) assures us that there exists a density operator ρ satisfying $\mu(\Pi) = \operatorname{tr}(\rho\, \Pi)$ for all projections $\Pi \in \mathcal{P}$. This implies the **trace formula** according to which, for every orthonormal set $G \in \mathcal{G}$, one has

$$P(G) = \mu(\Pi_{[G]}) = \operatorname{tr}(\rho\, \Pi_{[G]}) = \sum\nolimits_{m\in M} \operatorname{tr}(\rho\, \Pi_{[\{\mathfrak{h}_m\}]}) = \sum\nolimits_{m\in M} \langle \mathfrak{h}_m, \rho \mathfrak{h}_m\rangle \tag{20}$$

because $\Pi_{[G]} = \sum_{m\in M} \Pi_{[\{\mathfrak{h}_m\}]}$ for any orthonormal basis $\{\mathfrak{h}_m\}_{m\in M}$ of G, and the trace is linear.

For each foursome $(\sim, \mathcal{X}, K, 2^K)$ specified in Section 3.1, analogous trace formulae also apply to both the probability measure $J \mapsto q_{K\mathcal{X}}(J)$ defined by (16) and its marginal $B \mapsto \xi_{K\mathcal{X}}(B)$ defined by (18). Indeed, using (20) to substitute for each instance of the term $P(\{\mathfrak{e}\})$ in (16) gives

$$q_{K\mathcal{X}}(J) = \sum\nolimits_{\mathfrak{e}\in K} \ell(J_{\mathfrak{e}}|\mathfrak{e}) \operatorname{tr}(\rho\, \Pi_{[\{\mathfrak{e}\}]}) = \operatorname{tr}\left(\rho \sum\nolimits_{\mathfrak{e}\in K} \ell(J_{\mathfrak{e}}|\mathfrak{e}) \Pi_{[\{\mathfrak{e}\}]}\right) = \operatorname{tr}[\rho\, Q_{K\mathcal{X}}(J)] \tag{21}$$

because of (17). Using (18) and (21) with $J = K \times B$, then (19), yields the expected likelihood

$$\xi_{K\mathcal{X}}(B) = q_{K\mathcal{X}}(K \times B) = \operatorname{tr}[\rho\, Q_{K\mathcal{X}}(K \times B)] = \operatorname{tr}[\rho\, \Xi_{K\mathcal{X}}(B)] \tag{22}$$

of any observable Borel set $B \in \mathcal{X}$. It is the trace of the fixed density operator ρ — the quantum equivalent of a Bayesian prior — multiplied by the appropriate quantum operator $\Xi_{K\mathcal{X}}(B)$. Finally, for each action a, the quantum counterpart of (7) is the roulette lottery over consequences given by

$$Y \ni y \mapsto \eta_{K\mathcal{X}}(y; a) := \int_X \lambda(y; a, x)\, \xi_{K\mathcal{X}}(dx) = \operatorname{tr}\left[\rho \int_X \lambda(y; a, x)\, \Xi_{K\mathcal{X}}(dx)\right]. \tag{23}$$

4 Concluding Summary and Research Agenda

In our laboratory game, player **E**'s choice of experiment $e \in E$ gives rise to a physically compatible pair consisting of an indistinguishability relation $\sim_e$ on the space X of possible observations, along with an orthonormal basis K_e of a fixed complex separable Hilbert space $\mathcal{H}$ of quantum states. Together these determine a range of possible subjective likelihood laws mapping the measurable space $(K_e, 2^{K_e})$ into probability measures on the $\sim_e$-compatible σ-algebra $\mathcal{X}_e$ of X. When the trace formula (20) holds, quantum consistency implies there is a density operator ρ on $\mathcal{H}$ which, just as quantum Bayesians claim it should, fully characterizes player $\mathbf{D}$'s prior subjective beliefs over every orthonormal subset of $\mathcal{H}$. Indeed, the trace formula multiplies the operator ρ by a subjective likelihood operator to derive both: (i) the joint probability distribution over pairs consisting of quantum states and experimentally observed data; (ii) the marginal distribution of observed data alone.

Since separable Hilbert spaces permit only operators with discrete spectra, one important technical task is to extend the analysis to non-separable spaces where likelihood operators can have a much richer spectral decomposition consistent with continuous probability density functions.

More fundamentally, future work should consider extensive form laboratory games that explicitly allow sequences of observations and/or decisions to be made at different times. Such games will involve stochastic processes whose quantum state evolves according to Schrödinger's wave equation; this helps to motivate the otherwise rather counter-intuitive complex Hilbert space that quantum theory always uses. Then too "normal form invariance" may allow SEU theory to be justified using the "consequentialist perspective" of Hammond (1998a). Finally, considering extensive forms may also further elucidate interpretational issues such as how Bayesian updating of density operators relates to "collapsing" quantum states, as well as the vexing measurement problem.

For now, however, this paper has merely applied a standard Kolmogorov probability framework to the special space of experiment/state/outcome triples (e, s, x) that was introduced in Section 2.2. Then standard SEU theory, allied with Bayesian statistics, really is rich enough to analyse decision-making under uncertainty about what quantum behaviour could be observed in an experiment — at least when the quantum state space is separable. Nor is any special "quantum logic" needed beyond a structured family of Boolean σ-algebras, as in Griffiths (2003). Note too that those complex-valued latent variables, hitherto rather heavily disguised as "quantum probability amplitudes", now emerge as mere parameters of a subjective likelihood function defined over the relevant space of possible experimental outcomes. Moreover, there is no reason why these observed outcomes should be anything more obscure than data series in a space of entirely *classical* physical configurations.

References

Anscombe, F.J. and R.J. Aumann (1963) "A Definition of Subjective Probability," *Annals of Mathematical Statistics*, **34**, 199–205.

Arrow, K.J. (1971) *Essays in the Theory of Risk Bearing* (Amsterdam: North Holland).

Barnum, H., C.M. Caves, J. Finkelstein, C.A. Fuchs, and R. Schack (2000) "Quantum Probability from Decision Theory" *Proceedings of the Royal Society of London, A* 456: 1175–1182.

Boole, G. (1862) "On the Theory of Probabilities" *Philosophical Transactions of the Royal Society of London* 152: 225–52.

Caves, C.M., C.A. Fuchs, and R. Schack (2002) "Quantum Probabilities as Bayesian Probabilities" *Physical Review A* 65: 022305.

Danilov, V. and A. Lambert-Mogiliansky (2008) "Measurable Systems and Behavioral Sciences" *Mathematical Social Sciences* 55: 315–340.

Danilov, V.I. and A. Lambert-Mogiliansky (2010) "Expected Utility Theory under Non-Classical Uncertainty" *Theory and Decision* 68: 25–47.

Deutsch, D. (1999) "Quantum Theory of Probability and Decisions" *Proceedings of the Royal Society of London, A* 455: 3129–3137.

Feynman, R.P. (1951) "The Concept of Probability in Quantum Mechanics" *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability* (Berkeley, CA: University of California Press), 533–541.

Fishburn, P.C. (1982) *The Foundations of Expected Utility* (Dordrecht: D. Reidel).

Friedman, A. (1982) *Foundations of Modern Analysis* (New York: Dover).

Gleason, A.M. (1957) "Measures on Closed Subspaces of a Hilbert Space" *Journal of Mathematics and Mechanics* 6: 885–894.

Griffiths, R.B. (2003) *Consistent Quantum Theory* (Cambridge: Cambridge University Press).

Gyntelberg, J. and F. Hansen (2009) "Subjective Expected Utility Theory with 'Small Worlds'", Discussion Paper 09-26, University of Copenhagen, Department of Economics; `http://econpapers.repec.org/RePEc:kud:kuiedp:0926`.

Hammond, P.J. (1998a) "Objective Expected Utility: A Consequentialist Perspective" in S. Barberà, P.J. Hammond and C. Seidl (eds.) *Handbook of Utility Theory, Vol. 1: Principles* (Dordrecht: Kluwer Academic) ch. 5, pp. 143–211.

Hammond, P.J. (1998b) "Subjective Expected Utility" in S. Barberà, P.J. Hammond and C. Seidl (eds.) *Handbook of Utility Theory, Vol. 1: Principles* (Dordrecht: Kluwer Academic) ch. 6, pp. 213–271.

Hess, K. and W. Philipp (2005) "Bell's Theorem: Critique of Proofs with and without Inequalities" in *Proceedings of the Conference on the Foundations of Probability and Physics-3* 750: 150–155 (Melville, NY: American Institute of Physics).

Janssens, B. (2004) *Quantum Measurement: A Coherent Description* Master's Thesis, Radboud University, Nijmegen; `arXiv:quant-ph/0503009v1`.

Khrennikov, A.Yu. (2008) "Contextual Probabilistic Analysis of Bell's Inequality: Nonlocality, 'Death of Reality' or Non-Kolmogorovness?" in *Proceedings of the Second International Conference on Quantum, Nano and Micro Technologies* (ICQNM) (IEEE Computer Society: Washington, DC).

La Mura, P. (2009) "Projective Expected Utility: A Subjective Formulation" *Journal of Mathematical Psychology* 53: 408–414.

Parthasarathy, K.R. (1992) *An Introduction to Quantum Stochastic Calculus* (Basel: Birkhäuser Verlag).

Pitowsky, I. (1994) "George Boole's 'Conditions of Possible Experience' and the Quantum Puzzle" *British Journal for the Philosophy of Science* 45: 95–125.

Pitowsky, I. (2003) "Betting on the Outcomes of Measurements: A Bayesian Theory of Quantum Probability" *Studies in History and Philosophy of Modern Physics* 34: 395–414.

Savage, L.J. (1954, 1972) *Foundations of Statistics* (New York: John Wiley; and New York: Dover Publications).

Schack, R., T.A. Brun and C.M. Caves (2001) "Quantum Bayes Rule" *Physical Review A* 64: 014305.

Shafer, G. and V. Vovk (2001) *Probability and Finance: It's Only a Game!* (New York: John Wiley).

Slavnov, D.A. (2001) "Statistical Algebraic Approach to Quantum Mechanics" *Theoretical and Mathematical Physics* 129: 1385–1397.

Suppes, P. (1961) "Probability Concepts in Quantum Mechanics" *Philosophy of Science* 28: 378–389.

Suppes, P. (ed.) (1976) *Logic and Probability in Quantum Mechanics* (Dordrecht: D. Reidel).

Von Neumann, J. and O. Morgenstern (1944; 3rd edn. 1953) *Theory of Games and Economic Behavior* (Princeton: Princeton University Press).

Vorob′ev, N.N. (1962) "Consistent Families of Measures and Their Extensions" *Theory of Probability and its Applications* 7: 147–163.